Optimal control for a conservation law modeling the development of ovulation

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Outline

- Introduction of the model and related control problems
- Some numerical results
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- Numerical results based on PMP
- Optimal control result for PDE case
- Open problems
- References

The cell population in a follicle is represented by cell density functions $\rho_{j,k}(t,x,y)$ defined on each cellular phase $Q_{j,k}$ with age x and maturity y, which satisfy the following conservation laws

$$\frac{\partial \rho_{j,k}(t,x,y)}{\partial t} + \frac{\partial \rho_{j,k}(t,x,y)}{\partial x} + \frac{\partial (h(y,u)\rho_{j,k}(t,x,y))}{\partial y} = 0, \quad \text{in } Q_{j,k} \tag{1.1}$$

Here $k = 1, \cdots, N$, and N is the number of consecutive cell cycles. $j = 1, 2, 3$ denotes Phase 1, Phase 2 and Phase 3.

$$h(y,u) = -y^2 + (c_1 y + c_2)\, u, \tag{1.2}$$

with c_1 and c_2 given positive constants.

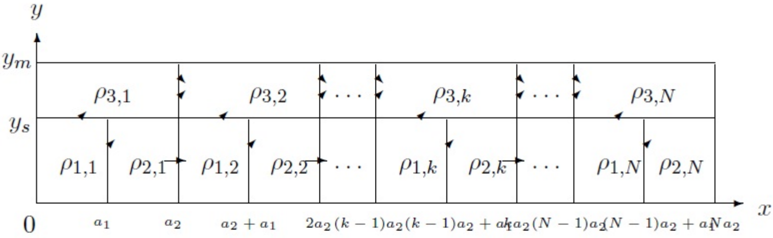

y
y_m
y_s
$\rho_{3,1}$
$\rho_{3,2}$
$\rho_{3,k}$
$\rho_{3,N}$
$\rho_{1,1}$
$\rho_{2,1}$
$\rho_{1,2}$
$\rho_{2,2}$
$\rho_{1,k}$
$\rho_{2,k}$
$\rho_{1,N}$
$\rho_{2,N}$
0
a_1
a_2
$a_2 + a_1$
$2a_2$
$(k-1)a_2$
$(k-1)a_2 + a_1$
ka_2
$(N-1)a_2$
$(N-1)a_2 + a_1$
Na_2
x

Figure 1:

related control problems

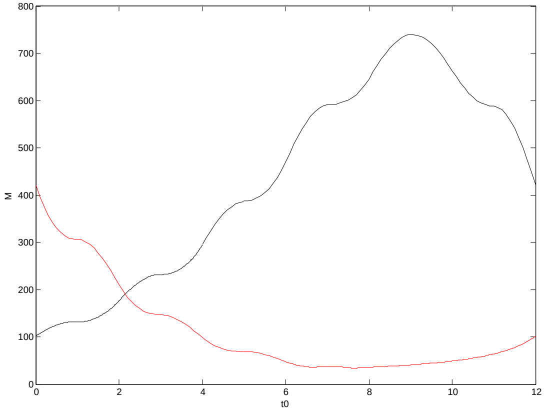
- We define

$$M(t) := \sum_{j=1}^{3} \sum_{k=1}^{N} \int_0^{+\infty} \int_0^{+\infty} y\, \rho_{j,k}(t,x,y)\, dx\, dy \qquad (1.3)$$

  as the follicular maturity.
- The control $u(M(t))$.
- Ovulation is triggered when the maturity reaches a given threshold value M_s. Hence, the optimal control problem is, for fixed observed time t_1 to maximize the maturity $M(t_1)$.
- Proliferative cells leave the cycle in an irreversible way, we get the restraint of control $u \in [w, 1]$ with $w \in (0, 1)$.

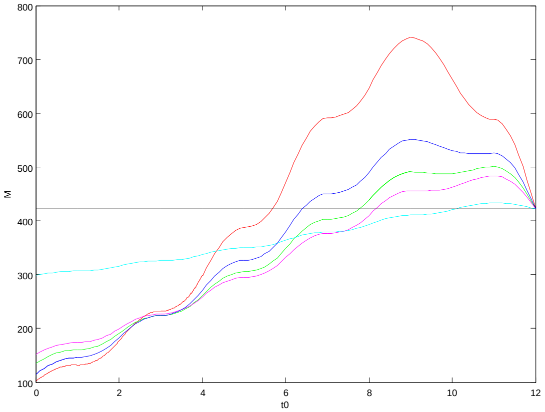
switching direction

black : $u = w \to u = 1$; red: $u = 1 \to u = w$



Numerical result of optimal bang-bang control

red: $u = w \to u = 1$, black: $u = w$



simplified model

Consider the following balance law

$$\rho_t + \rho_x + (h(y,u)\rho)_y = c_s\,\chi(y), \quad t \geq 0,\ x \geq 0,\ y \geq 0, \tag{3.1}$$

with c_s a given positive constant, we denote

$$a(y) = -y^2, \quad b(y) = c_1 y + c_2. \tag{3.2}$$

$$h(y,u) = a(y) + b(y)u. \tag{3.3}$$

$\chi(y)$ is a characteristic function

$$\chi(y) = \begin{cases} 1, & \text{if } y \in [0, y_s), \\ 0, & \text{if } y \in (y_s, \infty), \end{cases} \tag{3.4}$$

(BC)

$$\rho(t,0,y)=\rho(t,x,0)=0,\quad \forall x\geq 0,\ y\geq 0. \tag{3.5}$$

(IC)

The initial condition $\rho_0(x,y)$ is given as a positive Borel measure with compact support $\subset [0,1]^2$.

For any admissible control $u\in L^\infty([t_0,t_1];[w,1])$, the cost function is

$$J(u)=-\iint_{[0,+\infty)\times[0,+\infty)} y\,d\rho(t_1,x,y). \tag{3.6}$$

one Dirac mass

We consider the following optimal control problem $(\mathcal{P})$:

$$\begin{cases} \dot{x} = f(x,u), \quad u \in L^\infty([t_0,t_1];[w,1]), \quad t \in [t_0,t_1], \\ x(t_0) = x^0, \\ J(u) = \int_{t_0}^{t_1} (p(x,u) + q(x)\,\chi(x_2))\,dt \end{cases} \tag{3.7}$$

where

$$p(x,u) := -(a(x_2) + b(x_2)\,u)\,x_3, \quad q(x) := -c_s\,x_2\,x_3. \tag{3.8}$$

$$f = \begin{pmatrix} 1 \\ a(x_2) + b(x_2)\,u \\ c_s\,\chi(x_2)\,x_3 \end{pmatrix}$$

$x = (x_1, x_2, x_3)^{tr} \in R^3$, x_1 denotes the age, x_2 denotes the maturity and x_3 denotes the mass.

The main difficulty of problem $(\mathcal{P})$ is that both f and the integrand χ of the functional J are discontinuous.
First, we proved the existence of optimal control to problem $(\mathcal{P})$ by approximating method.

Theorem

The infimum of the functional J in $L^\infty([t_0, t_1]; [w, 1])$ is achieved, i.e., there exists $u \in L^\infty([t_0, t_1]; [w, 1])$ such that

$$J(u) = \inf_{u \in L^\infty([t_0,t_1];[w,1])} J(u).$$

Theorem

For any measurable optimal control u_* to problem $(\mathcal{P})$, the following property holds:
There exists $t' \in [t_0, t_1)$ such that

$$u_* = w \text{ in } (t_0, t') \text{ and } u_* = 1 \text{ in } (t', t_1). \tag{3.9}$$

Furthermore, under the assumption that

$$2y_s - c_1 > 0 \quad \text{and} \quad c_s > \frac{a(y_s) + b(y_s)}{y_s}, \tag{3.10}$$

this optimal switch time t' is the exit time $\hat{t}$.

Necessary optimal conditions for optimal control (PMP)

Necessary optimal conditions (A. I. Smirnov, 2008). Let us define the Hamilton-Pontryagin function and the Hamiltonian as

$$\mathcal{H}(x,u,\psi,\psi^0) := < f(x,u), \psi > + \psi^0 \Big(p(x,u) + q(x)\chi(x_2) \Big), \tag{3.11}$$

$$H(x,\psi,\psi^0) = \max_{u \in U} \mathcal{H}(x,u,\psi,\psi^0). \tag{3.12}$$

Let us denote $\hat{t}$ as $x_{*2}(\hat{t}) = y_s$.

Let $u_* \in L^\infty([t_0, t_1]; [w, 1])$ and $x_* = (x_{*1}, x_{*2}, x_{*3})^{tr}$ be the optimal control and the corresponding optimal trajectory in problem $(\mathcal{P})$. Then there exist a function $\psi = (\psi_1, \psi_2, \psi_3)^{tr}$, $\psi_1 \in W^{1,\infty}(t_0, t_1)$, $\psi_2 \in W^{1,\infty}(t_0, \hat{t}) \cup (\hat{t}, t_1)$, and $\psi_3 \in W^{1,\infty}(t_0, t_1)$ such that the following conditions hold:

(a) The function ψ is a solution to the adjoint system:

$$\dot{\psi}_1(t) = 0, \tag{3.13}$$

$$\dot{\psi}_2(t) = -\Big(\frac{\partial a(x_{*2}(t))}{\partial x_2} + \frac{\partial b(x_{*2}(t))}{\partial x_2} u_*(t)\Big) \psi_2(t) + \frac{\partial p(x_*(t), u_*(t))}{\partial x_2} + \frac{\partial q(x_*(t))}{\partial x_2} \chi(x_{*2}(t)), \ t \neq \hat{t}, \tag{3.14}$$

$$\dot{\psi}_3(t) = -c_s\, \chi(x_{*2}(t))\, \psi_3(t) + (a(x_{*2}(t)) + b(x_{*2}(t)) u_*(t)) + c_s\, x_{*2}(t)\, \chi(x_{*2}(t)). \tag{3.15}$$

(b) The jump when the $x_{*2} = y_s$:

$$\psi_2(\hat{t}+0) - \psi_2(\hat{t}-0) \in c_s x_3(\hat{t})\Big(\frac{y_s + \psi_3(\hat{t})}{a(y_s)+b(y_s)}, \frac{y_s + \psi_3(\hat{t})}{a(y_s)+b(y_s)w}\Big).$$

$$\psi_1(t_1) = \psi_2(t_1) = \psi_3(t_1) = 0.$$

(c) The maximum condition holds:

$$H\Big(x_*(t), \psi\Big) \overset{a.e.}{=} \mathcal{H}\Big(x_*(t), u_*(t), \psi\Big).$$

proof of the Theorem

Now, the Hamiltonian becomes

$$H(t) = (a(x_2)+c_s\chi x_2)\, x_3+\psi_1+a(x_2)\psi_2+c_s\chi\, x_3\, \psi_3+b(x_2)(x_3+\psi_2)\, u.$$

Noting that $b(x_2) > 0$, one has

$$u_*(t) = 1 \quad \text{if} \quad x_3(t) + \psi_2(t) > 0, \tag{3.16}$$

$$u_*(t) = w \quad \text{if} \quad x_3(t) + \psi_2(t) < 0. \tag{3.17}$$

Under the assumption that $2y_s - c_1 > 0$, we have

$$(x_3 + \psi_2)(\hat{t} - 0) \leqslant x_3(t_1)(1 - c_s \frac{y_s}{a(y_s) + b(y_s)}). \tag{3.18}$$

Under the assumption that $c_s > \dfrac{a(y_s) + b(y_s)}{y_s}$, we have

$$(x_3 + \psi_2)(\hat{t} - 0) < 0. \tag{3.19}$$

Hence,

$$(x_3 + \psi_2)(t) < 0, \ t \in [t_0, \hat{t}). \tag{3.20}$$

Moreover,

$$(x_3 + \psi_2)(t) > 0, \ t \in (\hat{t}, t_1]. \tag{3.21}$$

Above all, we have proved the Theorem.

proof of the necessary optimal conditions

Step 1: Mollifier the characteristic χ.
Step 2: Let u_*, x_* be an optimal pair in problem $(\mathcal{P})$. Take a sequence $\{z_i\}$, $i = 1, 2, \cdots$, of functions $z_i \in C^1[t_0, t_1]$ that satisfy the following conditions

$$z_i \to u_* \text{ in } L^2[t_0, t_1] \text{ as } i \to \infty, \tag{3.22}$$

$$\sup_{t_0 \leqslant t \leqslant t_1} \|z_i(t)\| \leqslant \|U\| + 1, \ i = 1, 2, \cdots, \tag{3.23}$$

$$\sup_{t_0 \leqslant t \leqslant t_1} \|\dot{z}_i(t)\| \leqslant \sigma_i < \infty. \tag{3.24}$$

We may assume without loss of generality that $\sigma_i \to \infty$ as $i \to \infty$.

Step 3: Now consider the following sequence of auxiliary optimal control problems $(\mathcal{P}_i)$

$$\begin{cases} \dot{x} = f_i(x,u), \quad u \in L^\infty([t_0,t_1];[w,1]), \quad t \in [t_0,t_1], \\ x(t_0) = x^0, \\ J_i(u) = \int_{t_0}^{t_1} \Big(p(x,u) + q(x)\chi_i(x_2) \Big) dt + \frac{1}{1+\sigma_i} \int_{t_0}^{t_1} \|u(t) - z_i(t)\|^2 dt. \end{cases} \tag{3.25}$$

Here

$$f_i = \begin{pmatrix} 1 \\ a(x_2) + b(x_2)\, u \\ c_s\, \chi_i(x_2)\, x_3 \end{pmatrix}$$

For any $i = 1, 2, \cdots$, problem $(\mathcal{P}_i)$ is a smooth optimal control problem. Hence, there exists an optimal control u_i in problem $(\mathcal{P}_i)$ (L. Cesari, 1983). Let x_i be the corresponding optimal trajectory. We have the following result

Lemma

The following relations hold as $i \to \infty$

$$u_i \to u_* \quad \textit{in} \quad L^2[t_0, t_1], \tag{3.26}$$

$$x_i \to x_* \quad \textit{in} \quad C^0[t_0, t_1]. \tag{3.27}$$

Suppose that x_i and u_i is an optimal pair in problem $(\mathcal{P}_i)$. Define the Hamilton-Pontryagin function and the Hamiltonian for problem $(\mathcal{P}_i)$ as follows

$$\mathcal{H}_i(t,x,u,\psi,\psi^0) = < f_i(x,u),\psi > +\psi^0(p(x,u)+q(x)\chi_i(x_2)) + \psi^0\Big(\frac{1}{1+\sigma_i}\|u(t)-z_i(t)\|^2\Big),$$

and

$$H_i(t,x,\psi,\psi^0) = \max_{u\in U} \mathcal{H}_i(t,x,u,\psi,\psi^0).$$

By Pontryagin's maximum principle, there exists a number $\psi_i^0 \leqslant 0$ and an absolutely continuous function ψ_i on $[t_0, t_1]$ such that

$$\begin{aligned}\dot{\psi}_i(t) \overset{a.e.}{=} & -\Big[\frac{\partial f_i(x_i(t), u_i(t))}{\partial x}\Big]^* \psi_i - \psi_i^0 \frac{\partial p(x_i(t), u_i(t))}{\partial x} \\ & - \psi_i^0 \Big(\frac{\partial q(x_i(t))}{\partial x} \chi_i(x_i(t)) + q(x_i(t)) \frac{\partial \chi_i(x_i(t))}{\partial x} \Big), \\ \psi_i(t_1) = & 0, \end{aligned} \tag{3.28}$$

and

$$H_i(t, x_i(t), \psi_i(t), \psi_i^0) \overset{a.e.}{=} \mathcal{H}_i(t, x_i(t), u_i(t), \psi_i(t), \psi_i^0). \tag{3.29}$$

Passing to the limit $i \to \infty$ in necessary optimal conditions for problem $(\mathcal{P}_i)$, finally we prove the necessary optimal conditions for problem $(\mathcal{P})$.

n Dirac masses

Using the same method as one Dirac mass, we get similar result that

Theorem

Under the assumption that

$$2y_s - c_1 > 0 \quad \textit{and} \quad c_s > \frac{a(y_s) + b(y_s)}{y_s}, \tag{3.30}$$

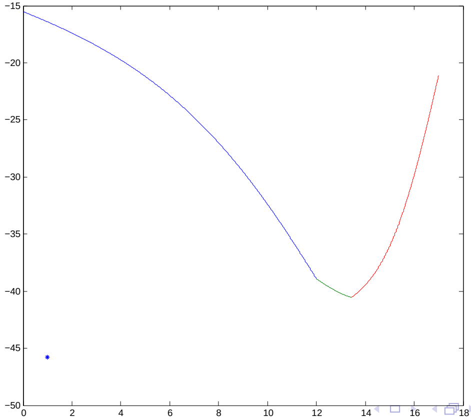
For any optimal control u_*, the following property holds
There exists $t' \in (t_0, t_1)$ such that

$$u_* = w \textit{ in } (t_0, t') \textit{ and } u_* = 1 \textit{ in } (t', t_1). \tag{3.31}$$

counter example with small c_s

Remark

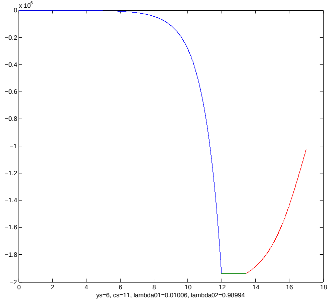
The assumption (3.30) is important to guarantee that the optimal switch time is once.

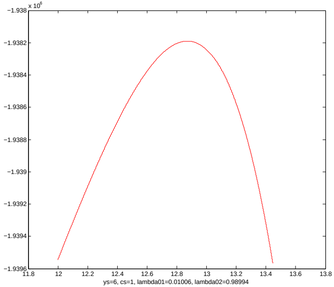


nonuniqueness of the optimal control for two Dirac masses

Remark

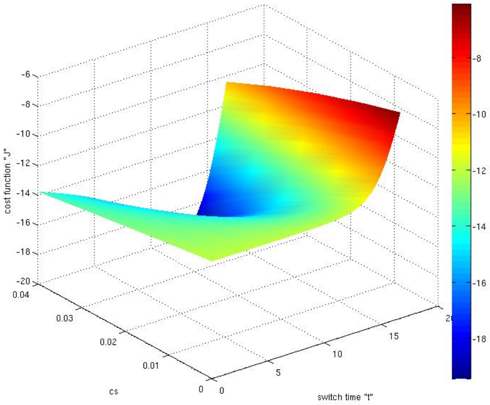
For more than one Dirac masses, the optimal control is not unique.





optimal control for one Dirac mass with different c_s

For one Dirac mass, when c_s is small, the optimal control is always $u = 1$; when c_s is large, the optimal control is $u = w \to u = 1$.



Recall the PDE case

$$\rho_t + \rho_x + (h(y,u)\rho)_y = c_s\,\chi(y), \quad t \geq 0,\ x \geq 0,\ y \geq 0. \tag{5.1}$$

The cost function is

$$J(u) = -\iint_{[0,+\infty)\times[0,+\infty)} y\,d\rho(t_1,x,y). \tag{5.2}$$

We have the following result

Theorem

Under the assumption that

$$2y_s - c_1 > 0 \quad \textit{and} \quad c_s > \frac{a(y_s) + b(y_s)}{y_s}, \tag{5.3}$$

we have that among all admissible controls $u \in L^\infty([t_0, t_1]; [w, 1])$, *there exists an optimal control* u_* *to* (5.2) *such that the following property holds*
There exists $t' \in (t_0, t_1)$ *such that*

$$u_* = w \textit{ in } (t_0, t') \textit{ and } u_* = 1 \textit{ in } (t', t_1). \tag{5.4}$$

Step 1: There exists a sequence

$$\rho_0^n = \sum_{i=1}^{n} \lambda_0^i \delta_{x_0^i, y_0^i}, \tag{5.5}$$

such that for any given $\varphi \in C^0(K)$ we have

$$(\rho_0^n - \rho_0)\,\varphi \to 0 \quad \text{as} \quad n \to \infty. \tag{5.6}$$

The cost function is

$$J(\rho_0^n, u) = -\sum_{i=1}^{n} y_i(t_1, u)\lambda_i(t_1, u). \tag{5.7}$$

For any $u \in L^\infty([t_0, t_1]; [w, 1])$, it is easy to prove that

$$\lim_{n\to\infty} J(\rho_0^n, u) = J(\rho_0, u). \tag{5.8}$$

Step 2: We assume that for each ρ_0^n, there exists an optimal control u_*^n such that

$$u_*^n := w \text{ in } (t_0, t_n'),\ u_*^n := 1 \text{ in } (t_n', t_1). \tag{5.9}$$

Without loss of generality, we may assume there exists $t' \in [t_0, t_1]$ such that

$$t_n' \to t' \text{ as } n \to \infty. \tag{5.10}$$

Let u_* be defined as

$$u_* := w \text{ in } (t_0, t'),\ u_* := 1 \text{ in } (t', t_1). \tag{5.11}$$

Then we prove that

$$\lim_{n\to\infty} J(\rho_0^n, u_*^n) = J(\rho_0, u_*). \tag{5.12}$$

Combining (5.8) and (5.12), we have proved that u_* defined an optimal control.

open problems

- For PDE case, can we get the result that each measurable optimal control is bang-bang control?
- For the moment, we consider the open loop problem, what about the close loop problem, e.x. $u(t) = u(t, M^1(t))$?

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Thanks for your attention!