Control problems in the coefficients with a nonlinear cost functional in the gradient

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Elliptic problem (model example)

Given $\alpha, \beta > 0$, $\Omega \subset \mathbb{R}^N$ open bounded, $f \in H^{-1}(\Omega)$

$F: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ Carathéodory function, such that

$$|F(x,s,\xi)| \leq C(h(x) + |s|^2 + |\xi|^2), \qquad h \in L^1(\Omega),$$

we consider

$$\inf \int_\Omega F(x,u,\nabla u)dx$$

$$\begin{cases} -\mathrm{div}\Big(\big(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}\big)\nabla u\Big) = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

$$\omega \subset \Omega, \text{ measurable}, \qquad |\omega| \leq \mu$$

This problem has not solution in general (F. Murat 1971)

Two reasons

- The solutions of a sequence of problems of the form

$$\begin{cases} -\mathrm{div}\Big(\big(\alpha\chi_{\omega_n} + \beta\chi_{\Omega\setminus\omega_n}\big)\nabla u_n\Big) = f \text{ in } \Omega \\ u_n = 0 \ \text{ on } \partial\,\Omega \end{cases}$$

  do not converge to the solution of a similar problem.

- The functional

$$u \longmapsto \int_\Omega F(x, u, \nabla u)dx$$

  is not continuous in general for the weak topology of $H^1(\Omega)$

Remark. The second difficulty does not hold if $F(x, s, \xi)$ is linear in ξ.

The first difficulty allow us to study the asymptotic behavior of

$$(1)\ \begin{cases} -\mathrm{div}(A_n(x)\nabla u_n) = f \text{ in } \Omega \\ u_n = 0 \text{ on } \partial\,\Omega \end{cases}$$

Theorem (G. Spagnolo 1968 symmetric case; F. Murat , L. Tartar non-symmetric case 1977)

$$A_n \in L^\infty(\Omega)^{N\times N}, \quad \alpha \le A_n(x)\xi\cdot\xi,\ \ \alpha \le A_n^{-1}(x)\xi\cdot\xi\ ,$$

Then, for a subsequence, $\exists A$ in the same conditions that A_n such that $\forall f \in H^{-1}(\Omega)$

$$u_n \rightharpoonup u \text{ in } H^1(\Omega), \quad A_n\nabla u_n \rightharpoonup A\nabla u\ in\ L^2(\Omega)^N.$$

with u solution of $\begin{cases} -\mathrm{div}(A(x)\nabla u) = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\,\Omega. \end{cases}$

We say that A_n H-converges to A.

Definition. For $p \in [0,1]$, , we denote by $\mathcal{K}(p)$ the set of constant matrices A such that

$$\exists \omega_n \subset \Omega, \;\; \chi_{\omega_n} \overset{*}{\rightharpoonup} p \text{ in } L^\infty(\Omega), \qquad \left(\alpha\chi_{\omega_n} + \beta\chi_{\Omega\setminus\omega_n}\right) \xrightarrow{H} A$$

Phisically, $\mathcal{K}(p)$ is the set of materials obtained mixing the materials corresponding to the diffusion matrices $\alpha I, \beta I$ with proportions $p, 1-p$.

Theorem (L.Tartar 1985; K. Lurie, A. Cherkaev 1986)

$$\underline{\lambda}(p) = \left(\frac{p}{\alpha} + \frac{1-p}{\beta}\right)^{-1}, \qquad \overline{\lambda}(p) = \alpha p + \beta(1-p)$$

$\mathcal{K}(p)$ agrees with the set of matrices A with eigenvalues $\lambda_1, \dots, \lambda_N$ satisfying $\underline{\lambda}(p) \le \lambda_1 \le \cdots \le \lambda_N \le \overline{\lambda}(p)$

$$\sum_{i=1}^{N} \frac{1}{\lambda_i - \alpha} \le \frac{1}{\underline{\lambda}(p) - \alpha} + \frac{N-1}{\overline{\lambda}(p) - \alpha}, \;\; \sum_{i=1}^{N} \frac{1}{\beta - \lambda_i} \le \frac{1}{\beta - \underline{\lambda}(p)} + \frac{N-1}{\beta - \overline{\lambda}(p)}$$

Theorem (F. Murat, L. Tartar 1985) A relaxation of

$$\inf \int_{\Omega} F(x,u)dx$$

$$-\mathrm{div}\Big(\big(\alpha\chi_{\omega} + \beta\chi_{\Omega\setminus\omega}\big)\nabla u\Big) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\,\Omega$$

$$\omega \subset \Omega, \text{measurable}, \qquad |\omega| \leq \mu$$

is given by

$$\inf \int_{\Omega} F(x,u)dx$$

$$-\mathrm{div}(A\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\,\Omega$$

$$\theta \in L^{\infty}(\Omega; [0,1]), \qquad \int_{\Omega} \theta dx \leq \mu, \qquad A \in \mathcal{K}(\theta) \text{ a.e.in } \Omega$$

In order to consider general functional

$$u \longmapsto \int_{\Omega} F(x, u, \nabla u)dx$$

we need to pass to the limit in

$$\int_{\Omega} F(x, u_n, \nabla u_n)dx,$$

with u_n solution of

$$-\mathrm{div}\Big(\big(\alpha\chi_{\omega_n} + \beta\chi_{\Omega\setminus\omega_n}\big)\nabla u_n\Big) = f \text{ in } \Omega, \quad u_n = 0 \text{ on } \partial\,\Omega$$

Periodic homogenization $Y = (0,1)^N$, $M \in L^\infty_\#(Y)^{N\times N}$, elliptic.

Then $M_\varepsilon(x) = M\left(\frac{x}{\varepsilon}\right) \overset{H}{\rightharpoonup} M_h$ defined as

$$M_h\xi = \int_Y M(\xi + \nabla w)dy, \qquad \forall\xi \in \mathbb{R}^N$$

with w solution of $-\mathrm{div}\big(M(\xi + \nabla w)\big) = 0$ in $\mathbb{R}^N$, $w \in H^1_\#(Y)$

So, if u_n solution of $-\mathrm{div}\left(M\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f$ in Ω, $u_\varepsilon = 0$ on $\partial\,\Omega$.

Then $u_\varepsilon \rightharpoonup u$ in $H^1(\Omega)$ with $-\mathrm{div}(M_h\nabla u) = f$ in Ω, $u = 0$ on $\partial\,\Omega$.

Moreover (corrector result) if u smooth, $w_u = w_u(x,y)$ solution of

$$-\mathrm{div}\Big(M(y)\big(\nabla_x u(x) + \nabla_y w_u(x,y)\big)\Big) = 0 \text{ in } \mathbb{R}^N,\ \ w_u \in L^2\left(\Omega; H^1_\#(Y)\right).$$

Then

$$u_\varepsilon - u - \varepsilon w_u\left(x, \frac{x}{\varepsilon}\right) \to 0 \text{ in } H^1(\Omega).$$

In particular

$$\int_{\Omega} F(x, u_\varepsilon, \nabla u_\varepsilon) dx \to \int_{\Omega} \int_{Y} F(x, u(x), \nabla u(x) + \nabla_y w(x, y)) dy dx$$

Theorem: (G. Dal Maso, R. Kohn) The matrices obtained by periodic homogenization are dense in the matrices obtained by general homogenization

Definition:

$$H(x,s,\xi,\eta,p) = \lim_{\delta \searrow 0} H_\delta(x,s,\xi)$$

$$\text{with} \quad H_\delta(x,s,\xi,\eta,p) = \inf \int_Y F(x,s,\xi + \nabla_{\mathrm{y}} w)dx$$

$$-\mathrm{div}\Big(\big(\alpha\chi_Z + \beta\chi_{Y\setminus Z}\big)\big(\xi + \nabla_{\mathrm{y}} w\big)\Big) = 0 \text{ in } Y \text{ with periodic conditions}$$

$$|Z| = p, \qquad \left|\int_Y \big(\alpha\chi_Z + \beta\chi_{Y\setminus Z}\big)\big(\xi + \nabla_{\mathrm{y}} w\big)dy - \eta\right| < \delta$$

Theorem (JCD, J. Couce-Calvo, J.D. Martín-Gómez)

A relaxation of

$$\inf \int_\Omega F(x,u,\nabla u)dx$$

$$-\mathrm{div}\Big(\big(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}\big)\nabla u\Big) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\,\Omega$$

$$\omega \subset \Omega, \text{measurable}, \qquad |\omega| \leq \mu$$

is given by

$$\inf \int_\Omega H(x,u,\nabla u, A\nabla u, \theta)dx$$

$$-\mathrm{div}(A\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\,\Omega$$

$$\theta \in L^\infty(\Omega;[0,1]), \qquad \int_\Omega \theta dx \leq \mu, \qquad A \in \mathcal{K}(\theta) \text{ a.e. in } \Omega$$

Related results (G.Allaire, J.C. Bellido, P-Pedregal, L. Tartar,…)

Remark. The problem only provides $A\nabla u$ and not A. Thus, it is not necessary to know $\mathcal{K}(\theta)$ but only

$$\mathcal{K}(p)\xi = \{A\xi : A \in \mathcal{K}(p)\}, \qquad \forall \xi \in \mathbb{R}^N, p \in [0,1].$$

In our case

$$\mathcal{K}(\theta)\xi = B\left(\frac{\overline{\lambda}(p) + \underline{\lambda}(p)}{2}\xi, \frac{\overline{\lambda}(p) - \underline{\lambda}(p)}{2}|\xi|\right)$$

The function H is not known in general by it is known in the boundary of its domain. Indeed

$$H(x, s, \xi, \eta, p) = pF\left(x, s, \frac{\beta\xi - \eta}{p(\beta - \alpha)}\right) + (1-p)F\left(x, s, \frac{\eta - \alpha\xi}{(1-p)(\beta - \alpha)}\right)$$

If $\eta \in \partial(\mathcal{K}(p)\xi)$.

Parabolic problem

$$\inf \int_0^T \int_\Omega F(t,x,u,\nabla u)dx$$

$$\begin{cases} \partial_t u - \operatorname{div}\Big(\big(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}\big)\nabla u\Big) = f \text{ in } (0,T)\times\Omega \\ u = 0 \text{ on } \partial\,\Omega\times(0,T), \qquad u(0,x) = u^0(x) \text{ in } \Omega \end{cases}$$

$$\omega \subset \Omega, \text{measurable}, \qquad |\omega| \le \mu.$$

It is known that the homogenized matrix for a parabolic problem with coefficients independent of t agrees with the elliptic one. Moreover the elliptic corrector provides a parabolic corrector.

Therefore, if u_ε solution of ($M \in L^\infty_\#(Y)^{N\times N}$)

$$\begin{cases} \partial_t u_\varepsilon - \mathrm{div}\left(M\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f \text{ in } \Omega \times (0,T) \\ u_\varepsilon = 0 \text{ on } \partial\,\Omega \times (0,T), \qquad u_\varepsilon(0,x) = u^0(x) \text{ in } \Omega \end{cases}$$

Then

$$\int_\Omega F(t,x,u_\varepsilon,\nabla u_\varepsilon)dx$$

$$\to \int_\Omega \int_0^T \int_Y F(t,x,u(t,x),\nabla u(t,x) + \nabla_{\mathrm{y}} w_u(t,x,y))dydx$$

$$-\mathrm{div}_{\mathrm{y}}\left(M(y)\left(\nabla_x u(t,x) + \nabla_y w_u(t,x,y)\right)\right) = 0 \text{ in } \mathbb{R}^N,$$

$$w \in L^2\left((0,T)\times\Omega; H^1_\#(Y)\right)$$

Definition: $H : \Omega \times L^2(0,T) \times L^2(0,T) \times \{(M,p) : M \in \mathcal{K}(p)\} \to \mathbb{R}$

$$H(x,s,\xi,M,p) = \lim_{\delta \searrow 0} H_\delta(x,s,\xi,M,p)$$

$$H_\delta(x,s,\xi,M,p) = \inf \int_0^T \int_Y F(x,t,s(t),\xi(t) + \nabla_y w(t,y))\,dt\,dy$$

$$-\mathrm{div}\Big(\big(\alpha\chi_Z + \beta\chi_{Y\setminus Z}\big)\big(\xi(t) + \nabla_y w(t,y)\big)\Big) = 0 \text{ in } Y$$

with periodic conditions

$$|Z| = p, \quad \left|\int_Y \big(\alpha\chi_Z + \beta\chi_{Y\setminus Z}\big)\big(\xi(t) + \nabla_y w(t,y)\big) - M\xi(t)\right| < \delta|\xi(t)|$$

Relaxed parabolic problem

$$\inf \int_{\Omega} H(x, u(x, .), \nabla u(x, .), A(x), \theta(x))dx$$

$$\begin{cases} \partial_t u - \operatorname{div}(A(x)\nabla u) = f \text{ in } (0, T) \times \Omega \\ u = 0 \text{ on } \partial\,\Omega \times (0, T), \qquad u(0, x) = u^0(x) \text{ in } \Omega \end{cases}$$

$$\theta \in L^\infty(\Omega; [0,1]), \qquad \int_{\Omega} \theta dx \leq \mu, \qquad A \in \mathcal{K}(\theta) \text{ a.e. in } \Omega$$

Difficulties:

- The functional is non-local
- It is necessary to know the whole set $\mathcal{K}(p)$
- To obtain the value of H in some points
- To construct admissible directions to solve numerically the relaxed problem