Geometric analysis issues in uncompressible fluid mechanics

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INVISCID INCOMPRESSIBLE FLOWS VIEWED AS MINIMIZING GEODESICS FOR VOLUME PRESERVING MAPS AND RELATED GEOMETRIC ANALYSIS ISSUES

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EULER'S EQUATIONS: GEOMETRIC DEFINITION

One can describe the motion of an incompressible fluid inside a bounded domain D in R^d by a time-dependent family $t \to M_t$ of maps, in the Hilbert space $H = L^2(D, R^d)$, valued in the subset VPM(D) of all Lebesgue measure-preserving maps

$$\boxed{VPM(D) = \{M \in H, \ \int_D q(M(x))dx = \int_D q(x)dx, \ \forall q \in C(R^d)\}}$$

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Solutions of the Euler equations, introduced in 1755, correspond to those curves $t \to M_t \in VPM(D)$ for which there exists a time dependent scalar function p_t, called 'pressure field', defined on D, such that

$$\boxed{\frac{d^2 M_t}{dt^2} + (\nabla p_t) \circ M_t = 0}$$

where ∇ is the gradient operator on R^d (with respect to the Euclidean norm $|\cdot|$).

THE PRINCIPLE OF LEAST ACTION

THEOREM **Assume D to be convex. Let (M_t, p_t) a solution of the Euler equations, with a constant λ such that**

$$\boxed{\sum_{i,j=1}^{d} \frac{\partial^2 p_t(x)}{\partial x_i \partial x_j} \xi_i \xi_j \le \lambda |\xi|^2, \quad \forall \xi \in \mathbf{R}^d, \quad \forall x \in D, \quad \forall t}$$

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Then, for every $t_0 < t_1$ so that $(t_1 - t_0)^2 \lambda < \pi^2$, M_t is the unique minimizer, among all curves along $VPM(D)$ that coincide with M_t at $t = t_0$, $t = t_1$, of the following ACTION

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see Arnold 1966, Ebin-Marsden 1970, Arnold-Khesin book 1998

THE DUAL ACTION

Minimizing the action can be written as a saddle point problem, just by using a time-dependent Lagrange multiplier to relax the constraint for M_t to belong to $VPM(D)$

$$\boxed{\inf_{M} \sup_{p} \int_{t_0}^{t_1} \int_{D} \{\frac{1}{2}|\frac{dM_t(x)}{dt}|^2 - p_t(M_t(x)) + p_t(x)\} dx dt}$$

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This is trivially bounded from below by

$$\boxed{\sup_p \inf_M \int_{t_0}^{t_1} \int_D \{\frac{1}{2}|\frac{dM_t(x)}{dt}|^2 - p_t(M_t(x)) + p_t(x)\}dxdt}$$

which naturally leads to a dual least action principle

THE DUAL LEAST ACTION PRINCIPLE

THEOREM Under exactly the same conditions (D convex and $(t_1 - t_0)^2\lambda < \pi^2$), the pressure p is the unique maximizer of the **CONCAVE DUAL ACTION**

$$\boxed{I[p] = \int_D J_p(M_{t_0}(x), M_{t_1}(x))dx + \int_{t_0}^{t_1}\int_D p_t(x)dxdt}$$

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As for the previous theorem, the proof is elementary and directly follows from the 1D Poincaré inequality, which explains the role of constant π. Notice that M_t is never assumed to be smooth or one-to-one and the case $d = 1$ is fine.

GEOMETRIC ANALYSIS ISSUES I

1) DENSITY OF DIFFEOMORPHISMS IN VPM(D) It is customary to consider the subset $\mathrm{SDiff}(D)$ of $\mathrm{VPM}(D)$ made of Lebesgue-measure preserving maps that are, in addition, orientation preserving diffeomorphisms. For $d \geq 2$, $\mathrm{VPM}(D)$ is precisely the L^2 closure of $\mathrm{SDiff}(D)$. This is a relatively easy result.

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The identification of the closure of SDiff(D) for the a priori finer geodesic distance induced by L^2 is a much more difficult issue. For simple (say contractile) domains D, this closure is still VPM(D) for $d \geq 3$ (but definitely not for $d = 2$) as shown by Shnirelman in his landmark paper (Math USSR Sb 1985).

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These results have striking consequences: in particular maps of form

$$\boxed{M(x) = (h(x_1), x_2, x_3)}$$

where h is any Lebesgue-measure preserving map of the unit interval, are in the closure of $\mathrm{SDiff}([0,1]^3)$.

GEOMETRIC ANALYSIS ISSUES II

2) DENSITY OF PERMUTATIONS IN VPM(D) **Another interesting subset of VPM$([0,1]^3)$ is made of all "permutations" of all dyadic divisions of the unit cube in sub-cubes of equal volumes.**

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The set of all such permutations, denoted $\mathrm{P}(\mathrm{D})$ turns out to be L^2 dense in $\mathrm{VPM}([0,1]^3)$ for all dimensions.
Combined with the previous result, this justifies the use of 1D calculations by (simple) combinatorial optimization methods to understand the geometry of 3D volume preserving maps.

GEOMETRIC ANALYSIS ISSUES III

3) GEODESIC COMPLETENESS This amounts to globally solving the initial value problem for the Euler equations. This is an outstanding problem for nonlinear evolution PDEs, which will not be discussed here.

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The case $d = 2$ is clearly linked to symplectic geometry and seems extremely difficult: a fascinating strategy has been developed by Shnirelman, by adding braid constraints to the minimization problem, which certainly deserves further investigations.

APPROXIMATE MINIMIZING GEODESICS

DEFINITION Let us assume D to be convex, fix $t_0 = 0,\ t_1 = 1$ and consider two maps $M_0, M_1 \in VPM(D)$. We say that $(M_t^\epsilon) \in SDiff(D)$ is an ϵ-minimizing geodesic if

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where $\frac{1}{2}d(M_0, M_1)^2$ denotes the maximal dual action. The existence of such approximations is in no way trivial and is a consequence of a key density results due to Shnirelman (GAFA 1994) that we will use later.

MINIMIZING GEODESICS: EXISTENCE OF A UNIQUE PRESSURE GRADIENT

MAIN THEOREM Let us assume D to be convex, with $d \geq 3$, fix $t_0 = 0,\ t_1 = 1$ and consider two maps $M_0, M_1 \in VPM(D)$. Then, there is a UNIQUE pressure-gradient ∇p_t such that for all (M_t^ϵ) ϵ-minimizing geodesics, we have in the sense of distributions

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This result essentially goes back to YB CPAM 1999, with important improvements in Ambrosio-Figalli ARMA 2008. It is a combination of solving the dual least action problem and using Shnirelman's density result for "generalized flows", GAFA 1994.

MINIMIZING GEODESICS: FINAL COMMENTS

1) UNIQUENESS OF THE PRESSURE GRADIENT This is a remarkable feature of the theory. There is no equivalent result for finite dimensional configuration spaces such as SO(3), on which geodesic curves (for appropriate metrics) correspond to the motion of solid bodies in classical mechanics. We believe this strange phenomenon to be the consequence of the "hidden convexity" of the problem in dimension 3 and more. Of course thr case d=2 is completely different .

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2) LIMITED REGULARITY OF THE PRESSURE GRADIENT The pressure gradient was proven first (YB CPAM 1999) to be a locally bounded measure. Recently, Ambrosio and Figalli have shown a better L^2 integrability with respect to the time variable (with measure values in space). Recently, I found an explicit example (that goes back to Duchon and Robert) of solutions with a pressure field semi-concave in the space variable and not more. The optimal regularity, and its dependence on the data, are clearly challenging analytic issues.

SOME REFERENCES

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L. Euler, opera omnia, seria secunda 12, p. 274 (in french, english translation available)

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