Numerical Methods for a Shape Optimization Problem [1]

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Dedicated to S. Kesavan on his 60th birthday

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Problem Statement

Shape Optimization Problem.

- domain $\Omega \subset \mathbb{R}^n$, $\quad 0 < \alpha < \beta, \quad 0 < m < |\Omega|$
- $B \subset \Omega$ measurable; $A = \Omega \setminus B$; $|B| = m$.
- $$\inf \{\lambda(B) : B \subset \Omega \text{ measurable}, |B| = m.\} \ .$$

$$\begin{aligned} -\mathrm{div}(\sigma \nabla u) &= \lambda(B) u \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \ . \end{aligned}$$

- $\sigma = \alpha\chi_A + \beta\chi_B$; $\lambda(B)$ the first eigenvalue.

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Questions of Interest

Existence

- General Domains - Open
- Case of a Ball
- Alvino, Lions, Trombetti - Nonlinear Analysis 1989 [1]
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- Existence of a radially symmetric solution.

Uniqueness

Open Question.

Characterization

Can we find some explicit solutions?

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- Conclusions-The Disk Case.
- Numerical Results.
- A Descent Algorithm.
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Asymptotic Expansion

- Case when $\beta = \alpha + \varepsilon, \quad \varepsilon > 0$ small
- $\sigma^\varepsilon = \alpha + \varepsilon \chi_B$

Theorem (Rellich)

The first eigenvalue λ^ε of

$$-\operatorname{div}(\sigma^\varepsilon \nabla u^\varepsilon) = \lambda^\varepsilon u^\varepsilon \quad \text{in } \Omega, \tag{3.1}$$

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{3.2}$$

is an analytic function of ε in a neighbourhood of $\varepsilon = 0$ and the positive eigenfunction u^ε satisfying the normalization condition

$$\int_\Omega (u^\varepsilon)^2 = 1 \tag{3.3}$$

is analytic with respect to ε.

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...Asymptotic Expansion

So, we can introduce the series expansion

$$u^\varepsilon = v_0 + \varepsilon v_1 + \dots, \tag{3.4}$$
$$\lambda^\varepsilon = \lambda_0 + \varepsilon \lambda_1 + \dots, \tag{3.5}$$

in equations (3.1)-(3.2) and gather terms of similar order in ε :

$$-\operatorname{div}(\alpha \nabla v_0) = \lambda_0 v_0 \text{ in } \Omega, \tag{3.6}$$
$$v_0 = 0 \text{ on } \partial\Omega. \tag{3.7}$$

$$-\operatorname{div}(\alpha \nabla v_1) - \lambda_0 v_1 = \operatorname{div}(\chi_B \nabla v_0) + \lambda_1 v_0 \text{ in } \Omega, \tag{3.8}$$
$$v_1 = 0 \text{ on } \partial\Omega. \tag{3.9}$$

Due to the Fredholm alternative, equation (3.8)-(3.9) has a solution if and only if

$$\int_\Omega \operatorname{div}(\chi_B \nabla v_0) v_0 + \lambda_1 \int_\Omega v_0^2 = 0.$$

...Asymptotic Expansion

As

$$\int_\Omega v_0^2 = 1$$

we obtain

$$\lambda_1 = \int_B |\nabla v_0|^2. \tag{3.10}$$

Theorem (Conca, Laurain, M.)

For sufficiently small $\varepsilon > 0$

$$\operatorname{argmin}_{|B|=m} \lambda^\varepsilon(B) = \operatorname{argmin}_{|B|=m} \lambda_1(B) \tag{3.11}$$

Under additional hypotheses, the optimal solution for the problem (1.1) *is of the form*

$$\{x : |\nabla v_0(x)| < c^*\}.$$

$$\operatorname{argmin}_{|B|=m} \lambda^\varepsilon(B) = \operatorname{argmin}_{|B|=m}(\lambda_0 + \varepsilon\lambda_1(B) + \dots)$$

Conclusions - The Disk Case

- $\Omega = B(0,1)$; 2- or 3- dimensional space.
- solution of evp (3.6)-(3.7) is radial $v_0(x) = w(|x|)$

$$r^2 w_0''(r) + (d-1) r w_0'(r) + r^2 \frac{\lambda_0}{\alpha} w_0(r) = 0, \tag{3.12}$$

$$w_0'(0) = 0, \; w_0(1) = 0. \tag{3.13}$$

In 2-D, $w_0(r) = J_0(\eta_d r)$ where J_0 is Bessel functions of the first kind and η_d is it's first zero.

- So $|\nabla v_0|^2(x) = (w_1(r))^2$. where $w_1(r) := -w_0'(r)$ and the solution is then

$$\{x : w_1(r) < c^*\}$$

where c^* is such that $|\{x : w_1(|x|) < c^*\}| = m$.

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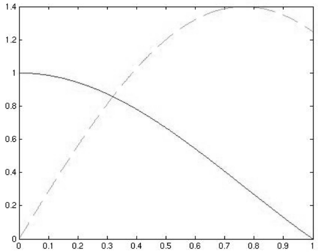
Theorem

The solution of (1.1) *is of two types. There exists* m_c *such that*

- *Type I: If* $m \leq m_c$ *then* $B^* = B(0, (m/\pi)^{1/2})$ *or,*
- *Type II: If* $m > m_c$ *then there exists* ξ^0 *and* ξ^1 *with* $(m/\pi)^{1/2} < \xi^0 < \xi^1 < 1$ *such that*

$$B^* = B(0, \xi^0) \cup \left(B(0,1) \setminus \overline{B(0,\xi^1)} \right) .$$



Small Conductivity Gap-Other Domains

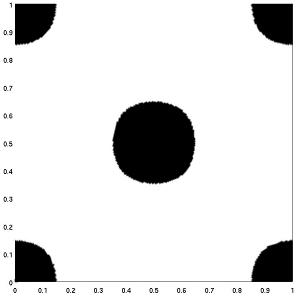


Figure: The optimal distribution in the square case.

...Small Conductivity Gap-Other Domains

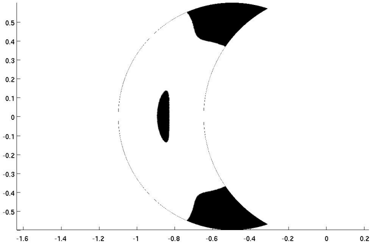


Figure: The optimal distribution in the crescent case.

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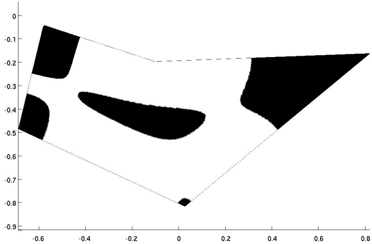

Figure: The optimal distribution in the polygon case.

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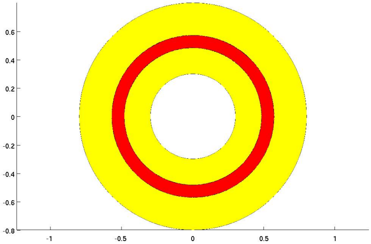

Figure: The optimal distribution in the ring case.

A Descent Algorithm-general α, β

Variational formulation for λ

$$\lambda = \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega \sigma |\nabla u|^2}{\int_\Omega u^2} = \min_{u \in H_0^1(\Omega), ||u||_2 = 1} \int_\Omega \sigma |\nabla u|^2. \tag{5.1}$$

A Descent Algorithm

- Initial measurable set B_0, $|B_0| = m$.
- $m(B_0, c) = |\{x : |\nabla u_{B_0}(x)| \leq c\}|$. Non-decreasing $m(B_0, c) \to 0$ as $c \to 0$ whereas, $m(B_0, c) \to |\Omega|$ as $c \to \infty$.
- $$c_0 := \inf\{c : m(B_0, c) \geq m\}. \tag{5.2}$$
- Under suitable conditions $|\{x : |\nabla u_{B_0}(x)| \leq c_0\}| = m$.
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$\lambda(B_1) \leq \lambda(B_0)$; equality holds if and only if $B_1 = B_0$ almost everywhere (under extra hypotheses). If B_0 is optimal, then B_0 is almost everywhere equal to the level set $\{x : |\nabla u_{B_0}(x)| \leq c_0\}$.

- **The disk case.** $\Omega = B(0, R)$. The optimal set B^* should include the origin.
- **The ring or torus case.** If again we have radial symmetry, then the gradient of u vanishes at one point along a radius of the domain and by radial symmetry, the gradient of u vanishes on a whole circle whose center is the center of the ring or torus. This circle is in the optimal set.
- **Domains with corners in two dimensions.** In this case the optimal set B^* contains a neighbourhood of the corners with angle smaller than π while its complement $A^* = \Omega \setminus B^*$ contains a neighbourhood of the corners with angle greater than π.

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The Disk Case

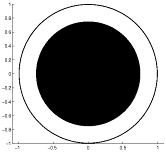

Figure: Initial domain $B_0 = B(0, 0.75)$

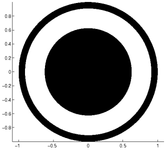

Figure: The optimal distribution in the disk case.

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