Relations between Convexity and Convection

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discussion session on Fluid Mechanics PDEs

Benasque Sept 2011

CONVEXITY AND CONVECTION

1. Multidimensional rearrangement with convex potetial

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3. Interpretation of the scheme in terms of incompressible fluid mechanics and the hydrostatic limit of the Navier-Sokes Boussinesq equations
4. Short time derivation and global existence of solutions for the hydrostatic Boussinesq model

Multidimensional rearrangement with convex potential

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and an L^2 map $x \in D \to z(x) \in R^d$,

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$\mathbf{z}^\sharp(\mathbf{x}) = \nabla \mathbf{p}(\mathbf{x})$,

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$p(x)$ lsc convex in $x \in R^d$, a.e. differentiable on D, such that

$$\boxed{\int_D f(\nabla p(x))dx = \int_D f(z(x))dx}$$

for all continuous function f such that $|f(x)| \leq cst(1 + |x|^2)$

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This is a typical result in optimal transport theory, see YB, CRAS Paris 1987 and CPAM 1991, Smith and Knott, JOTA 1987, cf. Villani's book, Topics in optimal transportation, AMS, 2003, see also books by Rachev-Rüschendorf, Evans, Villani, Ambrosio-Gigli-Savaré and many others contributions...

A rearrangement-scheme

Setting:

-a smooth bounded domain $\mathbf{x} \in \mathbf{D} \subset \mathbf{R}^{\mathbf{d}}$

-a vector-valued field: $\mathbf{y}(\mathbf{t}, \mathbf{x}) \in \mathbf{R}^{\mathbf{d}}$ (generalized temperature)

-a vector-valued source term: $\mathbf{G} = \mathbf{G}(\mathbf{x}) \in \mathbf{R}^{\mathbf{d}}$ with bounded derivatives

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Time discrete scheme:

-time step $h > 0$, $\quad y(t = nh, x) \sim y_n(x), \quad n = 0, 1, 2, \cdots$

-predictor: $y_{n+1/2}(x) = y_n(x) + h\, G(x)$

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-corrector: $y_{n+1} = y^\sharp_{n+1/2}$
as the unique rearrangement with convex potential $y_{n+1} = \nabla p_{n+1}$

Interpretation of the multi-d rearrangement scheme

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It turns out that the scheme can be interpreted as a singular limit of a Navier-Stokes Boussinesq model with (generalized) buoyancy forces.

The NS-Boussinesq model

Let D be a smooth bounded domain $D \subset R^3$ in which moves an incompressible fluid of velocity $v(t,x)$ at $x \in D, \quad t \geq 0$, subject to the Navier-Stokes equations

$$\boxed{\textbf{NSB} \quad \epsilon^2(\partial_t v + (v \cdot \nabla)v) - \nu\nabla^2 v + \nabla p = y \qquad \nabla \cdot v = 0}$$

with $\epsilon, \nu > 0$ and $v = 0$ along ∂D.

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The force field y is a "generalized buoyancy", vector-valued, force, subject to the advection equation

$$\boxed{\partial_t y + (v \cdot \nabla)y = G(x)}$$

where G is a given smooth source term with bounded derivatives.

Remark 1: In the concrete convection model we considered with Mike Cullen (UK met'office), there is no x_2 dependence and $G_1 = 0$. Then the force field y is vector-valued and combines both Coriolis (in the x_1 direction) and buoyancy (in the x_3 direction) effects.

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Remark 2: From the PDE viewpoint, global existence of weak solutions in 3D follows from Leray/Diperna-Lions theory, while global existence of smooth solutions in 2D follows from Hou-Li 2005 and Chae 2006. (See also recent work by Danchin-Paicu.)

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Remark 3: The smallness of ϵ is also equivalent to the action, on a long time interval, of a small "global change" source term , through the following rescaling:

$$\boxed{\mathbf{G} \to \epsilon\mathbf{G}(\mathbf{x}), \quad \mathbf{t} \to \mathbf{t}/\epsilon, \quad \mathbf{v} \to \epsilon\mathbf{v}(\mathbf{t}\epsilon, \mathbf{x}).}$$

Remark 4: for any suitable test function f we have INDEPENDENTLY of $\epsilon, \mathbf{v}, \nu$ the following key property

$$\boxed{\frac{d}{dt}\int_D f(y(t,x))dx = \int_D (\nabla f)(y(t,x)) \cdot G(x)dx}$$

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Remark 5: when both the source term and the initial force are gradients and the fluid initially is at rest

$$\boxed{G(x) = \nabla g(x), \quad y(0,x) = \nabla p_0(x), \quad v(0,x) = 0}$$

the system has a trivial but interesting "convection-free" solution, independently of ϵ, ν, namely

$$\boxed{v(t,x) = 0, \quad y(t,x) = \nabla p(t,x), \;\; p(t,x) = p_0(x) + tg(x)}$$

A natural convexity condition for the HB system

The Hydrostatic Boussinesq HB system formally obtained by setting ϵ, ν to zero

$$\boxed{\mathbf{HB}: \quad \partial_t \mathbf{y} + (\mathbf{v} \cdot \nabla)\mathbf{y} = \mathbf{G}(\mathbf{x}), \quad \nabla \cdot \mathbf{v} = \mathbf{0}, \quad \nabla \mathbf{p} = \mathbf{y}}$$

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Notice that, $(\mathbf{v}\cdot\nabla)\mathbf{y} = (\mathbf{D}^2_{\mathbf{x}}\mathbf{p}\cdot\mathbf{v})$ and $\mathbf{v} = \nabla\times\mathbf{A}$, for some divergence-free vector potential $\mathbf{A} = \mathbf{A}(\mathbf{t},\mathbf{x}) \in \mathbf{R}^3$, when $\mathbf{d} = \mathbf{3}$.

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This linear 'magnetostatic' system in A is elliptic whenever p is strongly convex: $\boxed{\mathbf{cst}\ \mathbf{Id} < \mathbf{D}_\mathbf{x}^2\mathbf{p}(\mathbf{t},\mathbf{x}) < \mathbf{cst}'\ \mathbf{Id}}$

Derivation of the HB model under strong convexity condition

Theorem

Assume $D = R^3/Z^3$, (y, p, v) to be a smooth solution of the HB hydrostatic Boussinesq model, with $\boxed{\text{cst Id} < D_x^2 p(t, x) < \text{cst}' \text{ Id}}$ Then, as $\nu = \epsilon \to 0$, any Leray solution $(y^\epsilon, p^\epsilon, v^\epsilon)$ to the full NSB Navier-Stokes Boussinesq equations, with same initial condition, converges to (y, p, v).

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Idea of the proof: **Do not try to estimate plain L^2 distances (which completely fails) but rather use the non-quadratic functional**

$$\int_D \{K(t, y^\epsilon(t, x), y(t, x)) + \frac{\epsilon^2}{2} |v^\epsilon - v|^2\} dx$$

$$K(t, y', y) = p^*(t, y') - p^*(t, y) - \nabla p^*(t, y) \cdot (y' - y) \sim |y - y'|^2,$$

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built on the Legendre-Fenchel transform $p^*(t, z) = \sup_{x \in D} x \cdot z - p(t, x)$ of the limit convex potential p.

Breakdown of convexity and global solutions

Solutions of the HB model cannot be expected to stay globally strictly convex. This is obvious, in particular, for solutions of form

$$\boxed{\mathbf{v}(\mathbf{t},\mathbf{x})=\mathbf{0},\quad \mathbf{y}(\mathbf{t},\mathbf{x})=\nabla\mathbf{p}(\mathbf{t},\mathbf{x}),\ \ \mathbf{p}(\mathbf{t},\mathbf{x})=\mathbf{p_0}(\mathbf{x})+\mathbf{t}\mathbf{g}(\mathbf{x})}$$

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Thus, in the limit, it seems reasonable to enforce (what is known as the Cullen-Purser condition for semi-geostrophic equations)

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in which case, the force field $y(t,x) = \nabla p(t,x)$ is completely determined by the knowledge of all 'observables'

$$\boxed{f \rightarrow \int_D f(y(t,x))dx}$$ **by MULTI-D REARRANGEMENT THEORY**

A concept of "entropy" solutions for the HB system

By analogy with hyperbolic conservation laws, we introduce the concept of "entropy" solution, formally self-consistent, for the HB system

DEFINITION

We say that $(t \to y(t,\cdot)) \in C^0(R_+, L^2(D, R^3))$ is a solution with convex potential to the HB system, if

$$\boxed{\frac{d}{dt}\int_D f(y(t,x))dx = \int_D (\nabla f)(y(t,x)) \cdot G(x)dx, \quad \forall f}$$

with $y(t,x) = \nabla p(t,x)$ for some CONVEX function p.

Global existence of "entropy" solutions

Theorem

As $h \to 0$, the multi-d rearrangement scheme has converging subsequences.
Each limit y belongs to the space $C^0(R_+,\ L^2(D, R^d))$ and has a convex potential: $y(t,\cdot) = \nabla p(t,\cdot)$ for each $t \geq 0$.

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See YB, JNLS 2009. Notice that the system is self-consistent, thanks to the rearrangement theorem. However, our global existence result does not imply stability with respect to initial conditions, except for $d = 1$, where we can use the theory of scalar conservation laws, or $d > 1$ and $G(x) = -x$, where we can use maximal monotone operator theory

Sketch of proof: consistency part

Take a smooth function f. Then

$$\int_D f(y_{n+1}(x))dx = \int_D f(y_{n+1/2}(x))dx$$

(because y_{n+1} is a rearrangement of $y_{n+1/2}$)

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(by definition of predictor $y_{n+1/2}$)

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$$= \int_D f(y_n(x))dx + h\int_D (\nabla f)(y_n(x)) \cdot G(x)dx \quad + o(h)$$

Open problems

Stability and singularities

Global "entropy" solutions are known to be stable with respect to initial conditions only in some special cases, such as $d = 1$ or $G(x) = -x$. Clearly, this needs to be extended to all cases. Moreover, strict convexity clearly breaks down in finite time for some data, but is it generically true? This is known only for $d = 1$ thanks to scalar conservation law theory.

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Convergence beyond singularities

It is much more challenging to prove, after strict convexity breaks down, that the "extended" solutions which obey the convexity principle, correctly describe the limit of the NSB solutions in the HB regime. They may be just crude (but relevant) approximations, in some suitable sense for which a right mathematical framework has to be found. A similar situation occurs in shallow water theory when shock waves ("hydraulic jumps") appear.

Some references

1-The 1D rearrangement scheme

a) convergence to the subdifferential equation in L^2:
YB Methods Appl. Anal. 2004 see also YB Arma 2009 and Bolley, B, Loeper J. Hyp. DE 2005,
b) convergence to Kruzhkov's solutions in L^1:
YB, CRAS 1981 and JDE 1983

2-The multi-D rearrangement scheme and its relationship with convection theory

a) General discussion: YB, JNLS 2009, b) Global existence theory see YB, JNLS 2009, following unpublished note 2002, in the case $G(x) = -x$ and Loeper SIMA 2008 in the case of semigeostrophic (SG) equations, namely $G(x) = Jx$, J symplectic
c) Local smooth solutions: G. Loeper 2008 (for SG equations)
d) Derivation from the NSB equations
YB and M. Cullen, CMS 2010 (derivation of the "xz" SG equations, to be fixed for domains with boundaries)