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Workshop 'Partial Differential Equations, Optimal Design and Numerics'

Cubic Schroedinger equation: controllability/non-controllability results by geometric control methods

Andrey Sarychev
Dipartimento di Matematica per le Decisioni,
Università di Firenze, Italia

Introduction

Main theme: Lie algebraic approach of geometric control to *controllability of nonlinear distributed parameter systems*.

Example of implementation of such approach - study of *approximate controllability and controllability in finite-dimensional projections* (cf. A.Agrachev, A.Sarychev, S.Rodrigues) for 2D Navier-Stokes/Euler equation of fluid motion controlled by *low-dimensional forcing*.

Extension onto 3D-case A.Shirikyan, H.Nersisyan

Introduction ctd.

Goal: developing similar technique for *cubic defocusing 2D Schroedinger equation*

$$-i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + F(t,x) \quad \text{(NLS)}$$

controlled via source term $F(t,x)$. Problem setting is distinguished by the *type of control*:

- it enters additively and is 'generated by few functions':

$$F(t,x) = \sum_{k\in\hat{\mathcal{K}}} v_k(t) F^k(x), \ \mathbb{Z}^2 \supset \hat{\mathcal{K}} \text{ - finite},$$

i.e. $\forall t \mapsto F(t,x)$ takes values in finite-dim. space $\text{span}\{F^k(x) | \ k \in \hat{\mathcal{K}}\}$.

In 2D periodic case, $x \in \mathbb{T}^2$, it is natural to choose

$$F^k(x) = e^{ik \cdot x},\ \ k \in \mathbb{Z}^2.$$

Control functions $v_k(t) \in L_\infty[0, T]$.

Controlled NLS equation: controllability problem settings

Let defocusing cubic NLS on $\mathbb{T}^2$

$$-i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + \sum_{k \in \mathcal{K}^1} v_k(t) e^{ik\cdot x}$$

evolve in (functional) Hilbert space H. We will study:

controllability in finite-dimensional projections -
- $\forall$ finite-dimensional subspace $\mathcal{L} \subset H$ proper controls $v_k(t)$ may steer the system in time $T > 0$ from $u_0 \in H$ *to a point with preassigned orthogonal projection on* $\mathcal{L}$;

approximate controllability, when set of 'points', attainable in time $T > 0$, from each $u_0 \in H$ is dense in H,

(exact) controllability, when set of 'points', attainable in time $T > 0$, from each $u_0 \in H$ coincides with H.

Few references

to other approaches to controllability of linear and semilinear Schroedinger equation controlled via bilinear or additive control.

Surveys [Z:Zuazua, CRM Lecture Notes, 2003],
[ILT: Illner,Lange,Teismann, ESAIM COCV, 2006]

Results on:

- exact controllability for linear Schroedinger equation with additive control (numerous publications starting from [Lebeau,1992]);

References -2

• controllability of linear Schroedinger equation with control entering bilinearly.

Results by [Beauchard 2005, B. & Coron 2006] on local (exact) controllability in H^7 of 1-D equation, obtained by 'return method' and Nash-Moser th.;

Criterion (obtained by geometric control methods) [Chambrion, Mason, Sigalotti, Boscain, 2009] for approximate controllability for the case of 'drift Hamiltonian' with discrete non-resonant spectrum.

Talk by J.-P. Puel at present workshop.

References -3

• exact controllability of semilinear Schroedinger equation by means of internal additive ('infinite-dimensional') control [Dehman, Gerard, Lebeau, 2006], [Rosier, B.-Y. Zhang 2009] for 2D and 1D cases.

In semilinear case: key tool - 'linearization principle', going back to [Lasiecka & Triggiani, App.Math.Optim., 1991].

In contrast our approach makes **direct and exclusive use of the nonlinear term**.

What regards approaches to non-controllability we mention classical paper[Ball,Marsden,Slemrod, SICON, 1982] on bilinear systems and [Shirykian, Physica D, 2008] on Euler equation.

Controllability of NLS equation: criterion of approximate controllability and controllability in projections

Theorem 1. **There exists a set** $\hat{\mathcal{K}} = \{k^1, k^2, k^3, k^4\} \subset \mathbb{Z}^2$ **of** 4 **modes, such that cubic defocusing Schroedinger equation**

$$-i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + \sum_{s=1}^{4} v_s(t) e^{ik^s \cdot x},$$

evolving in $H^{1+\sigma}(\mathbb{T}^2)$, $\sigma > 0$, **is controllable in finite-dimensional projections and approximately controllable.** $\square$

Controllability of NLS equation: negative result on exact controllability

Theorem 2.

Given 2D periodic defocusing Schroedinger equation

$$-i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + \sum_{k \in \widehat{\mathcal{K}}} v_k(t) e^{ik\cdot x},$$

with initial data in $H^{1+\sigma}(\mathbb{T}^2)$,

controlled via source term $F(t,x) = \sum_{k\in\widehat{\mathcal{K}}} v_k(t)e^{ik\cdot x}$ acting on *arbitrary finite set* $\widehat{\mathcal{K}} \subset \mathbb{Z}^2$ **of controlled modes,**

$\forall T > 0$, **the time-T attainable set $\mathcal{A}_{T,u^0}$ from u^0 is contained in a countable union of compact subsets of $H^{1+\sigma}$ and therefore the complement $H^{1+\sigma} \setminus \mathcal{A}_{T,u^0}$ is dense in $H^{1+\sigma}$.**$\square$

Remark.

Ball, Marsden , Slemrod 1982 result on lack of controllability regarded a bilinear control system

$$\dot{u} = (A + v(t)B)\, u,$$

with scalar control $v(t)$, A generator of a C^0-semigroup, B - bounded operator.

Preliminaries on existence, uniqueness and continuous dependence of trajectories of NLS

Consider semilinear equation

$$(-i\partial_t + \Delta)u = G(t,u),\ \ u(0) = \tilde{u}^0,$$

an its integral reformulation according to Duhamel formula

$$u(t) = e^{it\Delta}\left(u^0 + i\int_0^t e^{-i\tau\Delta}G(\tau, u(\tau))d\tau\right).$$

Local existence results are valid for the right-hand sides under some boundedness and Lipschitz conditions. To guarantee those we opt for *very regular setting*: NLS equation will be evolving in Sobolev space $H = H^{1+\sigma}(\mathbb{T}^2),\ \sigma > 0$.

Semilinearities - such as polynomials in $u, \bar{u}$ with integrable coefficients $c(t)$ - are 'well behaving' in this space due to

'Product Lemma' (cf. [T.Tao, Nonlinear Dispersive Equations, AMS, 2006]) *For Sobolev spaces $H^s(\mathbb{T}^d)$, $s > d/2$ of functions there holds: for $s > d/2$:* $\|fg\|_{H^s} \leq (C'(s,d)\|f\|_{H^s}\|g\|_{H^s}$. $\square$

Controlled source term F is trigonometric polynomial in x with measurable essentially bounded in t controlled coefficients $v_k(t)$.

Local existence of solutions in regular setting is standard and can be established by fixed point argument for a contracting map in $C([0,T]; H^{1+\sigma}(\mathbb{T}^2))$.

Preliminaries-2

Global existence/uniqueness result for cubic NLS with source term:

$$-i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + F(t,x), \qquad (1)$$

Proposition 2. For the source term $F(t,x)$ from $L_\infty([0,T], H^{1+\sigma}(\mathbb{T}^2))$. for each $\tilde{u} \in H^{1+\sigma}$ the Cauchy problem with the initial condition $u(0) = \tilde{u}$ possesses **unique strong solution** $u(\cdot) \in C([0,T], H^{1+\sigma}(\mathbb{T}^2))$. $\square$

Preliminaries-3 Consider semilinear equation

$$(-i\partial_t + \Delta)u = G(t,u),\ \ u(0) = \tilde{u}^0. \tag{2}$$

and its semilinear 'perturbation':

$$(-i\partial_t + \Delta)u = G(t,u) + \phi(t,u), u(0) = u^0. \tag{3}$$

Proposition 3 (continuity in the right-hand side) Let $\tilde{u}(t) \in C([0,T],H)$ be solution of (2). Then $\exists\delta > 0,\ c > 0$ such that whenever

$$\|u^0 - \tilde{u}^0\| + \int_0^T \sup_{\|u\|\leq b} \|\phi(t,u)\|_H dt < \delta, \tag{4}$$

then solution $u(t)$ of the perturbed equation (3) exists on the interval $[0,T]$, is unique and admits an upper bound

$$\sup_{t\in[0,T]} \|u(t) - \tilde{u}(t)\| \leq c\left(\|u^0 - \tilde{u}^0\| + \int_0^T \sup_{\|u\|\leq b} \|\phi(t,u)\|_H dt\right). \ \square$$

For our construction we will need a stronger version of continuity in the r.-h. side, where $\phi(t, u)$ is fast oscillating in t and condition (4) is substituted by smallness of ϕ in *relaxation metric*.

Controllability proof by geometric control approach

Study of controllability of NLS equation is based (as well as earlier work on Navier-Stokes/Euler equation) on method of iterated Lie extensions.

Lie extension of control system $\dot{x} = f(x, u), \ u \in U$ is a way to add vector fields to the right-hand side of the system guaranteeing (almost) invariance of its controllability properties.

The additional vector fields are expressed via Lie brackets of $f(\cdot, u)$ for various $u \in U$. If after a series of extensions one arrives to a controllable system, then the controllability of the original system will follow.

Controlled NLS equation

$$-i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + \sum_{k\in\widehat{\mathcal{K}}} v_k(t) e^{ik\cdot x}$$

is a particular type of infinite-dimensional control-affine system

$$\partial_t u = f^0(u) + \sum_{k\in\widehat{\mathcal{K}}} f^k(u) v_k(t).$$

For each Lie extension following Lie brackets are significant:

$$[f^m,[f^m,f^0]], \ \ [f^n,[f^m,[f^m,f^0]]], \ \ m,n \in \widehat{\mathcal{K}}.$$

The 3rd-order Lie brackets $[f^m,[f^m,f^0]]$ are obstructions to controllabilit
which have to be 'compensated'.

The 4th-order Lie bracket $[f^n,[f^m,[f^m,f^0]]]$ are constant vector fields or directions along which the extended control acts.

Geometric control in infinite dimensions

Obstacles:

- instead of flows one often has to deal with semigroups of operators;

- r.-h. sides of equations ('vector fields') include unbounded operators; lack of adequate infinite-dimensional differential geometry for this case;

Lie-algebraic computations are used as a guiding principle for establishing controllability.

To justify their application in infinite-dimensional setting we use fast-oscillating controls, which underly Lie extensions method. Specially designed resonances between such controls result in a motion which provides (*approximates*) motion in *extending* direction, along a Lie bracket. We also manage to compensate 'in average' the obstructions.

We arrive to final result proceeding with (finite) sequence of **elementary extensions**.

Cubic Schroedinger equation on $\mathbb{T}^2$ as infinite-dimensional system of ODE

We invoke Fourier Ansatz seeking solution of NLS equation as series expansion

$$u(t,x) = \sum_{k\in\mathbb{Z}^2} q_k(t) e^{i(kx+|k|^2 t)}.$$

with respect to **modes** $\mathbf{e}_k = e^{i(k\cdot x+|k|^2 t)}$.

The source term will be

$$F(t,x) = \sum_{k\in\hat{\mathcal{K}}\subset\mathbb{Z}^2} e^{i(kx+|k|^2 t)} v_k(t),$$

notation $v_k(t)$ is kept for controls.

Note that $(-i\partial_t + \Delta)\mathbf{e}_k = 0$.

Substituting the expansions of u and F into NLS equation we get infinite system of ODE's for the coefficients $q(t)$:

$$-i\dot{q}_k(t) = \underbrace{-q_k|q_k|^2 + 2q_k \sum_{j\in\mathbb{Z}^2} |q_j|^2 + \sum_{k_1-k_2+k_3=k;k\neq k_1,k_3} q_{k_1}\bar{q}_{k_2}q_{k_3}e^{i\omega(K)t}}_{S_k(q,t)} +$$

$$+\chi_{\widehat{\mathcal{K}}}(k)v_k(t),$$

$$\omega(K) = |k_1|^2 - |k_2|^2 + |k_3|^2 - |k|^2.$$

Controls $v_k(t)$ appear in the equations, indexed by $k \in \widehat{\mathcal{K}}$.

This is also infinite-dimensional control-affine system

$$\dot{q} = f^0(q,t) + \sum_{k \in \widehat{\mathcal{K}}} f^k(q) v_k(t),$$

with $S_k(q,t)$, being components of the drift vector field $f^0(q,t)$, and with constant controlled v.f. $f^k = i\mathbf{e}_k = i\frac{\partial}{\partial q_k},\ k \in \widehat{\mathcal{K}}$.

Computing Lie bracket $[f^n,[f^m,[f^m,f^0]]]$ for $m,n \in \widehat{\mathcal{K}}$ we get linear combination of vector fields f^m, f^n, f^{2m-n}. In the case, where $2m-n \notin \widehat{\mathcal{K}}$ we get a new direction $\mathbf{e}_{2m-n}$, or an **extension**.

Extension design

Pick two of the controlled modes is $\{m, n\} \in \hat{\mathcal{K}}$, such that $2m-n \notin \hat{\mathcal{K}}$. 'Modulating' in clever way controls in the modes $\mathbf{e}_m, e_n$, one manages to get an extended control for the mode $\mathbf{e}_{2m-n}$ and 'affects little' (in average, on given time interval) all other modes.

Feed into the r.-h. side of the ODEs for q_m, q_n control functions $\dot{v}_m(t) + \tilde{v}_n(t), \dot{v}_m(t) + \tilde{v}_n(t)$ respectively, where $v_m(t), v_n(t)$ are Lipschitzian functions. We get

$$-i\dot{q}_m(t) = S_m(q,t) + \dot{v}_m(t) + \tilde{v}_m,$$
$$-i\dot{q}_n(t) = S_n(q,t) + \dot{v}_n(t) + \tilde{v}_n.$$

Introduce new variables q_ℓ^* by relations

$$q_m = q_m^* - iv_r(t), q_n = q_n^* - iv_n(t), \;\; q_k^* = q_k, \;\; \text{for } k \neq m, n,$$

or

$$q = q^* + iV(t),\ V(t) = v_m(t)\mathbf{e}_m + v_n(t)\mathbf{e}_n.$$

The equations for components of q^* are:

$$-i\dot{q}_j^*(t) = \begin{cases} S_j(q + V(t), t) + \tilde{v}_j(t), & j \in \{m, n\}; \\ S_j(q + V(t), t), & j \neq m, n. \end{cases}$$

S_j are cubic polynomials in $v_m, v_n, \bar{v}_m, \bar{v}_n, q_k, \bar{q}_k,\ k \in \mathbb{Z}^2$.

We impose 'isoperimetric condition' $V(0) = V(T) = 0$, to preserve the end-points of the trajectory:

$$q(0) = q^*(0),\ q(T) = q^*(T).$$

Time-T controllability of equations for q^* $\Rightarrow$ controllability of the original equation.

Fast oscillations

Now we introduce fast-oscillations, choosing the controls $v_m(t), v_n(t)$ of the form

$$v_m(t) = e^{i(t/\varepsilon+\rho(t))}\widehat{v}_m(t), v_n(t) = e^{i2t/\varepsilon}\widehat{v}_n(t), \ (OSC)$$

where $\widehat{v}_m(t), \widehat{v}_n(t)$ are real-valued Lipschitzian functions, $\rho(t)$ and $\varepsilon > 0$ will be specified later.

The monomials of $S_j(q^* + V(t))$ are classified in *resonant* and *non-resonant*. We call a monomial *non-resonant* if, after substitution of **(OSC)** into it, we get a fast-oscillating factor $e^{i\beta t/\varepsilon}, \ \beta > 0$. All other, *resonant*, monomials are classified as *bad resonances - obstructions*, and *good resonances - extending controls.*

Non-resonant monomials are present in each ODE.

Obstructions

For each $j \in \mathbb{Z}^2$ r.-h. side $S_j(q+V(t),t)$ ODE contains term

$$-i\dot{q}_j^*(t) = \cdots + 2q_j^* \left(|\hat{v}_m(t)|^2 + |\hat{v}_n(t)|^2 \right) + \cdots ,$$

corresponding to Lie brackets $[f^m,[f^m,f^0]], \ [f^n,[f^n,f^0]]$ mentioned above. These are obstructions; motion 'along' obstructing v.f. is *unilateral*. BUT for Schroedinger equation it is *unilateral ROTATION*.

We can get rid of the obstructing term by time-variant substitution for the variables

$$q^\star = q^* e^{-2iR(t)}, \ \ R(t) = \int_0^t \left(|\hat{v}_m|^2 + 2|\hat{v}_n|^2 \right)(\tau)d\tau.$$

In order to guarantee $q^\star(T) = q^*(T) = q(T)$ one can impose additional *(isoperimetric)* conditions on $\hat{v}_m(t), \hat{v}_n(t)$:

$$\int_0^T |\hat{v}_m(t)|^2 dt = \pi N_m, \ \int_0^T |\hat{v}_n(t)|^2 dt = \pi N_n, \ \ N_m, N_n \in \mathbb{Z}.$$

Extending control via resonance

Now we study **resonance cubic monomial** which appears at the r.-h. side of ODE for $q^{\star}_{2m-n}$ and equals

$$v_m^2(t)v_n(t)e^{-2iR(t)} = e^{2i(\rho(t)-R(t)-|m-n|^2t)}\hat{v}_m^2(t)\hat{v}_n(t), \quad (\diamondsuit)$$

$$R(t) = \int_0^t \left(|\hat{v}_m|^2 + 2|\hat{v}_n|^2\right)(\tau)d\tau.$$

Lemma. *For each* $w(t) \in L_\infty[0,T]$ *and any* $\varepsilon > 0$ *one can find Lipschitzian* $\hat{v}_m(t), \hat{v}_n(t), \rho(t)$ *which satisfy all the introduced 'isoperimetric conditions' and* $(\diamondsuit)$ ε*-approximates* $w(t)$ *in* $L_1[0,T]$*-metric.* $\square$

Thus we can approximately simulate any extending control in the ODE for $q^{\star}_{2m-n}$.

Effect of non-resonant terms

We managed to **extend the original two-control system**

$$(-i\partial_t + \Delta)u(t,x) = |u(t,x)|^2 u(t,x) + v_m(t)\mathbf{e_m} + v_n(t)\mathbf{e_n}$$

to **three-control system**

$$-i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + \quad \mathbf{(OSC)}$$
$$+\tilde{v}_m(t)\mathbf{e_m} + \tilde{v}_n(t)\mathbf{e_n} + w(t)\mathbf{e}_{2m-n} + \underline{\phi^{\varepsilon}(t,x,u)}$$

'burdened' with (noised by) *fast-oscillating term* $\phi^{\varepsilon}(t,x,u)$, which is Nemytskii-type operator, 2nd degree polynomial in $u, \bar{u}$ with coefficients $a(t,x)$, which are sums of terms $e^{i\beta_r t/\varepsilon} e^{i\ell \cdot x} w(t)$ with $\beta_r \neq 0$ and $w(t)$ are equibounded in norm $W_{11}[0,T]$.

Introduce 'limit equation' for (OSC)

$$(-i\partial_t + \Delta)u(t,x) = |u(t,x)|^2 u(t,x) +$$
$$+\tilde{v}_m(t)\mathbf{e_m} + \tilde{v}_n(t)\mathbf{e_n} + w(t)\mathbf{e_{2m-n}}. \quad \mathbf{(LIM)}$$

We wish to prove that, when *rate of oscillation grows ($\varepsilon \to 0$), solutions of the noised equation (OSC) converge to solutions of the limit equation (LIM)*.

This fact is part of **relaxation result** for semilinear evolution equations*

Relaxation seminorm $\|\cdot\|_b^{\mathrm{rx}}$ is defined by formula:

$$\|\phi\|_b^{\mathrm{rx}} = \max_{t,t' \in [0,T], x, \|u\| \leq b} \left\| \int_t^{t'} \phi(\tau, x, u) d\tau \right\|.$$

Fast-oscillating (in t) functions have small seminorms $\|\cdot\|^{rx}$.

*compare with results by H.Frankowska (1990), H.Fattorini (1994), N.Ahmed (1987), on relaxation of evolution equations.

The following theorem affirms **continuous dependence of trajectories in r.-h. side w.r.t. the relaxation seminorm.**

Theorem. *Let solution* $\tilde{u}(t)$ *of the (LIM) equation exist on* $[0, T]$*, belongs to* $C([0, T], H)$ *and* $\sup_{t \in [0,T]} \|u(t)\| < b$*. Then* $\forall \varepsilon > 0 \exists \delta > 0$ *such that whenever* $\|\phi\|_b^{\mathrm{rx}} < \delta$*, then the solution* $u(t)$ *of the perturbed equation exists on* $[0, T]$*, is unique and*

$$\sup_{t \in [0,T]} \|u(t) - \tilde{u}(t)\| < \varepsilon. \ \square$$

The 'extension technique' shows that controllability properties that NLS equation with controls applied to the modes $\mathbf{e_m}, \mathbf{e_n}$ and $\mathbf{e_m}, \mathbf{e_n}, \mathbf{e_{2m-n}}$ are 'approximately the same'.

Saturation and approximate controllability

The extension design can be repeated in iterative way: starting from a set $\widehat{\mathcal{K}} = \mathcal{K}^1$ we construct a sequence of expanding sets

$$\mathcal{K}^j = \left\{ 2m - n | \ m, n \in \mathcal{K}^{j-1}, \right\}; j = 2, \ldots, \ \mathcal{K}^\infty = \bigcup_{j=1}^{\infty} \mathcal{K}^j.$$

We call $\mathcal{K}^1$ **saturating** if $\mathcal{K}^\infty = \mathbb{Z}^2$.

It is easy to prove that

whenever set $\mathcal{K}^1$ of controlled modes is saturating one can conclude controllability of NLS in each finite-dimensional projection and approximate controllability in H^2.

It is not difficult to describe some classes of saturating sets.

Lemma. *If* $k, \ell \in \mathbb{Z}^2$ *and* $k \wedge \ell = \pm 1$, *then the 4-element set* $\{0, k, \ell, k+\ell\}$ *is saturating.*

Example. *Set* $\{(0,0),(1,0),()0,1),(1,1)\}$ *is saturating and controls, applied to this modes, guarantee controllability in fin.-dim. projections and approximate controllability.*

Lack of exact controllability: sketch of the proof

Consider again NLS equation

$$-i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + \sum_{k\in\hat{\mathcal{K}}} \dot{v}_k(t)\mathbf{e_k}(\mathbf{t},\mathbf{x}) \quad \text{(NLS)}$$

with *integrable* controls $\dot{v}_k(t)$ applied to *any* finite set $\hat{\mathcal{K}}$ of modes.

Introducing function $V(t,x) = v_k(t)\mathbf{e_k}(\mathbf{t},\mathbf{x})$, and proceeding with time-variant substitution $u = u^* + iV(t,x)$ we transform the equation (NLS) into the form

$$-i\partial_t u^* + \Delta u^* = |u + iV|^2(u^* + iV)(REDU)$$

which can be seen as semilinear control system with absolutely-continuous inputs V (entering nonlinearly).

Lemma. *Input-trajectory map* $E^* : V(\cdot) \mapsto u^*(\cdot)$ *and time-*T *map* $E^*_T : V(\cdot) \mapsto u^*(T)$ *for the equation (REDU) are Lipschitzian in the space* $W_{1,1}([0,T],\mathbb{C}^\kappa)$ *of inputs* $V(t)$*, endowed with* $L_1([0,T],\mathbb{C}^\kappa)$*-metric, if the space of trajectories is endowed with* $C([0,T],H^{1+\sigma})$*-metric.* $\square$

Each ball in $W_{1,1}([0,T],\mathbb{C}^\kappa)$ is precompact in L_1-metric and so is its image under E^*_T. Hence the attainable set, which is image of time-T map is contained in a union of countable family of compacts.