Navier-Stokes system for a compressible, barotropic flow: Some open questions

Eduard Feireisl

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

Benasque, 2-nd September 2011

Navier-Stokes system

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + a \nabla_x \varrho^\gamma = \mu \Delta \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u}$$

$$\mathbf{u}|_{\partial \Omega} = 0$$

Adiabatic exponent

Available bounds

$$\varrho \in L^\infty(0,T;L^\gamma(\Omega)),\ \mathbf{u} \in L^2(0,T;W^{1,2}(\Omega;R^3))$$

$$\varrho \in L^{\gamma+\theta}((0,T)\times\Omega),\ \theta = \frac{2}{3}\gamma - 1$$

Renormalized continuity equation:

$$\partial_t b(\varrho) + \mathrm{div}_x (b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0$$

- DiPerna-Lions theory requires $\varrho \in L^2 \implies \gamma \geq 9/5$

Oscillation defect measure:

$$\mathbf{osc}_q[\varrho_n \to \varrho] = \sup_{k \geq 0} \limsup_{n \to \infty} \int_0^T \int_\Omega \Big| T_k(\varrho_n) - T_k(\varrho) \Big|^q \mathrm{d}t$$

- It is enough $\mathbf{osc}_q[\varrho_n \to \varrho] < \infty$ for $q > 2$. Can be shown for $q = \gamma + 1$.

Convective term:

$$\underbrace{\varrho}_{L^\gamma} \underbrace{\mathbf{u}}_{L^6} \otimes \underbrace{\mathbf{u}}_{L^6} \implies \gamma > 3/2$$

Anisotropic viscosity

- Needed

$$\partial_{x_i} \Delta^{-1}[\Delta u^i] = \mathrm{div}_x \mathbf{u}$$

Anisotropic viscosity:

$$\Delta_{\mathbf{u}} = \alpha \Delta_y \mathbf{u} + \beta \Delta_z \mathbf{u}$$

Vacuum problem

$$0 < \underline{\varrho} \leq \varrho_0(x) \leq \overline{\varrho}$$

Do we have conjecture (C)?

- $$0 < \underline{\varrho}(t) \leq \varrho(t,x) \text{ for all finite } t$$

$$\mathrm{div}_x \mathbf{u} \in L^1(0,T;L^\infty(\Omega)) \implies \text{(C)}$$

Do we have equivalence ?

$$\varrho \leq \overline{\varrho}(t) \iff \varrho \geq \underline{\varrho}(t) > 0 \iff \mathrm{div}_x \mathbf{u} \in L^1(0,T;L^\infty(\Omega))$$

Density dependent viscosity

VISCOUS STRESS TENSOR

$$\mathbb{S} = \mu(\varrho)\Big(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\Big) + \eta(\varrho)\mathrm{div}_x \mathbf{u}$$

Partial results

- Vaigant and Kazhikhov [1995], $N = 2$, periodic BC, $\mu = \mathrm{const}$
- Bresch and Desjardins (Mellet, Vasseur) [2006], special relation between μ and η, periodic BC

Convergence to equilibria

Driving force:

$$\varrho \mathbf{f} = \varrho \nabla_x F$$

EQUILIBRIUM SOLUTIONS

$$a \nabla_x \tilde{\varrho}^{\gamma} = \tilde{\varrho} \nabla_x F$$

There are equilibria $\tilde{\varrho}$ that vanish on some part of Ω!

Known results:

- $$\{ x \in \Omega \mid F(x) > k \} \text{ connected for any } k$$

  $$\Longrightarrow$$

  $$\varrho \to \tilde{\varrho}, \ \varrho \mathbf{u} \to 0 \text{ as } t \to \infty$$

Losing mass "at infinity"?

$$\int_\Omega \varrho(t,\cdot)\ \mathrm{d}x = M$$

$$\varrho(t,\cdot) \to \tilde{\varrho} \text{ in } L^\gamma(\Omega)$$

- Do we have

$$\int_\Omega \tilde{\varrho}\ \mathrm{d}x = M\ ?$$

Self-gravitation, plasma

$$\mathbf{f} = \nabla_x \Phi,\ \ \Delta\Phi = \alpha\varrho + g$$