A new proof of HLS inequalities

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in collaboration with
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Outline

1. A new proof of HLS inequalities
    - A New Proof of the HLS inequality with Equality Cases

Two Inequalities, One Equation

Sharp Hardy-Littlewood-Sobolev inequality

[a] It states that for all non-negative measurable functions f on $\mathbb{R}^d$, and all $0 < \lambda < d$,

$$\frac{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)\frac{1}{|x-y|^\lambda}f(y)\,\mathrm{d}x\,\mathrm{d}y}{\|f\|_p^2} \le \frac{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(x)\frac{1}{|x-y|^\lambda}h(y)\,\mathrm{d}x\,\mathrm{d}y}{\|h\|_p^2}$$

where

$$h(x) = \left(\frac{1}{1+|x|^2}\right)^{(2d-\lambda)/2} .$$

and $p = \frac{2d}{2d-\lambda}$.

Moreover, there is equality if and only if for some $x_0 \in \mathbb{R}^d$ and $s \in \mathbb{R}_+$, f is a non-zero multiple of $h(x/s - x_0)$.

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Sharp Gagliardo-Nirenberg-Sobolev inequality

[a] It states that for all locally integrable functions f on $\mathbb{R}^d$, $d \geq 2$, with a square integrable distributional gradient, and p with $1 < p < d/(d-2)$

$$\frac{\|\nabla f\|_2^\theta \|f\|_{p+1}^{1-\theta}}{\|f\|_{2p}} \geq \frac{\|\nabla g\|_2^\theta \|g\|_{p+1}^{1-\theta}}{\|g\|_{2p}}$$

where

$$g(x) = \left(\frac{1}{1+|x|^2}\right)^{1/(p-1)} .$$

and

$$\theta = \frac{d(p-1)}{p(d+2-(d-2)p)} .$$

Moreover, there is equality if and only if for some $x_0 \in \mathbb{R}^d$ and $s \in \mathbb{R}_+$, f is a non-zero multiple of $g(x/s - x_0)$.

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Two Inequalities, One Equation

The Fast Diffusion Equation

The equality cases are steady states for nonlinear Fokker-Planck equations related to the fast diffusion equation:

$$\frac{\partial}{\partial t}u(x,t) = \Delta u^m(x,t)\ .$$

$u(x,t)$ solves the fast diffusion equation if and only if

$$v(x,t) := e^{td}u(e^t x, e^{\alpha t})$$

with $\alpha = 2 - d(1-m)$ satisfies

$$\alpha\frac{\partial}{\partial t}v(x,t) = \Delta v^m(x,t) + \nabla\cdot[xv(x,t)]\ .$$

$$v_\infty(x) := \left(\frac{2m}{(1-m)}\right)^{1/(1-m)}\left(\frac{1}{1+|x|^2}\right)^{1/(1-m)}$$

is a stationary solution with certain mass M.

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The connection

- The relation between the HLS optimizers, the GNS optimizers and the FD equation is more than a superficial coincidence, at least for certain λ, p and m.
- For $d \geq 3$, the $\lambda = d - 2$ case of the sharp HLS inequality can be proved by using the fast diffusion flow for $m = d/(d+2)$ to deform any reasonably nice trial function into a multiple of h.

Motivation: We choose m for the fast diffusion equation and λ for the HLS inequality in order to have h as stationary solution/equality case. Thus,

$$\frac{1}{1-m} = \frac{d+2}{2} \qquad \Leftrightarrow \qquad m = \frac{d}{d+2}.$$

The exact value of the exponent for the GNS inequality

$$1 < p = \frac{d+1}{d-1} < \frac{d}{d-2}$$

will be dictated by the evolution of the HLS functional along the flow of the FD equation. Note p is in the correct range of the sharp GNS inequalities.

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The proof

The *HLS functional*

For $\lambda = d-2$, the HLS inequality can be rewritten as $\mathcal{F}[f] \geq 0$ for all $f \in L^{2d/(d+2)}(\mathbb{R}^d)$ where

$$\mathcal{F}[f] := C_S \left(\int_{\mathbb{R}^d} f^{2d/(d+2)}(x)\mathrm{d}x \right)^{(d+2)/d} - \int_{\mathbb{R}^d} f(x) \left[(-\Delta)^{-1} f \right](x)\, \mathrm{d}x\,,$$

with

$$C_S := \frac{4}{d(d-2)} |\mathbb{S}^d|^{-2/d}\,.$$

The *GNS inequality*

The GNS inequality with $p = (d+1)/(d-1)$ can be written as $\mathcal{D}[g] \geq 0$ for all g with a square integrable distributional gradient on $\mathbb{R}^d$ where

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Dissipation of the HLS functional by the FD equation

Let $f \in L^1(\mathbb{R}^d) \cap L^{2d/(d+2)}(\mathbb{R}^d)$ be non-negative, and suppose that

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} h(x)\,\mathrm{d}x = M$$

Let $u(x,t)$ be the solution of the FD equation with $u(x,1) = f(x)$. Then, for all $t > 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[u(\cdot,t)] = -2\mathcal{D}[u^{(d-1)/(d+2)}(\cdot,t)]\,.$$

HLS inequality via FD flow

Let $f \in L^1(\mathbb{R}^d) \cap L^{2d/(d+2)}(\mathbb{R}^d)$ be non-negative, and suppose that

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Then $\mathcal{F}[f] \geq 0$.

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Then $\mathcal{F}[f] \geq 0$.

The proof

Same steps as in the log-HLS case. The only difference is that the scaling relation implies now for all $t > 0$,

$$\mathcal{F}[v(\cdot,t)] = e^{t(d-2)}\mathcal{F}[u(\cdot,e^{4t/(d+2)})]\,.$$

And thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t(d-2)}\mathcal{F}[v(\cdot,t)]\right) = -\frac{8}{d+2}e^{t4/(d+2)}\mathcal{D}[u^{(d-1)/(d+2)}(\cdot,e^{4t/(d+2)})]\,.$$

from which

$$\mathcal{F}[f] - \left(e^{-t(d-2)}\mathcal{F}[v(\cdot,t)]\right) = \frac{8}{d+2}\int_0^t e^{s4/(d+2)}\mathcal{D}[u^{(d-1)/(d+2)}(\cdot,e^{4s/(d+2)})]\,\mathrm{d}s\,.$$

Passing to the limit $t\to\infty$ and using Bonforte-Vazquez result with the assumptions on f, gives

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By the sharp GNS inequality, the right hand side is non-negative.

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$$\frac{\partial}{\partial t} u(x,t) = \Delta u^{d/(d+2)}(x,t)$$

we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[u] = C_S \frac{d+2}{d}\left(\int_{\mathbb{R}^d} u^{2d/(d+2)}\,\mathrm{d}x\right)^{2/d}\int_{\mathbb{R}^d}\frac{2d}{d+2}u^{(d-2)/(d+2)}\Delta u^{d/(d+2)}\,\mathrm{d}x$$
$$-2\int\left(\Delta u^{d/(d+2)}\right)(-\Delta)^{-1}u\,\mathrm{d}x$$
$$= -2C_S\left(\int_{\mathbb{R}^d} u^{2d/(d+2)}\,\mathrm{d}x\right)^{2/d}\int_{\mathbb{R}^d}\left(\frac{d-2}{d+2}\right)\left(\frac{d}{d+2}\right)u^{-6/(d+2)}|\nabla u|^2\,\mathrm{d}x$$
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Define

$$g = u^{(d-1)/(d+2)} .$$

Then one computes

$$\int_{\mathbb{R}^d} u^{-6/(d+2)} |\nabla u|^2 \,\mathrm{d}x = \left(\frac{d+2}{d-1}\right)^2 \int_{\mathbb{R}^d} |\nabla g|^2 \,\mathrm{d}x .$$

Now rewriting the right hand side in terms of g, one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[u] = -2C_S \left(\int_{\mathbb{R}^d} g^{2d/(d-1)} \,\mathrm{d}x\right)^{2/d} \frac{d(d-2)}{(d-1)^2} \int_{\mathbb{R}^d} |\nabla g|^2 \,\mathrm{d}x + 2\int g^{(2d+2)/(d-1)} \,\mathrm{d}x$$

$$= -2\left[\frac{4|S^d|^{2/d}}{(d-1)^2} \int_{\mathbb{R}^d} |\nabla g|^2 \,\mathrm{d}x \left(\int_{\mathbb{R}^d} g^{2d/(d-1)} \,\mathrm{d}x\right)^{2/d} - \int g^{(2d+2)/(d-1)} \,\mathrm{d}x\right] .$$

Expressing this in terms of $\mathcal{D}$, we have the result.

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