# Convergence of an inverse problem for discrete wave equations.

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# Outline

# An inverse problem for the wave equation

Let us consider the wave equation in a smooth bounded domain $\Omega$ :

$$\begin{cases} \partial_{tt} y - \Delta_x y + p(x) y = f, & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = g(t,x), & (t,x) \in (0,T) \times \partial\Omega \\ (y(0,x), \partial_t y(0,x)) = (y^0(x), y^1(x)), & x \in \Omega. \end{cases}$$

Unknown : the potential $p = p(x)$.

Known quantities :

- Initial data : $(y^0, y^1)$ ; source terms $f, g$.
- *An additional information $\partial_\nu y(t,x)$ on $(0,T) \times \partial\Omega$.*

Goal : Find the potential $p$.

$\rightsquigarrow$ Applications : geology, medical imagery, oil prospection,...

*Is it possible to retrieve the potential* $p = p(x),\ x \in \Omega$
*from measurement of the flux* $\partial_\nu y(t,x)$ *on* $(0,T) \times \partial\Omega$ *?*

- Several related questions
  - Uniqueness : Given two potentials $p_1 \neq p_2$, can we guarantee $\partial_\nu y_1 \neq \partial_\nu y_2$ ?
  - Stability : Given two potentials $p_1, p_2$, if $\partial_\nu y_1 \simeq \partial_\nu y_2$, can we guarantee that $p_1 \simeq p_2$ ?
  - Reconstruction : Given $\partial_\nu y$, can we compute $p$ ?

- Known results : Uniqueness (Klibanov 92) and stability (Yamamoto 99, Imanuvilov Yamamoto 01), using Carleman estimates.

- Main Open Problem : Reconstruction, how to compute the potential from the boundary measurement ?

# A natural idea for reconstruction

Given a continuous measure $m(t,x) = \partial_\nu y[p]|_{(0,T)\times\partial\Omega}(t,x)$

- Discretize the wave equation

$$\begin{cases} \partial_{tt} y_h - \Delta_h y_h + p_h y_h = f_h, \\ y_h|_{(0,T)\times\partial\Omega} = g_h \simeq g, \\ (y_h, \partial_t y_h)(t=0) = (y_h^0, y_h^1) \simeq (y^0, y^1). \end{cases}$$

- Solve the following discrete inverse problem : Find a potential $p_h$ so that the corresponding discrete solution $y_h[p_h]$ approximates at best the measure :

$$\partial_h y_h[p_h]|_{(0,T)\times\partial\Omega} \simeq m(t,x)$$

Question : Do we get $p_h \simeq p$ ?

*Goal : Analyze the convergence of the discrete inverse problems and propose a convergent numerical method.*

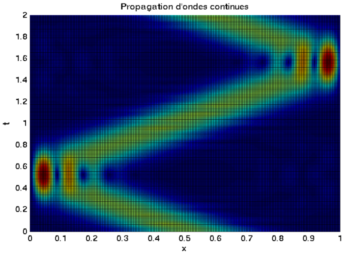
*Remarks :*

- Natural question for all inverse problems in infinite dimensions : Finding a source term, a conductivity...
- Depends *a priori* on the numerical scheme employed.

*Difficulties :*

- Non-linear problem in $p$.
- The wave equation and its discrete approximations have different dynamics, cf Ervedoza - Zuazua 11 :
  $\rightsquigarrow$ Numerical artefacts : High-frequency spurious waves, generated by the schemes.

# Wave propagation, continuous and discrete media



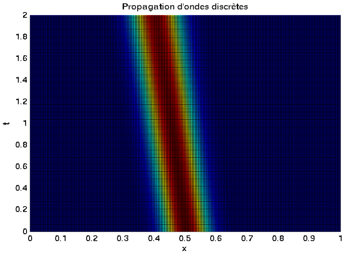




FIGURE: Wave propagation in continuous and discrete media

Continuous dynamics $\neq$ Discrete dynamics

# Stability result for the continuous wave equation

Let $x_0 \in \mathbb{R}^N \setminus \Omega$ and let $\Gamma_0$ and $T$ satisfy

$$\{x \in \partial\Omega,\ (x - x_0)\cdot\nu(x) > 0\} \subset \Gamma_0, \quad T > \sup_{x\in\Omega}\{|x - x_0|\}.$$

One can prove uniqueness and local Lipschitz stability for $p \in L^\infty(\Omega)$, $\|p\|_{L^\infty} \leq m$ under the assumptions

$$\inf_{\Omega}\{|y^0(x)|\} \neq 0, \quad y[p] \in H^1(0,T;L^\infty(\Omega)).$$

*Remark :* In 1D, the quantity $\|y[p]\|_{H^1(0,T;L^\infty(0,1))}$ can be uniformly bounded for $p \in L^\infty_{\leq m}(0,1)$ by taking smooth data such as $(y^0, y^1) \in H^2 \times H^1$, $g \in H^2((0,T)\times\{0,1\})$, $f \in W^{1,1}(0,T;L^2(0,1))$ + CC.

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# Theorem (Yamamoto 99, revisited LB 01)

Assume the above geometric constraints on $(\Gamma_0, T, \Omega)$.

Let $m > 0$ and $q \in L^\infty_{\leq m}(\Omega)$. Then $\exists C > 0$ depending only on

$$\inf_\Omega \{|y^0(x)|\} \ (\neq 0) \quad \text{and} \quad \|y[p]\|_{H^1(0,T;L^\infty(\Omega))},$$

such that for all $q \in L^\infty_{\leq m}(\Omega)$,

$$\frac{1}{C}\|p-q\|_{L^2(\Omega)} \leq \|\partial_\nu y[p] - \partial_\nu y[q]\|_{H^1((0,T);L^2(\Gamma_0))} \leq C\|p-q\|_{L^2(\Omega)}.$$

$\rightsquigarrow$ In our setting, it will be important to have an idea of how this proof works.

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# Weights for a Carleman estimate

For the wave operator with potential

$$L = \partial_{tt} - \Delta_x + q(x),$$

let $\lambda > 0$ and $\beta \in (0,1)$ and define $\psi = \psi(x,t)$, $\varphi = \varphi(x,t)$ by

$$\begin{aligned} \psi(x,t) &= |x - x_0|^2 - \beta t^2 + C_0, \\ \varphi(x,t) &= e^{\lambda \psi(x,t)} \end{aligned}$$

where $C_0 > 0$ is such that $\psi \geq 1$ on $\Omega \times [0,T]$.

# Carleman Estimate (Zhang 99, Imanuvilov 02)

Assuming $p \in L^\infty_{\leq m}(\Omega)$, $L = \partial_{tt} - \Delta_x + q(x)$ and

$$\{x \in \partial\Omega,\ (x - x_0) \cdot \nu(x) > 0\} \subset \Gamma_0,$$

$\exists s_0 > 0$, $\lambda > 0$ and $M = M(s_0, \lambda, T, \beta, x_0, m) > 0$ such that :

$$s \int_{-T}^{T} \int_\Omega e^{2s\varphi}(|\partial_t w|^2 + |\nabla w|^2) + s^3 \int_{-T}^{T} \int_\Omega e^{2s\varphi}|w|^2$$
$$\leq M \int_{-T}^{T} \int_\Omega e^{2s\varphi}|Lw|^2\, dxdt + Ms \int_{-T}^{T} \int_{\Gamma_0} e^{2s\varphi}\, |\partial_\nu w|^2\ dtd\sigma$$

for all $s > s_0$ and $w$ satisfying

$$\begin{cases} Lw \in L^2(\Omega \times (-T, T)) \\ w \in L^2(-T, T; H^1_0(\Omega)), \\ w(x, \pm T) = \partial_t w(x, \pm T) = 0,\ \forall x \in \Omega. \end{cases}$$

# How to use it ?

Let $z = y[p] - y[q]$. Then $z$ solves

$$\begin{cases} \partial_{tt} z - \Delta_x z + q(x) z = (q-p) y[p], & (t,x) \in (0,T) \times \Omega, \\ z(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega \\ (z(0,x), \partial_t z(0,x)) = (0,0), & x \in \Omega. \end{cases}$$

Set $Z = \partial_t z$ and extend the equation in negative time :

$$\begin{cases} \partial_{tt} Z - \Delta_x Z + q(x) Z = (q-p) \partial_t y[p], & (t,x) \in (-T,T) \times \Omega, \\ Z(t,x) = 0, & (t,x) \in (-T,T) \times \partial\Omega \\ (Z(0,x), \partial_t Z(0,x)) = (0, (q-p) y^0), & x \in \Omega. \end{cases}$$

- Apply the Carleman estimate to $w = \eta(t)Z$, where $\eta$ is a cut-off function vanishing close to $t = \pm T$.

$$\begin{cases} \partial_{tt}w - \Delta_x w + q(x)w = \eta(q-p)\partial_t y[p] + \eta'\partial_t Z + \eta'' Z, \\ w(t,x) = 0, \\ (w(0,x), \partial_t w(0,x)) = (0, (q-p)y^0). \end{cases}$$

$\rightsquigarrow$ that creates a source term localized in Supp $\eta' \cup$ Supp $\eta''$ localized close to $\pm T$.

- But assuming $T > \sup_{x\in\Omega}\{|x - x_0|\}$,

$$\sup_{x\in\Omega} \psi(\pm T, x) = \sup\{|x - x_0|^2\} - \beta T^2 + C_0 \leq C_0 \leq \inf_{x\in\Omega} \psi(0,x).$$

- We conclude by an energy estimate.

# Discretized setting

Consider the 1D wave equation observed at $x = 1$ (ie $x_0 < 0$).

Discretization using the finite-difference method :

$$\begin{cases} \partial_{tt} w_j - (\Delta_h w_h)_j + p_j w_j = f_j, \quad j \in \{1, \cdots, N\} \\ w_0(t) = w_{N+1}(t) = 0, \\ (w_h, \partial_t w_h)(t = \pm T) = 0, \end{cases}$$

where $N + 1 = 1/h$,

$$(\Delta_h w_h)_j = \frac{1}{h^2} \left( w_{j+1} - 2w_j + w_{j-1} \right).$$

The observation is then denoted by

$$(\partial_h^- w)_{N+1} = \frac{w_{N+1}(t) - w_N(t)}{h} = -\frac{w_N(t)}{h}.$$

# A Lax type argument

We shall develop a two steps proof of convergence

- Consistency : For any potential $q$, one can find discrete potentials $q_h$ so that $q_h \underset{h\to 0}{\longrightarrow} q$ in $L^2(\Omega)$ and

$$\partial_t(\partial_h^- y_h[q_h])_{N+1} \underset{h\to 0}{\longrightarrow} \partial_{tx} y[q](t,1) \text{ in } L^2(0,T).$$

- Uniform Stability : There exists a constant $C$ independent of $h > 0$ such that for all $p_h, q_h$,

$$\|p_h - q_h\|_{L^2(\Omega)} \leq C \left\| \partial_t(\partial_h^- y_h[p_h])_{N+1} - \partial_t(\partial_h^- y_h[q_h])_{N+1} \right\|_{L^2(0,T)}.$$

The convergence follows from the same ideas than Lax theorem for numerical analysis.

$\rightsquigarrow$ We shall focus on the uniform stability estimates.

# Discrete Carleman estimate

There exists $s_0 > 0$, $\lambda > 0$, $\varepsilon > 0$ and a constant $M = M(s_0, \lambda, \varepsilon, T, \beta, x_0) > 0$ such that for all $h \in (0,1)$, $s \in (s_0, \varepsilon/h)$

$$s \int_{-T}^{T} \int_{[0,1)} e^{2s\varphi} (|\partial_t w_h|^2 + |\partial_h^+ w_h|^2) dt + s^3 \int_{-T}^{T} \int_{(0,1)} e^{2s\varphi} |w_h|^2 dt$$
$$\leq M \int_{-T}^{T} \int_{(0,1)} e^{2s\varphi} |f_h|^2 \, dt + Ms \int_{-T}^{T} e^{2s\varphi(t,1)} \left|(\partial_h^- w_h)_{N+1}\right|^2 \, dt$$
$$+ Ms \int_{-T}^{T} \int_{[0,1)} e^{2s\varphi} |h \partial_h^+ \partial_t w_h|^2 \, dt.$$

and for all $w_h$ satisfying $w_{0,h}(t) = w_{N+1,h}(t) = 0$ on $(-T, T)$ and $w_h(\pm T) = \partial_t w_h(\pm T) = 0$.

# Remarks

- Similarities with the continuous Carleman estimate : the weight function $\varphi$ and the powers of $s$ are the same.
- The range of $s$ is limited to $s \leq \varepsilon/h$, expected in view of Boyer - Hubert - Le Rousseau 09,10.
- On the extra term in $s\, e^{2s\varphi}|h\partial_h^+ \partial_t w_h|^2$
  - Needed ! Otherwise, this would imply observability for the discrete waves, uniformly with respect to $h > 0$, see Zuazua 05 SIREV. Sharp scale.
  - Concentrated at high-frequencies : $(h\partial_h^+ w_h)_j = w_{j+1} - w_j$ and $h\partial_h^+ = o(1)$ for frequencies in $o(1/h)$,
    $h\partial_h^+ \simeq 1$ for frequencies larger than $1/h$.

# Consequence

Instead of getting the stability estimate

$$\|p_h - q_h\|_{L^2(\Omega)} \leq C \left\| \partial_t(\partial_h^- y_h[p_h])_{N+1} - \partial_t(\partial_h^- y_h[q_h])_{N+1} \right\|_{L^2(0,T)}$$

we get the following one :

$$\begin{aligned} &\|p_h - q_h\|_{L^2(\Omega)} \\ &\quad \leq C \left\| \partial_t(\partial_h^- y_h[p_h])_{N+1} - \partial_t(\partial_h^- y_h[q_h])_{N+1} \right\|_{L^2(0,T)} \\ &\qquad + C \left\| h\partial_h^+ \partial_{tt}(y_h[p_h] - y_h[q_h]) \right\|_{L^2((0,T);L^2([0,1)))}. \end{aligned}$$

This is enough to our purpose, since the added term weakly converges to $0$.

# Observation operator

We introduce, for $h > 0$, the following operator

$$\begin{array}{rccl} \Theta_h : & L^\infty_{h,\leq m}(0,1) & \rightarrow & L^2(0,T) \times L^2(0,T;L^2(0,1)) \\ & p_h & \mapsto & \left(\partial_t(\partial_h^- y_h)_{N+1}[p_h], h\partial_h^+\partial_{tt} y_h[p_h]\right), \end{array}$$

and its continuous analogous

$$\begin{array}{rccl} \Theta_0 : & L^\infty_{\leq m}(0,1) & \rightarrow & L^2(0,T) \times L^2(0,T;L^2(0,1)) \\ & p & \mapsto & \left(\partial_t \partial_x y[p](\cdot,1), 0\right). \end{array}$$

$m > 0$ is fixed, and we know *a priori* that $p \in L^\infty_{\leq m}(0,1)$.

# Convergence result

## Theorem

*Under some regularity, mild convergence and positivity assumptions on the data, let* $q_h \in L^\infty_{h,\leq m}(0,1)$ *be such that*

$$\Theta_h(q_h) \underset{h\to 0}{\longrightarrow} \Theta_0(p) \quad \textit{strongly in } L^2(0,T) \times L^2((0,T)\times(0,1)).$$

*Then*

$$q_h \underset{h\to 0}{\longrightarrow} p \quad \textit{in } L^2(0,1).$$

$\rightsquigarrow$ The proof is based on the following consistency result

# Consistency results

## Theorem

*Under the same assumptions, for any potential* $p \in L^\infty_{\leq m}(0,1)$*, there exists* $p_h \in L^\infty_{h,\leq m}(0,1)$ *such that*

$$p_h \underset{h\to 0}{\longrightarrow} p \quad \text{in } L^2(0,1) \quad \text{and}$$

$$\Theta_h(p_h) \underset{h\to 0}{\longrightarrow} \Theta_0(p) \quad \text{in } L^2(0,T) \times L^2((0,T)\times(0,1)).$$

*Moreover*

$$\sup_{h\in(0,1)} \|y_h[p_h]\|_{H^1(0,T;L^\infty_h(0,1))} < \infty.$$

# Further comments and open problems

- Proofs can easily be adapted to fully discrete case or (more tedious) to 2D uniform mesh in a square ;
- An extra Tychonoff term appears. Consistent with the theory of observability of discrete waves ;
- Finding $q_h$ such that $\Theta_h(q_h) \to \Theta_0(q)$ is a difficult problem since $\Theta_h$ is highly non-linear.
  $\rightsquigarrow$ a Carleman based approach for waves is being developed with M. de Buhan, F. de Gournay and S. Ervedoza ;
- Other discretization schemes ;
- Obtaining convergence rates ;
- Non-uniform meshes.

Thank you for your attention !