Null Controllability of Coupled Parabolic Degenerate Equations

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Coupled Parabolic Systems

$$u_t - (a_1(x)u_x)_x + c_1(t,x)u + b_1(t,x)v = h_1 1_\omega,$$

$$v_t - (a_2(x)v_x)_x + c_2(t,x)v + b_2(t,x)u = 0,$$

$+$ Boundary Conditions

$$u(0,x) = u_0(x),\ v(0,x) = v_0(x), \quad x \in (0,1),$$

- $$\exists ?\ h_1 : \qquad u(T) = v(T) = 0.$$
- Nondegenerate Case : $a_i(x) \geq m > 0$
- Gonzalez-Burgos, De Teresa, Ammar-Khodja, Benabdellah, Dupaix, Zuazua, ...

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Degenerate Coupled Parabolic Systems

- $a_i(0) = 0$ e.g. $a_i(x) = x^{\alpha_i}$
- $a_i(0) = a_i(1) = 0$ e.g. $a_i(x) = x^{\alpha_i}(1-x)^{\beta_i}$

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Cannarsa and De Teresa : $a_1 = a_2 =: a, \qquad b_1 = 0$

$$
\begin{aligned}
&u_t - (a(x)u_x)_x + c_1(t,x)u = h_1 1_\omega,\\
&v_t - (a(x)v_x)_x + c_2(t,x)v + b_2(t,x)u = 0,\\
&u(t,1) = v(t,1) = 0\\
&u(t,0) = v(t,0) = 0 \qquad (Weak\ deg.)\\
&\text{or}\\
&(au_x)(0) = (av_x)(0) = 0 \quad (Strong\ deg.)\\
&u(0,x) = u_0(x),\ \ v(0,x) = v_0(x), \quad x \in (0,1),
\end{aligned}
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- $$a(x) = x^{\alpha}$$
- (Weak deg.) $\iff$ $0 \leq \alpha < 1.$
- (Strong deg.) $\iff$ $1 \leq \alpha < 2.$

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Degenerate Parabolic Cascade Systems

- $$a(x) = x^\alpha$$
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Adjoint Degenerate Cascade Systems

$$U_t - (a_1(x)U_x)_x + c_1(t,x)U + b_2(t,x)V = 0,$$
$$V_t - (a_2(x)V_x)_x + c_2(t,x)V = 0,$$
$$u(t,1) = v(t,1) = 0$$

$$u(t,0) = 0 \ \ (Weak\ deg.) \text{ or } (au_x)(0) = 0 \ \ (Strong\ deg.)$$
$$v(t,0) = 0 \ \ (Weak\ deg.) \text{ or } \ (av_x)(0) = 0 \ \ (Strong\ deg.)$$

$$U(0,x) = U_0(x), \ \ V(0,x) = V_0(x), \quad x \in (0,1),$$

$$a_1 = x^{\alpha_1}, \qquad a_2 = x^{\alpha_2}.$$

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&U(0,x) = U_0(x), \ \ V(0,x) = V_0(x), \quad x \in (0,1),
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$$
a_1 = x^{\alpha_1}, \qquad a_2 = x^{\alpha_2}.
$$

Theorem 1

$\exists\, C > 0, \quad \exists\, s_0 > 0 \quad / \quad \forall (U, V)\ , \ \forall s \geq s_0\ :$

$$\int_0^T \int_0^1 \left[s\Theta a_1 U_x^2 + s^3 \Theta^3 \frac{x^2}{a_1(x)} U^2 \right] e^{2s\varphi_1} dxdt$$
$$+ \int_0^T \int_0^1 \left[s\Theta a_2 V_x^2 + s^3 \Theta^3 \frac{x^2}{a_2(x)} V^2 \right] e^{2s\varphi_2} dxdt$$
$$\leq C \int_0^T \int_\omega \left[U^2 + V^2 \right] e^{2s\Phi_2} dxdt.$$

Weight functions

$$\Theta(t) := \frac{1}{[t(T-t)]^4},$$

$$\psi_i(x) := \lambda_i \left(\int_0^x \frac{y}{a_i(y)} dy - d_i \right) = \quad \lambda_i \left(x^{2-\alpha_i} - d_i \right)$$

$$\varphi_i(t,x) = \Theta(t)\psi_i(x), \quad i = 1, 2,$$

$$\Psi_i(x) := e^{r_i \zeta_i(x)} - e^{2\rho_i},$$

$$\zeta_i(x) := \int_x^1 \frac{1}{\sqrt{a_i(y)}} dy, \qquad \rho_i := r_i \zeta_i(0),$$

$$\Phi_i(t,x) := \Theta(t)\Psi_i(x) \quad i = 1, 2.$$

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Proof

$$\omega = (a,b) \subset\subset (0,1).$$

Estimate in $(0,a)$

ξ = cut-off function ;

$w := \xi U, \qquad z := \xi V,$

$$w_t - (a_1 w_x)_x + c_1 w = b_2 z - \xi_x a_1 U_x - (a_1 \xi_x U)_x =: f_1,$$

$$z_t - (a_2 z_x)_x + c_2 z = -\xi_x a_2 V_x - (a_2 \xi_x V)_x =: f_2,$$

+ Boundary conditions

$w(0,x) = w_0(x), z(0,x) = z_0(x).$

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Proof

$$\int_0^T \int_0^1 [s\Theta a_1 w_x^2 + s^3\Theta^3 \frac{x^2}{a_1(x)} w^2] e^{2s\varphi_1} dxdt$$
$$\leq C \int_0^T \int_0^1 [b_2^2 z^2 + (\xi_x a_1 U_x + (a_1 \xi_x U)_x)^2] e^{2s\varphi_1} dxdt$$

$$\int_0^T \int_0^1 [s\Theta a_2 z_x^2 + s^3\Theta^3 \frac{x^2}{a_2(x)} z^2] e^{2s\varphi_2} dxdt$$
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- $d_i \geq max\{\frac{1}{a_i(1)(2-K)}, 4\int_0^1 \frac{y}{a_i(y)}dy\}$,
- $e^{\rho_2} \geq 4\frac{d_2-\int_0^1 \frac{y}{a_2(y)}dy}{d_2-4\int_0^1 \frac{y}{a_2(y)}dy}$,
- $\rho_1 = 2\rho_2$,
- $\lambda_1 = \frac{e^{2\rho_1}-1}{d_1-\int_0^1 \frac{y}{a_1(y)}dy}, \qquad \lambda_2 = \frac{4}{3d_2}(e^{2\rho_2} - e^{\rho_2})$.

Weight functions comparaison

For this choice we have :

- $\varphi_1 \leq \varphi_2$,
- $\Phi_1 \leq \Phi_2$,
- $\varphi_i \leq \Phi_i$.

Hardy-Poincaré Inequality

$$\int_0^1 \frac{a_i(x)}{x^2} g^2(x)dx \;\leq\; C \int_0^1 a_i(x)|g_x|^2 dx$$

$$\int_0^1 x^{\gamma-2} g^2(x)dx \;\leq\; C \int_0^1 x^{\gamma}|g_x|^2 dx$$

$$0 \leq \gamma < 2, \gamma \neq 1$$

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Non Cascade degenerate Systems

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New Carleman estimate of parabolic equations

$$y_t - (x^\alpha y_x)_x = f,$$
$$+\text{ boundary conditions}$$
$$y(0,x) = y_0, x \in (0,1).$$

Previous

$$\psi(x) = c_1(x^{2-\alpha} - c_2).$$

New

$$\psi(x) = c_1(x^{2-\beta} - c_2).$$

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New Carleman estimate of parabolic equations

Let $0 \leq \alpha < 1$.

$\forall \beta \in [\alpha, 1), \quad \exists C, s_0 > 0 :$

$$\int_0^T \int_0^1 (s\Theta(t)x^{2\alpha-\beta}y_x^2 + s^3\Theta^3(t)x^{2+2\alpha-3\beta}y^2)e^{2s\varphi}dxdt$$
$$\leq C \int_0^T \int_0^1 f^2(t,x)e^{2s\varphi(t,x)}\,dxdt$$
$$+ \int_0^T s\Theta(t)y_x^2(t,1)e^{2s\varphi(t,1)}dt, \qquad \forall\ s \geq s_0$$

$\beta \in [max(\alpha_1, \alpha_2), 1).$

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Difficulties and Open Problems

Null controllability for systems

- Weak-Strong : e.g. $\alpha_1 < 1$ and $\alpha_2 \geq 1$.
- Strong-Strong : $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$.
- General $a_i, i = 1, 2$.
- High dimension

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Boundary approx. control. for systems

$$u_t - (x^{\alpha_1} u_x)_x + c_1(t,x)u + b_1(t,x)v = 0$$

$$v_t - (x^{\alpha_2} v_x)_x + c_2(t,x)v + b_2(t,x)u = 0,$$

$$u(t,1) = v(t,1) = 0$$
$$u(t,0) = g_1(t), \qquad v(t,0) = g_2(t)$$
$$u(0,x) = u_0(x), \;\; v(0,x) = v_0(x).$$

Inverse problem for parabolic systems

$$u_t - (x^{\alpha_1} u_x)_x + c_1(t,x)u + b_1(t,x)v = f,$$

$$v_t - (x^{\alpha_2} v_x)_x + c_2(t,x)v + b_2(t,x)u = g,$$

$$+ \text{ Boundary Conditions}$$

$$u(0,x) = u_0(x), \;\; v(0,x) = v_0(x).$$

Choukran=Gracias=Thank you