Geometric Control Condition and Systems of Coupled Wave Equations

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The Goal

The Geometric Control Condition (GCC) of Bardos, Lebeau and Rauch (92) is equivalent to the controllability of the scalar wave equation.

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- Provide some Geometric Control Condition (s) as sharp as possible allowing exact controllabilty for sytems of coupled wave equations.
- More precisely, we seek for (sharp) **microlocal** conditions.
- Boundary or internal control.
- 2 cases: **Lamé system (boundary coupling)** and **a system of two wave equations with internal coupling.**
- Stabilization.
- Joint works with: M.Léautaud, J.Le Rousseau and J-P. Raymond.

The Bardos-Lebeau-Rauch theorem

Ω a bounded open set of $\mathbb{R}^d$ or a compact Riemannian manifold.

$$\begin{cases} \partial_t^2 u - \Delta u = \quad 1_\omega g \quad \text{in} \quad]0,+\infty[\times\Omega \\ u = 0 \qquad \text{on} \quad]0,+\infty[\times\partial\Omega \\ (u(0),\partial_t u(0)) = (u_0,u_1) \in H_0^1(\Omega)\times L^2(\Omega) \end{cases}$$

$$\begin{cases} \partial_t^2 u - \Delta u = \quad 0 \quad \text{in} \quad]0,+\infty[\times\Omega \\ u = 1_\Sigma h \qquad \text{on} \quad]0,+\infty[\times\partial\Omega \\ (u(0),\partial_t u(0)) = (u_0,u_1) \in L^2(\Omega)\times H^{-1}(\Omega) \end{cases}$$

Geometry

a) In the interior: In $T^*(\mathbb{R}\times\Omega)$, the wave front (C^∞ or H^s) propagates along the bicharacteristic curves of the wave operator.

$$p(t,x,\tau,\xi) = -\tau^2 + |\xi|^2$$

The null bicharacteristic issued from $\rho_0 = (t_0, x_0, \tau_0, \xi_0)$ is the integral curve of the hamiltonian field of p.

$$\begin{cases} \gamma'(s) = H_p(\gamma(s)) \\ \gamma(0) = \rho_0 \end{cases}$$

In this case, $\tau_0^2 = |\xi_0|^2$, $(\tau,\xi) = (\tau_0,\xi_0) \neq (0,0)$, and

$$\gamma(s) = (t_0 - 2\tau_0 s, x_0 + 2s\xi_0, \tau_0, \xi_0)$$

These are straight lines.
The projection of a bicharacteristic on Ω is a geodesic.

b) Near the boundary

Assumption: $\partial\Omega$ has no infinite order contacts with its tangents.

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We have to work in the compressed contangent bundle of Melrose and propagate along the generalized bicharacteristic flow of Melrose-Sjöstrand. Denote $M = \mathbb{R} \times \Omega, \quad \partial M = \mathbb{R} \times \partial\Omega.$

$$T_b^* M = T^* M \cup T^* \partial M$$

Local coordinates: $(t, x) = y = (y_{0,\dots,} y_n) = (y', y_n), \quad \eta = (\eta', \eta_n)$

$$\partial M = \{y_n = 0\}, \quad M = \{y_n > 0\}$$

$$P = D_{y_n}^2 - R(y, D_{y'})$$

R is a second order tangential differential operator with principal symbol $r(y, \eta')$

$$p = \eta_n^2 - r(y, \eta')$$

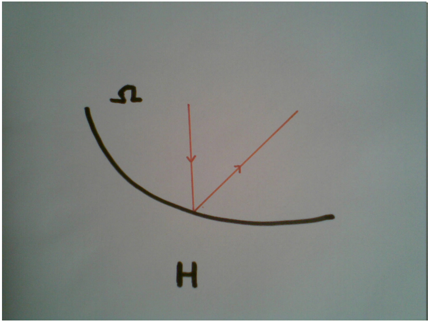
$T^*\partial M$ is subdivided into three subsets:

$$\mathcal{E} = \{(y', \eta') \in T^*\partial M, \quad r(y', 0, \eta') < 0\} \quad \text{elliptic set}$$

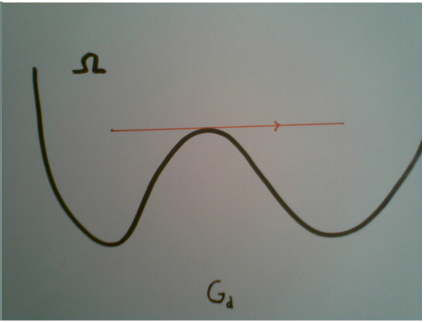
$$\mathcal{H} \quad = \{(y', \eta') \in T^*\partial M, \quad r(y', 0, \eta') > 0\} \quad \text{hyperbolic set}$$

$$\mathcal{G} \quad = \{(y', \eta') \in T^*\partial M, \quad r(y', 0, \eta') = 0\} \quad \text{glancing set}$$

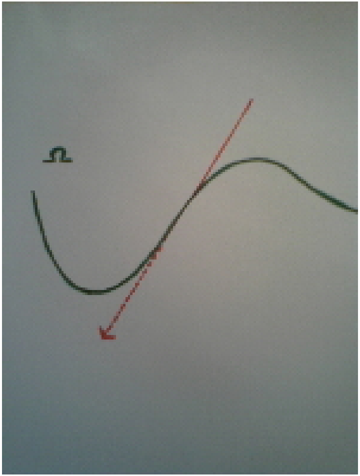
Theorem (Melrose-Sjöstrand 82'): The wave front up to the boundary $WF_b(u)$ propagates along the generalized bicharacteristic curves.



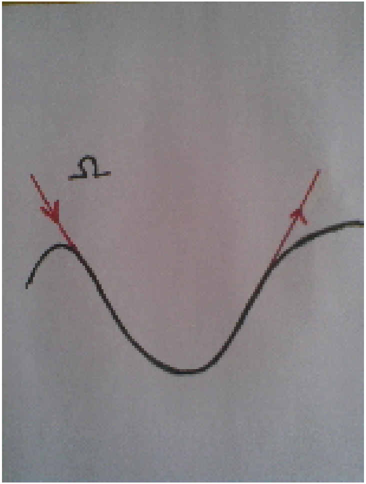
Hyperbolic



Diffractive



Glancing Non diffractive



Glancing ray

The geometric control condition

Consider Ω an open subset of $\mathbb{R}^d$, and $T > 0$,

$\rightarrow \omega \subset \Omega$, an open subset,

$\rightarrow \Sigma$ an open subset of the boundary $\partial\Omega$.

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The couple (ω, T) satisfies (G.C.C) if every geodesic of Ω, travelling with speed 1 and issued at $t = 0$ enters the open set ω before the time T.

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b) (G.C.C) for boundary control

The couple (Σ, T) satisfies (G.C.C) if every generalized bicharacteristic of the wave operator, travelling with speed 1 and issued at $t = 0$ intersects the set Σ at a non diffractive point, before the time T.

$\rightarrow$ Almost equivalent to exact controllability (Burq-Gérard (98)).

Theorem

Internal Control

Assume that (ω, T) *satisfies (GCC); then for every* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *there exists* $g \in L^2(]0, T[\times\Omega)$, *supp* $g \subset \omega$, *s.t the unique solution of*

$$(W) \qquad \begin{cases} \partial_t^2 u - \Delta u = \quad g \quad in \quad]0,+\infty[\times\Omega \\ u = 0 \qquad on \quad]0,+\infty[\times\partial\Omega \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \end{cases}$$

satisfies $(u(T), \partial_t u(T)) = (0, 0)$.

Theorem

Boundary Control

Assume that (Σ, T) *satisfies (GCC); then for every* $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, *there exists* $h \in L^2(]0, T[\times\partial\Omega)$, *supp* $h \subset \Sigma$, *s.t.the unique solution of*

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A brief Idea on the proof (Boundary control)

Contradiction argument

$$\|(v_0, v_1)\|^2_{H^1_0 \times L^2} \leq C \int_0^T \int_\Sigma |\partial_n v|^2 \, dxdt$$

$$\left\|(v_0^k, v_1^k)\right\|_{H^1_0 \times L^2} = 1, \qquad \int_0^T \int_\Sigma \left|\partial_n v^k\right|^2 dxdt \leq 1/k$$

(v^k) is bounded in $H^1(]0, T[\times\Omega)$ and the weak limit v satisfies

$$\begin{cases} \partial_t^2 v - \Delta v = \quad 0 \quad \text{in} \quad]0, T[\times\Omega \\ v = \quad 0 \quad \text{on} \quad]0, T[\times\partial\Omega \\ \partial_n v = 0 \quad \text{on} \quad]0, T[\times\Sigma \end{cases}$$

So $v \equiv 0$ (unique continuation).
μ a microlocal defect measure attached to (v^k) in $H^1(]0, T[\times\Omega)$. Every non diffractive point of $T^*(]0, T[\times\Sigma)$ is not in suppμ.
Then, by propagation, $\mu = 0 \quad \text{in} \quad]0, T[\times\Omega$, and $v^k \to 0$ strongly !

The main tools

$\longrightarrow$ A unique continuation property.

$\longrightarrow$ A propagation result (regularity/compactness).

$\longrightarrow$ A lifting lemma (boundary control).

The Lamé system

Ω bounded open subset of $\mathbb{R}^3$, with smooth boundary $\partial\Omega$.

$$(L) \qquad \left\{ \begin{array}{l} \partial_t^2 u - \Delta_e u = 0 \qquad \text{in} \quad]0,+\infty[\times\Omega \\ u = 0 \qquad \text{on} \quad]0,+\infty[\times\partial\Omega \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in H = (L^2(\Omega))^3 \times (H^{-1}(\Omega))^3 \end{array} \right.$$

Δ_e is the elasticity operator, i.e. the 3×3 matrix with differential operator coefficients

$$\Delta_e u = \mu\Delta u + (\lambda + \mu)\, \nabla \operatorname{div} u, \quad u =^t (u_1, u_2, u_3),$$

The Lamé coefficients μ and λ are constant and $\quad > 0$

Unique solution in $C^0(\mathbb{R}_+, (L^2)^3) \cap C^1(\mathbb{R}_+, (H^{-1})^3)$.

Decoupling the solution

$$(L) \qquad \begin{cases} \partial_t^2 u - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u = & 0 \quad \text{in} \quad]0,+\infty[\times\Omega \\ u = 0 \qquad \text{on} \quad]0,+\infty[\times\partial\Omega \end{cases}$$

One can split the solution u into

$$u = u_T + u_L$$

with

$$\begin{cases} (\partial_t^2 - c_L^2\Delta)u_L = 0, \quad rot\, u_L = 0, & c_L^2 = \lambda + 2\mu \\ (\partial_t^2 - c_T^2\Delta)u_T = 0, \quad \operatorname{div} u_T = 0, & c_T^2 = \mu \\ u = u_L + u_T = 0 \quad \text{on } \partial\Omega \end{cases}$$

Moreover, u_L and u_T are of bounded energy

$$\|u_L\|^2_{H^1((0,T)\times\Omega)} + \|u_T\|^2_{H^1((0,T)\times\Omega)} \leq C(T)Eu(0)$$

u_L is the longitudinal wave and u_T is the transversal wave.

The Geometry of the Lamé System

$$P = \partial_t^2 - \Delta_e$$

$$\det p(t, x; \tau, \xi) = (\mu|\xi|^2 - \tau^2)^2 \left((\lambda + 2\mu)|\xi|^2 - \tau^2\right).$$

We have to deal with two characteristic manifolds:

$$p_L(t, x; \tau, \xi) = c_L^2|\xi|^2 - \tau^2 \quad \text{and} \qquad p_T(t, x; \tau, \xi) = c_T^2|\xi|^2 - \tau^2$$

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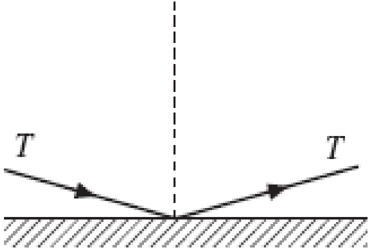
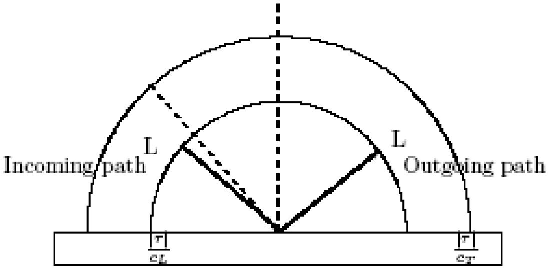
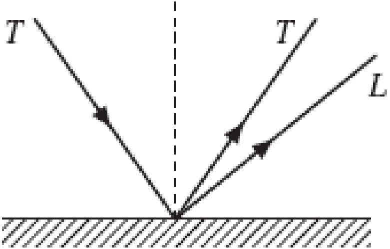
In the interior: The two waves u_L and u_T propagate independently.

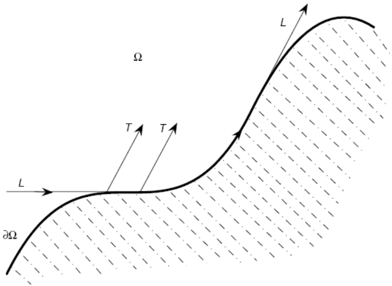
At the boundary: Possibility of energy transfert.





$0 \leq r_L < r_T$


T
T
L


L
Ω
T
T
L
∂Ω

γ^- : incident half-bicharacteristic at the point ρ of the boundary.

γ^+ : reflected half-bicharacteristic at the point ρ of the boundary.

The possible paths are:

$$\gamma_T^- \to \gamma_T^+, \quad \gamma_L^- \to \gamma_L^+, \quad \gamma_T^- \to \gamma_L^+, \quad \gamma_L^- \to \gamma_T^+$$

Definition: The bicharacteristic path

A generalized bicharacteristic path Γ is a curve in

$$T_b^* M = T^* M \cup T^* \partial M$$

constituted by generalized bicharacteristics of P, with the possibility of moving from a bicharacteristic manifold to another, at each point of $T^*(\partial M)$, in the way indicated above.
The projection of a generalized bicharacteristic path on $\overline{\Omega}$ is a geodesic path.

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The projection of a generalized bicharacteristic path on $\overline{\Omega}$ is a geodesic path.

Definition: Non diffractive point

We say that a point $\rho \in T^*(\partial M)\backslash 0$, is non diffractive iff none of the two corresponding bicharacteristic rays γ_T or γ_L is glancing diffractive at ρ.

In other words, the projection on the (t, x) space of both (free) rays γ_T and γ_L crosses the boundary ∂M at this point.

The Geometric Control Condition for Lamé

Take $T > 0$, ω an open subset of Ω and Σ and open subset of $\partial\Omega$.

The condition (G.C.L) for internal control
We say that (ω, T) satisfies the (G.C.L) if every geodesic path issued from Ω at time $t = 0$, intersects the region ω before time T.

The condition (G.C.L) for boundary control
We say that (Σ, T) satisfies the (G.C.L) if every generalized bicharacteristic path Γ issued from Ω at time $t = 0$, intersects the region Σ before time T, at a non diffractive point.

The Control Results (with J-P. Raymond)

Theorem

Internal Control: *Assume that* (ω, T) *satisfies (G.C.L); then for every* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *there exists* $g \in L^2(]0, T[\times\Omega)$, *supp* $g \subset \omega$, *s.t the unique solution of*

$$(L) \qquad \begin{cases} \partial_t^2 u - \Delta_e u = \quad g \quad in \quad]0, +\infty[\times\Omega \\ u = 0 \qquad on \quad]0, +\infty[\times\partial\Omega \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \end{cases}$$

satisfies $(u(T), \partial_t u(T)) = (0, 0)$.

Theorem

Boundary Control: *Assume that* (Σ, T) *satisfies (G.C.L); then for every* $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, *there exists* $h \in L^2(]0, T[\times\partial\Omega)$, *supp* $h \subset \Sigma$, *s.t the unique solution of*

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satisfies $(u(T), \partial_t u(T)) = (0, 0)$.

Comments

1. Exact controllability result by J-L Lions (88) under the (local) $\Gamma-$condition.
2. Analogy

Scalar wave	Lamé system
geodesic rays	geodesic paths
$(G.C.C)$	$(G.C.L)$

3. Is (G.C.L) necessary for control ? Open problem.
4. In the same setting, one can prove a stablization result with internal damping.

Theorem

Assume that $(\{a(x) > 0\}, T)$ *satisfies (G.C.L). then there exsit two positive constants* C *and* α *s.t every solution* u *of*

$$\left\{ \begin{array}{c} \partial_t^2 u - \Delta_e u + a(x)\partial_t u = 0 \qquad in \quad]0,+\infty[\times\Omega \\ u = 0 \qquad on \quad]0,+\infty[\times\partial\Omega \\ (u\,(0,x)\,,\partial_t u\,(0,x)) = (u_0\ ,u_1) \in H_0^1(\Omega)\times L^2(\Omega) \end{array} \right.$$

satisfies

$$\begin{array}{rcl} E\,(u)\,(t) & = & \int_\Omega (\mu\,|\nabla u|^2 + (\lambda+\mu)\,|\mathrm{div}\; u|^2 + |\partial_t u|^2)(t,x)dx \\ & \leq & C\exp(-\alpha t)E(u)(0), \qquad t \geq 0 \end{array}$$

Some references

1. K.Yamamoto (89'): propagation of the wave front up to the boundary for the Lamé system.
2. C.Bardos-T.Masrour-F.Tatout (95).
3. G.Lebeau-E.Zuazua (99'): Thermoelasticity system.
4. N.Burq-G.Lebeau (01'): microlocal defect measures for systems and propagation results for Lamé.
5. M.Bellassoued (01') : Carleman estimates and approximate controllability for Lamé.
6. M.Bellassoued (08') : Internal stabilization in a bounded domain with Neumann boundary condition.
7. M.Daoulatli-B.D-M.Khénissi (10'): stabilization on an exterior domain of $\mathbb{R}^3$ (outside a trapping obstacle).

Proof: The main ingredients

Regularity estimate

$$\left\{\begin{array}{c}\partial_t^2 v - \Delta_e v = 0 \qquad \text{in} \quad]0,+\infty[\times\Omega \\ v = 0 \qquad \text{on} \quad]0,+\infty[\times\partial\Omega \\ \left(v\left(0,x\right),\partial_t v\left(0,x\right)\right) = \left(v_0\ ,v_1\right) \in H_0^1(\Omega)\times L^2(\Omega)\end{array}\right.$$

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$$\mu \int_0^T \int_{\partial\Omega} \left| \frac{\partial v}{\partial n} \right|^2 d\sigma dt + (\mu + \lambda) \int_0^T \int_{\partial\Omega} (\mathrm{div}\, u)^2 d\sigma dt \leq C(T) E v(0)$$

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$$\mu \int_0^T \int_{\partial\Omega} \left|\frac{\partial v}{\partial n}\right|^2 d\sigma dt + (\mu+\lambda)\int_0^T \int_{\partial\Omega} (\mathrm{div}\, u)^2 d\sigma dt \leq C(T) Ev(0)$$

$\longrightarrow$ Take the inner product of the equation by $H(x,\partial_x)v = (\sum_1^3 a_j(x)\partial_{x_j})v$ and integrate by parts.

The regularity lifting lemma

Denote by $\pi : T^*(\partial M) \to \partial M$ the canonical projection.

Lemma Let ρ be a non diffractive point of $T^*(\partial M)\backslash 0$ and u a (vector) distribution defined near $\pi(\rho)$ in M, solution of

$$\partial_t^2 u - \Delta_e u \in C^\infty(\pi(\rho)), \qquad u_{|\partial M} \in H^s_\rho, \qquad \frac{\partial u}{\partial n}\,|_{\partial M} \in H^{s-1}_\rho$$

Then $\rho \notin WF_b^s(u)$.

$\longrightarrow$ Bardos-Lebeau-Rauch (88) for scalar waves.

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Then $\rho \notin WF^s_b(u)$.

$\to$ Bardos-Lebeau-Rauch (88) for scalar waves.

Lemma In the same setting, if $\rho \notin WF(u_{|\partial M})$, $\qquad \rho \notin WF(\frac{\partial u}{\partial n}\,|_{\partial M})$ then $\rho \notin WF_b(u)$.

$\to$ Andersson-Melrose (77) for scalar waves.

Microlocal defect measures for Lamé

We remind that

$$\begin{cases} (\partial_t^2 - c_L^2\Delta)u_L = 0, \quad rot\, u_L = 0, \quad c_L^2 = \lambda + 2\mu \\ (\partial_t^2 - c_T^2\Delta)u_T = 0, \quad \mathrm{div}\, u_T = 0, \quad c_T^2 = \mu \\ u = u_L + u_T = 0 \quad \text{on } \partial\Omega \end{cases}$$

$$p_L = c_L^2|\xi|^2 - \tau^2 \quad \text{and} \qquad p_T = c_T^2|\xi|^2 - \tau^2$$

Let (u^k) be a sequence of solutions to the Lamé system.

$$u^k = u_T^k + u_L^k$$

If $u^k \rightharpoonup 0$ in $H^1(]0,T[\times\Omega)$, so do $u_{T,L}^k$.

Denote by μ_T and μ_L the m.d.m's associated to these sequences (they are orthogonal in the sense of the measures).

Remark: In our case, $\mathcal{G}_L$ and $\mathcal{H}_L$ are contained in $\mathcal{H}_T$.

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Theorem (D-D-K): At a point ρ of the boundary,

1. If $\rho \in \mathcal{E}_L$, then $\mu_L = 0$ near ρ and
   1. if $\rho \in \mathcal{E}_T$, then $\mu_T = 0$ near ρ
2. If $\rho \in \mathcal{G}_L$, then $(\gamma_L^- \cap \mathrm{supp}\mu_L) = \emptyset \Rightarrow \mu_T$ propagates from γ_T^- to γ_T^+.
3. If $\rho \in \mathcal{H}_L$, then

$$(\gamma_{L,T}^- \cap \mathrm{supp}\mu_{L,T}) = \emptyset \Rightarrow \mu_{T,L} \text{ propagates from } \gamma_{T,L}^- \text{ to } \gamma_{T,L}^+.$$

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    2. if $\rho \in \mathcal{G}_T \cup \mathcal{H}_T$, then μ_T propagates from γ_T^- to γ_T^+.
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$$(\gamma_{L,T}^- \cap \mathrm{supp}\mu_{L,T}) = \emptyset \Rightarrow \mu_{T,L} \text{ propagates from } \gamma_{T,L}^- \text{ to } \gamma_{T,L}^+.$$

Corollary

$$(\gamma_T^- \cap \mathrm{supp}\mu_T) \cup (\gamma_L^- \cap \mathrm{supp}\mu_L) = \emptyset$$

$$\Updownarrow$$

$$(\gamma_T^+ \cap \mathrm{supp}\mu_T) \cup (\gamma_L^+ \cap \mathrm{supp}\mu_L) = \emptyset$$

In this case, we get $\rho \notin \mathrm{supp}\mu_{T,L}$.

Remark: This is the "measures" analogous of the Yamamoto propagation theorem for C^∞ singularities:

$$(\gamma_L^- \cup \gamma_T^-) \cap WF_b(u) = \emptyset \Leftrightarrow (\gamma_L^+ \cup \gamma_T^+) \cap WF_b(u) = \emptyset$$

Remark: Propagation of the Sobolev wave front set (Yamamoto 06) ???

A lifting lemma for the measures

Denote $M_T =]0,T[\times\Omega$ and $\partial M_T =]0,T[\times\partial\Omega$.

Lemma

Let (v^k) be a sequence weakly converging to 0 in $H^1(M_T)$ satisfying

$$\partial_t^2 v^k - \Delta_e v^k = 0 \text{ in } M_T, \qquad v^k{}_{|\partial M_T} = 0, \qquad \frac{\partial v^k}{\partial n}\,|_{\partial M_T} \to 0 \text{ in } L^2(\partial M_T)$$

and $\mu_{L,T}$ the m.d.m respectively attached to the sequence $(v^k_{L,T})$.
Then if ρ is a non diffractive point of $T^*(\partial M_T)\backslash 0$, one has
$\rho \notin supp(\mu_{L,T})$.

Proof of observability from the boundary

$$\begin{cases} \partial_t^2 v - \Delta_e v = 0 \quad \text{in} \quad M_T \\ v = 0 \quad \text{on} \quad \partial M_T \\ (v(0,x), \partial_t v(0,x)) = (v_0 \ , v_1) \in H_0^1(\Omega) \times L^2(\Omega) \end{cases}$$

$$\|(v_0 \ , v_1)\|_{H_0^1 \times L^2}^2 \leq C \int_0^T \int_\Sigma \left| \frac{\partial v}{\partial n} \right|^2 d\sigma dt$$

Contradiction argument:

$$\begin{cases} \partial_t^2 v^k - \Delta_e v^k = 0 \ , \quad \text{in } M_T \\ {v^k}_{|\partial M_T} = 0, \quad \frac{\partial v^k}{\partial n} |_{\partial M} \to 0 \text{ in } L^2(]0, T[\times \Sigma) \\ \|(v_0^k \ , v_1^k)\|_{H_0^1 \times L^2} = 1 \end{cases}$$

Killing the weak limit

$$\begin{cases} \partial_t^2 v - \Delta_e v = 0 \ , \quad \text{in } M_T \\ v_{|\partial M_T} = 0, \quad \frac{\partial v}{\partial n}\,|_{\partial M} = 0 \text{ on }]0,T[\times\Sigma \end{cases} \qquad (*)$$

Killing the weak limit

$$\begin{cases} \partial_t^2 v - \Delta_e v = 0 \ , & \text{in } M_T \\ v_{|\partial M_T} = 0, \quad \frac{\partial v}{\partial n}\,|_{\partial M} = 0 \text{ on }]0,T[\times\Sigma \end{cases} \qquad (*)$$

The space

$$G = \{v \in H = H_0^1 \times L^2, \quad v \text{ solution of } (*)\}$$

is constituted of smooth functions and closed in H. So dim $G < \infty$. Moreover $\partial/\partial t$ operates on G and has no eigenvalues.

Therefore $G = \{0\}$.

End of the proof: $v_k \to 0$ in $H^1(]0,T[\times\Omega)$ strongly.

Assume that some $q \in T^*(]0,T[\times\Omega)$ lies in $\mathrm{supp}\mu_T \cup \mathrm{supp}\mu_L$.
Consider then γ (γ_T or γ_L) a bicharacteristic issued from q.
Denote by ρ a point at which γ hits the boundary $\partial\Omega$.

- One of the two reflected half- bicharacteristics γ_T^+ (or γ_L^+) is in $\mathrm{supp}\mu_T$ (or $\mathrm{supp}\mu_L$).
- We follow this half- bicharacteristic.
- We construct by iterating this process a bicharacteristic path Γ contained in $\mathrm{supp}\mu_T \cup \mathrm{supp}\mu_L$.
- By (G.C.L), Γ intersects Σ at a non diffractive point ρ_0 before the time T.
- $\rho_0 \notin (\mathrm{supp}\mu_T \cup \mathrm{supp}\mu_L)$. Contradiction.

Two wave equations with internal coupling

$$\begin{cases} \Box u_1 + b(x) u_2 = 0 & \text{in} \quad]0,+\infty[\times M \\ \Box u_2 = b^{\omega}(x) f & \text{in} \quad]0,+\infty[\times M \\ \text{Initial Data} \end{cases} \qquad \text{(S)}$$

$\rightarrow$ (M, g) compact Riemannian manifold without boundary.
$\rightarrow$ $\Box = \partial_t^2 - \Delta_g$.
$\rightarrow$ f is the control.
$\rightarrow$ ω open subset of $M \approx \{x \in M, b^{\omega}(x) \neq 0\}$.
$\rightarrow$ $b(x)$, $b^{\omega}(x)$ both real and smooth, and $b(x) \geq 0$.
$\rightarrow$ Energy space: $(H^2 \times H^1) \times (H^1 \times L^2)$.
$\rightarrow$ Notice the shift between the two energy levels.
$\rightarrow$ Denote

$$\begin{array}{ll} O = \{x \in M, b(x) > 0\} & \text{coupling set} \\ \omega = \{x \in M, b^{\omega}(x) \neq 0\} & \text{control set} \end{array}$$

Some References

- F.Alabau-Boussouira (starting from 99').
- F.Alabau-Boussouira-M.Leautaud (11') (symmetric systems, long control time).
- L.Rosier - L.de Teresa (11') (1-D , geometric but not sharp control time).

The crucial point: what are the geometric constraints on the open sets O and ω ?

Preliminary remarks: Exact controllabilty fails if

- The propagation speeds of the two waves are different.
- One of the open sets O or ω does not satisfy (GCC).

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$\gamma^2 \neq 1$,

$$\begin{cases} (\partial_t^2 - \gamma^2 \Delta) u_1 = -b(x) u_2 & H^2 \times H^1 \\ (\partial_t^2 - \Delta) u_2 = b^\omega(x) f & H^1 \times L^2 \end{cases}$$

For $f \in L^2\ (]0, T[\times M)$,

$$u_2 \in H^2 \text{ outside } \{\tau^2 = |\xi|^2\}$$

Therefore, starting from $(0,0)$, $u_1 \in C(]0, T[, H^3)$.
$\Rightarrow$ We can not reach any state in $H^2 \times H^1$!

Theorem

For any bicharacteristic curve Γ *of the wave operator* $\Box$, *there exists a finite energy function* u *solution of* $\Box u = 0$ *in* $]0, T[\times M$ *such that* $WFu \subset \Gamma$ *(resp.* $WF^s u \subset \Gamma$*).*

Theorem

There exists a sequence of solutions u^k *, such that*

$$\liminf \left\| \left(u^k(0) \ , \partial_t u^k(0) \right) \right\|^2_{L^2 \times H^{-1}} \geq 1, \qquad u^k \rightharpoonup 0 \quad in\ L^2(]0, T[\times M)$$

and the m.d.m μ *of* u^k *satisfies supp(*μ*)*$\subset \Gamma$.

Assumption

$$(O, T_O) \quad \text{and} \quad (\omega, T_\omega) \quad \text{satisfy (GCC)}$$

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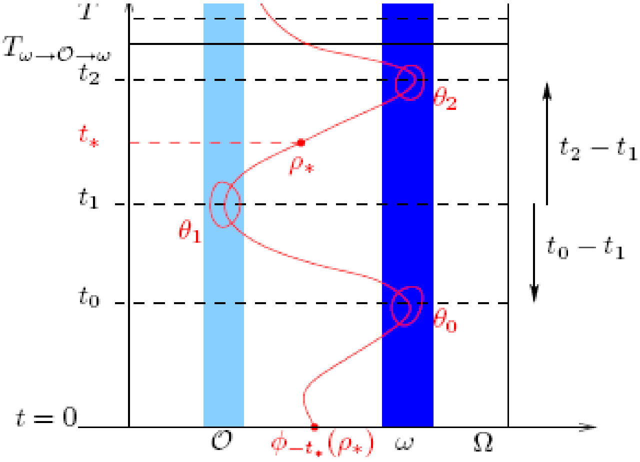
The optimal control time

Definition

$T_{\omega \to O \to \omega}$ is the infimum of the times $T > 0$ satisfying the following: Every geodesic travelling with speed 1 in M meets ω in a time $t_0 < T$, then meets O in a time $t_1 \in (t_0, T)$ and meets again ω in a time $t_2 \in (t_1, T)$.



Remarks

In general

- $T_{\omega\to\mathcal{O}\to\omega} \neq T_{\mathcal{O}\to\omega\to\mathcal{O}}$.
- $\max(T_\omega, T_\mathcal{O}) \leq T_{\omega\to\mathcal{O}\to\omega} \leq 2T_\omega + T_\mathcal{O}$.

Theorem

(D-Le Rousseau-Leautaud)
Assume that ω and O both satisfy (GCC). Then system (S) is controlable

if $T > T_{\omega \to O \to \omega}$ and is not controlable if $T < T_{\omega \to O \to \omega}$.

Theorem

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Remark: Case of a smooth domain of $\mathbb{R}^d$: work in progress.

The observabilty estimate

The adjoint system

$$\begin{cases} \Box v_1 = 0 \quad \text{in} \quad]0, +T[\times M \\ \Box v_2 + b(x) v_1 = 0 \quad \text{in} \quad]0, T[\times M \\ \text{I. D} \quad \text{in } (H^{-1} \times H^{-2}) \times (L^2 \times H^{-1}) \end{cases} \tag{S*}$$

$$E_{-1}(v_1) + E_0(v_2) \leq c \int_0^T \int_M |b^\omega v_2|^2 \, dxdt$$

where

$$E_{-1}(v) = \|(v, \partial_t v)(0)\|^2_{H^{-1} \times H^{-2}}$$

Change of functions

$$w_1 = (1-\Delta)^{-1/2} v_1, \qquad w_2 = v_2$$

$$\begin{cases} \Box w_1 = 0 \quad \text{in} \quad]0,+T[\times M \\ \Box w_2 + b(x)(1-\Delta)^{1/2} w_1 = 0 \quad \text{in} \quad]0,T[\times M \\ \text{I. D} \quad \text{in } (L^2 \times H^{-1}) \times (L^2 \times H^{-1}) \end{cases}$$

Observability Estimate

$$E_0(w_1) + E_0(w_2) \leq c_1 \int_0^T \int_M |b^\omega w_2|^2 \, dxdt + c_2 (E_{-1}(w_1) + E_{-1}(w_2))$$

Contradiction argument

$$\begin{cases} E_0(w_1^k) + E_0(w_2^k) = 1 \\ \int_0^T \int_M \left| b^\omega w_2^k \right|^2 dxdt + E_{-1}(w_1^k) + E_{-1}(w_2^k) \leq 1/k \end{cases}$$

(w_j^k) is bounded in $L^2(]0, T[\times M)$ and converges to 0 in H^{-1}.
Hence

$$w_j^k \rightharpoonup 0 \quad \text{in} \quad L^2(]0, T[\times M) \quad \text{weakly}$$

It remains to:
a) Prove the strong convergence (propagation of the m.d.m's).
b) Drop the compact term in the RHS of the relaxed observabilty estimate. (unique continuation).

Proof of b) assuming a).

The weak limit $(w_1,\ w_2)$ satisfy

$$\begin{cases} \Box w_1 = 0 \quad \text{in} \quad]0, +T[\times M \\ \Box w_2 + b(x)(1-\Delta)^{1/2} w_1 = 0 \quad \text{in} \quad]0, T[\times M \\ w_2 = 0 \quad \text{in} \quad]0, T[\times \omega \end{cases} \tag{L.S}$$

$$\mathcal{N}(T) = \{(w_1, w_2) \in L^2 \times H^{-1}, \text{solution of (L.S)}\}$$

$\mathcal{N}(T)$ is of finite dimension and stable by the action of $\partial/\partial t$.

$$\begin{cases} \Delta w_1 = \lambda^2 w_1 \\ \\ \Delta w_2 - b(x)(1-\Delta)^{1/2} w_1 = \lambda^2 w_2 \end{cases}$$

$$\int_M b(x) \left| (1-\Delta)^{1/2} w_1 \right|^2 = 0$$

So $w_1 = w_2 \equiv 0$. Contradiction.

Proof of a)

$$\begin{cases} \Box w_1^k = 0 \\ \\ \Box w_2^k + b(x)(1-\Delta)^{1/2} w_1^k = 0 \\ \\ w_2^k \to 0 \qquad \text{in} \qquad L^2(]0,T[\times\omega) \end{cases}$$

μ_1 a microlocal defect measure attached to (w_1^k) in $L^2(]0,T[\times M)$
μ_2 .... to (w_2^k) in ....
μ_{12} .... to (w_1^k, w_2^k) in ....

$$\begin{cases} H_p \mu_1 = 0 \\ \\ H_p \mu_2 = 2b\,|\eta|\,\operatorname{Im} \mu_{12} \\ \\ H_p \operatorname{Im} \mu_{12} = b\,|\eta|\,\mu_1 \\ \\ H_p \operatorname{Re} \mu_{12} = 0 \end{cases}$$

And

$$\mu_2 = \mu_{12} = 0 \quad \text{over }]0, T[\times\omega$$

Take $\rho \in B \subset S^*(]0, T[\times O)$ where B is a small borelian set of $S^*(]0, T[\times M)$.

$$\operatorname{Im} \mu_{12}(\Phi_{-T_1}(B)) - \operatorname{Im} \mu_{12}(\Phi_{T_2}(B))) = \int_{-T_1}^{T_2} b\,|\eta|\,\mu_1(\Phi_s(B))ds = 0$$

Hence $\rho \notin$supp(μ_1).....and conclude by (GCC).