A thin film approximation of the Muskat problem

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The Muskat problem I

Model for the motion of two immiscible fluids with different densities and viscosities in a porous medium (intrusion of water into oil).

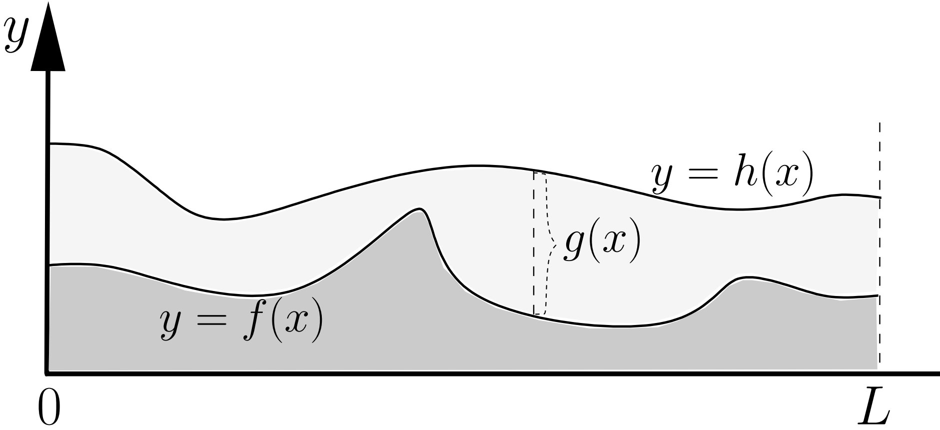
- Bottom of the porous medium: $y = 0$,
- Height of the lower fluid: $y = f(t, x)$, $\Gamma(f) := \{y = f\}$
- Domain occupied by the lower fluid:

$$\Omega(f) := \{(x, y) \in (0, L) \times (0, \infty) \ : \ y < f(t, x)\},$$

- Height of the upper fluid: $y = h(t, x)$, $\Gamma(h) := \{y = h\}$
- Domain occupied by the upper fluid:

$$\Omega(f, h) := \{(x, y) \in (0, L) \times (0, \infty) \ : \ f(t, x) < y < h(t, x)\}.$$

The Muskat problem II



The Muskat problem III

$$
\begin{array}{llll}
\Delta u_+ & = 0 & \text{in} & \Omega(f,g), \\
\Delta u_- & = 0 & \text{in} & \Omega(f), \\
\partial_t h & = -\mu_+^{-1} \langle \nabla u_+, (-\partial_x h, 1) \rangle & \text{on} & \Gamma(h), \\
u_+ & = G\rho_+ h - \gamma_d \kappa_{\Gamma(h)} & \text{on} & \Gamma(h), \\
\partial_t f & = -\mu_\pm^{-1} \langle \nabla u_\pm, (-\partial_x f, 1) \rangle & \text{on} & \Gamma(f), \\
u_+ - u_- & = G(\rho_+ - \rho_-) f + \gamma_w \kappa_{\Gamma(f)} & \text{on} & \Gamma(f), \\
\partial_y u_- & = 0 & \text{on} & \{y = 0\}.
\end{array}
$$

with initial data $0 < f_0 < h_0$ and

- $\rho_\pm, \mu_\pm$: density and viscosity of the fluid $\pm$,
- G: gravity constant,
- $u_\pm = p_\pm + G\rho_\pm y$, $\mathbf{v}_\pm = -\mu_\pm^{-1} \nabla u_\pm$ (Darcy's law),
- $\gamma_w, \kappa_{\Gamma(f)}$: surface tension and curvature of the interface $\Gamma(f)$,
- $\gamma_d, \kappa_{\Gamma(h)}$: surface tension and curvature of the interface $\Gamma(h)$.

Thin fluid layers: $h \ll L$

Scaling:

$$x = \tilde{x}, \quad y = \varepsilon\tilde{y}, \quad f = \varepsilon\tilde{f}, \quad h = \varepsilon\tilde{h}, \quad u_\pm = \tilde{u}_\pm, t = \tilde{t}/\varepsilon.$$

Formal asymptotic expansion in powers of ε:

$$\begin{aligned}
\partial_t f &= \mu_-^{-1} G(\rho_- - \rho_+)\partial_x(f\partial_x f) + \mu_-^{-1} G\rho_+\partial_x(f\partial_x h) \\
&\quad -\mu_-^{-1}\gamma_w\partial_x(f\partial_x^3 f) - \mu_-^{-1}\gamma_d\partial_x(f\partial_x^3 h) \\
\partial_t h &= \mu_-^{-1} G(\rho_- - \rho_+)\partial_x(f\partial_x f) + \mu_-^{-1} G\rho_+\partial_x(f\partial_x h) \\
&\quad +\mu_+^{-1} G\rho_+\partial_x((h-f)\partial_x h) - \mu_+^{-1}\gamma_d\partial_x((h-f)\partial_x^3 h) \\
&\quad -\mu_-^{-1}\gamma_w\partial_x(f\partial_x^3 f) - \mu_-^{-1}\gamma_d\partial_x(f\partial_x^3 h)
\end{aligned}$$

[Escher, Matioc & Matioc (2011)]

Thin film system

Neglecting the curvature terms ($\gamma_w = \gamma_d = 0$) and rescaling time give:

$$\begin{aligned} \partial_t f &= \partial_x(f\partial_x f) + R\partial_x(f\partial_x h), \\ \partial_t h &= \partial_x(f\partial_x f) + R\partial_x(f\partial_x h) + R_\mu\partial_x((h-f)\partial_x h), \end{aligned}$$

for $(t, x) \in (0, \infty) \times (0, L)$, supplemented with homogeneous Neumann boundary conditions and initial conditions $0 < f_0 < h_0$, where

$$R := \frac{\rho_+}{\rho_- - \rho_+} > 0 \quad \text{and } R_\mu := \frac{\mu_-}{\mu_+} R.$$

[Escher, Matioc & Matioc (2011)]

Strong solutions

- Assume that $\rho_- > \rho_+$ and $f_0, h_0 \in H^2_N(0,L)$, $0 < f_0 < h_0$ in $(0,L)$. Then there exists a local strong solution (f,h) satisfying $0 < f < h$ in $(0,T) \times (0,L)$.
- Energy functional: $2\mathcal{E}(f,h) := \|f\|_2^2 + R\,\|h\|_2^2$ with

$$\frac{d}{dt}\mathcal{E}(f,h) = -\int_0^L f\,(\partial_x f + R\partial_x h)^2\,dx - RR_\mu\int_0^L (h-f)(\partial_x h)^2\,dx \leq 0.$$

- Steady states are of the form (f_*, h_*) with constants $0 \leq f_* \leq h_*$.
- If $0 < f_* < h_*$ are constants, (f_*, h_*) is asymptotically exponentially stable (by the principle of linearised stability).

[Escher, Matioc & Matioc (2011)]

Alternative formulation

Define $g := h - f > 0$. Then (f, g) solves

$$
\begin{aligned}
\partial_t f &= (1+R)\partial_x(f\partial_x f) + R\partial_x(f\partial_x g) \quad \text{in} \quad (0,\infty)\times(0,L), \\
\partial_t g &= R_\mu \partial_x(g\partial_x f) + R_\mu \partial_x(g\partial_x g) \quad \text{in} \quad (0,\infty)\times(0,L), \\
\partial_x f &= \partial_x g = 0 \quad \text{on} \quad (0,\infty)\times\{0,L\}, \\
(f,g)(0) &= (f_0, g_0) \quad \text{in} \quad (0,L),
\end{aligned}
$$

with initial conditions $f_0 \geq 0$ and $g_0 \geq 0$ ($R > 0$, $R_\mu > 0$).

Degenerate parabolic system with full diffusion matrix

Properties

- $f \geq 0$ and $g \geq 0$ by the comparison principle,
- $\|f(t)\|_1 = \|f_0\|_1$ and $\|g(t)\|_1 = \|g_0\|_1$,
- Energy functional: $\mathcal{E}_2(f,g) := \|f\|_2^2 + R\|f+g\|_2^2$ with

$$\frac{d}{dt}\mathcal{E}_2(f,g) = -\int_0^L f\ (\partial_x f + R\partial_x h)^2\ dx - RR_\mu \int_0^L g(\partial_x h)^2\, dx \leq 0.$$

- Entropy functional:

$$\mathcal{E}_1(f,g) := \int_0^L (f\ln f - f + 1)\ dx + \frac{R}{R_\mu} \int_0^L (g\ln g - g + 1)\ dx$$

with

$$\frac{d}{dt}\mathcal{E}_1(f,g) = -\int_0^L \left(|\partial_x f|^2 + R\,|\partial_x h|^2\right)\ dx \leq 0.$$

Weak solutions: existence

Given $f_0, g_0 \in L^2(0,L)$, $f_0, g_0 \geq 0$, there is a global weak solution (f,g) satisfying

- $f \geq 0$ and $g \geq 0$,
- $f, g \in L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H^1(0,L))$,
- $\|f(t)\|_1 = \|f_0\|_1$ and $\|g(t)\|_1 = \|g_0\|_1$,
- "Weak entropy inequality":

$$\mathcal{E}_1(f(t),g(t)) + \int_0^t \left(\frac{1}{2} \|\partial_x f\|_2^2 + \frac{R}{1+2R} \|\partial_x g\|_2^2 \right) ds \leq \mathcal{E}_1(f_0,g_0)$$

- Energy inequality.

[Escher, L. & Matioc (2011)]

Weak solutions: difficulties

Difficulties: non-uniform parabolic system and full diffusion matrix.

General principle [Amann]:

positive lower bounds

$+$ $\longrightarrow$ global existence of strong solutions

$L^\infty(0, T; H^1(0, L))$ – bounds

regularisation

Weak solutions: regularised system

$$\begin{aligned}
\partial_t f_\varepsilon &= (1+R)\partial_x\left(f_\varepsilon \partial_x f_\varepsilon\right) + R\partial_x\left((f_\varepsilon - \varepsilon)\partial_x G_\varepsilon\right), \\
\partial_t g_\varepsilon &= R_\mu \partial_x\left((g_\varepsilon - \varepsilon)\partial_x F_\varepsilon\right) + R_\mu \partial_x\left(g_\varepsilon \partial_x g_\varepsilon\right),
\end{aligned}$$

with

$$F_\varepsilon := (1-\varepsilon^2\partial_x^2)^{-1} f_\varepsilon, \qquad G_\varepsilon := (1-\varepsilon^2\partial_x^2)^{-1} g_\varepsilon.$$

and supplemented with homogeneous Neumann boundary conditions and regularised initial conditions.

1. Comparison principle: $f_\varepsilon \geq \varepsilon$ and $g_\varepsilon \geq \varepsilon$.
2. Similar entropy and energy inequalities.
3. Coupling terms of lower order $\longrightarrow L^\infty(0,T;H^1(0,L))$-bounds.

(1)+(2)+(3) $\longrightarrow$ global existence for $(f_\varepsilon, g_\varepsilon)$ / (2) $\longrightarrow$ compactness

Weak solutions: large time behaviour

Exponential stability: there are $C > 0$ and $\omega > 0$ such that

$$\|f(t) - \bar{f}_0\|_2 + \|g(t) - \bar{g}_0\|_2 \leq C e^{-\omega t}, \qquad t \geq 0,$$

where

$$\bar{f}_0 := \frac{1}{L} \int_0^L f_0(x)\, dx \quad \text{and} \quad \bar{g}_0 := \frac{1}{L} \int_0^L g_0(x)\, dx.$$

Proof: compute

$$\frac{d}{dt} \left[\mathcal{E}_1(f,g) - \mathcal{E}_1(\bar{f}_0, \bar{g}_0) + \mathcal{E}_2(f,g) - \mathcal{E}_2(\bar{f}_0, \bar{g}_0) \right] \leq -C \left(\|\partial_x f\|_2^2 + \|\partial_x g\|_2^2 \right)$$

and use Poincaré-Wirtinger inequality.

Alternative approach

$$\begin{aligned}
\partial_t f &= \partial_x \left[f \, \partial_x \left((1+R) f + R g \right) \right] \quad \text{in} \quad (0,\infty) \times \mathbb{R}, \\
\partial_t g &= R_\mu \partial_x \left[g \, \partial_x (f+g) \right] \quad \text{in} \quad (0,\infty) \times \mathbb{R}, \\
(f,g)(0) &= (f_0, g_0) \quad \text{in} \quad \mathbb{R},
\end{aligned}$$

with integrable initial conditions $f_0 \geq 0$ and $g_0 \geq 0$ ($R > 0$, $R_\mu > 0$).

Energy functional: $\mathcal{E}_2(f,g) := \|f\|_2^2 + R\|f+g\|_2^2$

If $\|f_0\|_1 = \|g_0\|_1 = 1$, the above system is formally a gradient flow of $\mathcal{E}_2$ with respect to the 2-Wasserstein distance W_2 in $\mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R})$.

Notations

- $\mathcal{P}_2(\mathbb{R})$ is the set of probability measures in $\mathbb{R}$ with finite second moment.
- Given μ, ν in $\mathcal{P}_2(\mathbb{R})$, let $\Pi(\mu, \nu)$ be the set of probability measures γ in $\mathbb{R}^2$ with marginals μ and ν, that is,

$$\gamma(A \times \mathbb{R}) = \mu(A) \quad \text{and} \quad \gamma(\mathbb{R} \times A) = \nu(A)$$

  for all Borel sets A of $\mathbb{R}$.
- Given μ, ν in $\mathcal{P}_2(\mathbb{R})$, the 2-Wasserstein distance W_2 is defined by

$$W_2(\mu, \nu)^2 := \inf_{\gamma \in \Pi(\mu,\nu)} \int_{\mathbb{R}^2} |x - y|^2 \, d\gamma(x, y).$$

Time discrete scheme

Define

$$K := \left\{ (f,g) \in \mathcal{P}_2(\mathbb{R};\mathbb{R}^2) \ : \ (f,g) \in L^2(\mathbb{R};\mathbb{R}^2) \right\}.$$

Given $\tau \in (0,1)$ and $(f_0, g_0) \in K$, define the functional

$$\mathcal{F}_\tau(u,v) := \frac{1}{2\tau} \left(W_2^2(u,f_0) + \frac{R}{R_\mu} W_2^2(v,g_0) \right) + \mathcal{E}_2(u,v)$$

for $(u,v) \in K$ and solve the minimisation problem

$$\inf_{(u,v)\in K} \mathcal{F}_\tau(u,v).$$

Minimisers are actually in $H^1(\mathbb{R})$ [Matthes, McCann & Savaré (2009)]
(Work in progress with B.-V. Matioc)