Dispersive properties for Schrödinger equations

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Outline

1. Introduction

2. Discrete Schrödinger equations

3. Schrödinger equation on trees

Outline

Motivation

To build convergent numerical schemes for nonlinear PDE.

Example: Schrödinger equation

Similar problems for other dispersive equations: Korteweg de Vries, wave equation,...

Goal: To cover the classes of NONLINEAR Schrödinger equation that can be solved nowadays with fine tools from PDE theory and Harmonic analysis.

Key point: To handle nonlinearities one needs to use hidden properties of the underlying linear differential operators (Strichartz, Ginibre, Velo, Cazenave, Bourgain, Kenig, Ponce, Vega, Burq, Gérard, Linares, ...)

This has been done successfully for the PDE models.

Nonlinear problems

Nonlinear problems are solved by using fixed point arguments on the variation of constants formulation of the PDE:

$$u_t(t) = Au(t) + f(u(t)),\ t > 0,\ u(0) = u_0.$$

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)} f(u(s))ds.$$

Assuming $f : H \to H$ is locally Lipschitz, allows proving local existence and uniqueness in

$$u \in C([0, T]; H)$$

But, often in applications, the property that $f : H \to H$ is locally Lipshitz FAILS.

For instance $H = L^2(\Omega)$ and $f(u) = |u|^p u$, with $p > 0$.

Then, one needs to discover other properties of the underlying linear equation (smoothing, dispersion): If $e^{At}\varphi \in X$, then look for solutions of the nonlinear problem in

$$C([0,T;H]) \cap X.$$

One then needs to investigate whether

$$u \to e^{At}u_0 + \int_0^t e^{A(t-s)} f(u(s))ds$$

is a contraction in $C([0,T;H]) \cap X$.

Typically in applications $X = L^q(0,T;L^r(\Omega))$. This allows enlarging the class of solvable nonlinear PDE in a significant way.

If working in $C([0,T;H]) \cap X$ is needed for solving the PDE, for proving convergence of a numerical scheme we will need to make sure that it fulfills similar stability properties in X (or X_h)

THIS OFTEN FAILS!

Linear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = 0,\ x \in \mathbb{R},\ t \neq 0, \\ u(0,x) = \varphi(x),\ x \in \mathbb{R}, \end{cases}$$

Conservation of the L^2-norm

$$\|S(t)\varphi\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}$$

Dispersive estimate

$$|S(t)\varphi(x)| \leq \frac{1}{(4\pi|t|)^{1/2}} \|\varphi\|_{L^1(\mathbb{R})}$$

Space time estimates

The admissible pairs

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{r}$$

Strichartz estimates for admissible pairs (q, r)

$$\|S(\cdot)\varphi\|_{L^q(\mathbb{R}, L^r(\mathbb{R}))} \leq C(q, r)\|\varphi\|_{L^2(\mathbb{R})}$$

Local Smoothing effect

$$\sup_{x_0, R} \frac{1}{R} \int_{B(x_0,R)} \int_{-\infty}^{\infty} |(-\Delta)^{1/4} e^{it\Delta}\varphi|^2 dt dx \leq C\|\varphi\|^2_{L^2(\mathbb{R})}$$

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Nonlinear Schrödinger Equation

$$\begin{cases} iu_t + \Delta u = |u|^p u, \, x \in \mathbb{R}, \, t \neq 0 \\ u(0,x) = \varphi(x), \, x \in \mathbb{R} \end{cases}$$

For initial data in $L^2(\mathbb{R})$, Tsutsumi '87 proved the global existence and uniqueness for $p < 4$

$$u \in C(\mathbb{R}, \, L^2(\mathbb{R})) \cap L^q_{loc}(\mathbb{R}, \, L^r(\mathbb{R}))$$

This result can not be proved by methods based purely on energy arguments.

A first numerical scheme for NSE

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = |u^h|^2 u^h, \quad t \neq 0, \\ u^h(0) = \varphi^h \end{cases}$$

$$(\Delta_h u)_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$$

Questions

- Does u^h converge to the solution of NSE?
- Is u^h uniformly bounded in $L^q_{loc}(\mathbb{R}, l^r(h\mathbb{Z}^d))$?
- Local Smoothing ?

A conservative scheme for LSE

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = 0, \quad t > 0, \\ u^h(0) = \varphi^h. \end{cases}$$

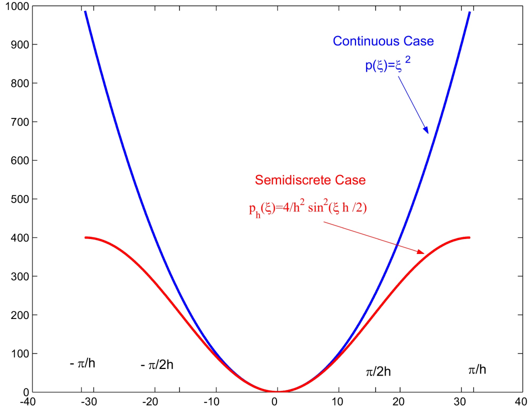
In the Fourier space the solution $\widehat{u}^h$ can be written as

$$\widehat{u}^h(t,\xi) = e^{-itp_h(\xi)}\widehat{\varphi}^h(\xi),\ \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right],$$

where

$$p_h(\xi) = \frac{4}{h^2}\sin^2\left(\frac{\xi h}{2}\right).$$

The two symbols in dimension one



- Lack of uniform $l^1 \to l^\infty$: $\xi = \pm\pi/2h$
- Lack of uniform local smoothing effect: $\xi = \pm\pi/h$

Lemma

(Van der Corput) Suppose ψ is real-valued and smooth in (a, b), and that $|\psi^{(k)}(x)| \geq 1$ for all $x \in (a, b)$. Then

$$\left| \int_a^b e^{i\lambda\psi(x)} dx \right| \leq c_k \lambda^{-1/k}$$

In dimension one:

$$\frac{\|u^h(t)\|_{l^\infty(h\mathbb{Z})}}{\|u^h(0)\|_{l^1(h\mathbb{Z})}} \lesssim \frac{1}{t^{1/2}} + \frac{1}{(th)^{1/3}}.$$

These slight changes on the shape of the symbol are not an obstacle for the convergence of the numerical scheme in the $L^2(\mathbb{R})$ sense for LSE. But produce the lack of uniform (in h) dispersion of the numerical scheme and consequently, makes the scheme useless for nonlinear problems.

Various remedies have been proposed L.I and E. Zuazua (2003-2010)

- Filtering the high frequencies
- Artificial numerical viscosity
- Two-grid methods
- Error estimates for rough initial data
- Wave packet analysis by A. Marica and E. Zuazua

Outline

Discrete Schrodinger equations

We consider

$$\begin{cases} iu_t + \Delta_d u = 0, & j \in \mathbb{Z}, t \neq 0, \\ u(0) = \varphi. \end{cases} \quad (1)$$

where

$$(\Delta_d u)_j = u_{j+1} - 2u_j + u_{j-1}$$

Theorem (Stefanov 2005, LI & Zuazua 2005)

For any $\varphi \in l^1(\mathbb{Z})$ the following holds

$$\|u(t)\|_{l^\infty(\mathbb{Z})} \leq \langle t \rangle^{-1/3} \|\varphi\|_{l^1(\mathbb{Z})} \quad (2)$$

where $\langle t \rangle = t + 1$.

Discrete Schrodinger equations

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Theorem (Stefanov 2005, LI & Zuazua 2005)

For any $\varphi \in l^1(\mathbb{Z})$ the following holds

$$\|u(t)\|_{l^\infty(\mathbb{Z})} \leq \langle t \rangle^{-1/3} \|\varphi\|_{l^1(\mathbb{Z})} \tag{2}$$

where $\langle t \rangle = t + 1$.

A simple proof

$$u(t,j) = (K_t * \varphi)(j) = \sum_{k\in\mathbb{Z}} K_t(j-k)\varphi(k),$$

where

$$K_t(j) = \int_{-\pi}^{\pi} e^{-4it\sin^2\frac{\xi}{2}} e^{ij\xi} d\xi.$$

It remains to prove that

$$|K_t(j)| \leq t^{-1/3}.$$

Apply Van der Corput and the fact that $\psi = 4\sin^2\frac{\xi}{2}$ satisfies

$$|\psi''| + |\psi'''| \geq C > 0.$$

DLSE with Dirichlet boundary condition

We consider the following equation

$$\begin{cases} iu_t(t,j) + (\Delta_d u)(t,j) = 0, & j \geq 1, \\ u(t,0) = 0, & \\ u(0,j) = \varphi(j), & j \geq 1. \end{cases} \tag{3}$$

In the matrix formulation we have $iU_t + AU = 0$ where

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}$$

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Theorem

For any $\varphi \in l^2(\mathbb{Z}^+)$ *there exists a unique solution* $u \in C([0,\infty), l^2(\mathbb{Z}^+))$ *of problem* (3) *given by the following formula*

$$u(t,j) = \sum_{k\geq 1}(K_t(j-k) - K_t(j+k))\varphi(k), \quad j \geq 1.$$

Moreover

$$\|u(t)\|_{l^\infty(\mathbb{Z}^+)} \leq \langle t\rangle^{-1/3}\|\varphi\|_{l^1(\mathbb{Z}^+)}.$$

Proof: Use odd extension of the function u to reduce the DLSE on the whole Z.

DLSE with Neumann boundary conditions

We consider the system

$$\begin{cases} iu_t(j) + (\Delta_d u)(j) = 0 & j \geq 1, \\ u(t,0) = u(t,1), & t > 0, \\ u(0,j) = \varphi(j), & j \geq 1. \end{cases} \tag{4}$$

In the matrix formulation we have $iU_t + AU = 0$ where

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & ... & ... & ... & 0 \\ 0 & 0 & 0 & ... & ... & ... \end{pmatrix}$$

Theorem

For any $\varphi \in l^2(\mathbb{Z}^+)$ *there exists a unique solution* $u \in C([0,\infty), l^2(\mathbb{Z}^+))$ *of problem* (4) *given by the following formula*

$$u(t, j+1) = \sum_{k\geq 1} (K_t(k-j-1) + K_t(k+j))\varphi(k).$$

Moreover

$$\|u(t)\|_{l^\infty(\mathbb{Z}^+)} \leq \langle t \rangle^{-1/3} \|\varphi\|_{l^1(\mathbb{Z}^+)}.$$

Proof: Use the even extension of u.

Coupled DLSE

The equation we analyze is the following

$$\begin{cases} iu_t(j) + (\Delta_d u)(j) = 0 & j \leq -1, \\ iv_t(j) + (\Delta_d v)(j) = 0 & j \geq 1, \\ u(t,0) = v(t,0), & t > 0, \\ u(t,-1) - u(t,0) = v(t,0) - v(t,1), & t > 0 \\ u(0,j) = \varphi(j), & j \leq -1, \\ v(0,j) = \varphi(j), & j \geq 1. \end{cases} \tag{5}$$

Theorem

For any $\varphi \in l^2(\mathbb{Z}^*)$ *there exist a unique solution* $(u,v) \in C([0,\infty, l^2(\mathbb{Z}^*))$ *of equation* (5) *which satisfies the dispersive estimate*

$$\|(u,v)(t)\|_{l^\infty(\mathbb{Z}^*)} \leq c(t+1)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z}^*)}. \tag{6}$$

A simple proof

Define

$$S(j) = \frac{v(j) + u(-j)}{2}, j \geq 0, \; D(j) = \frac{v(j) - u(-j)}{2}, j \geq 0.$$

Observe that

$$(u, v) = ((S - D)(-\cdot), S + D)$$

Key point: D and S satisfy two DLSE on the half line with Dirichlet, respectively Neumann, boundary condition:

$$\begin{cases} iD_t(j) + (\Delta_d D)(j) = 0 & j \geq 1, \\ D(t,0) = 0, & \\ D(0,j) = \frac{\varphi(j) - \varphi(-j)}{2}, & j \geq 1 \end{cases} \tag{7}$$

and

$$\begin{cases} iS_t(j) + (\Delta_d S)(j) = 0 & j \geq 1, \\ S(t,0) = S(t,1), & t > 0, \\ S(0,j) = \frac{\varphi(j) + \varphi(-j)}{2}, & j \geq 1. \end{cases} \tag{8}$$

Matrix formulation

Set $U = (u, v)^T$ where $u = (u(j))_{j\leq -1}$ and $v = (v(j))_{j\geq 1}$. It turns out that U solves the following system

$$\begin{cases} iU_t + AU = 0, \quad t > 0, \\ U(0) = \varphi, \end{cases} \tag{9}$$

where the operator A is given by

$$A = \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}.$$

Open Problem

How we can obtain the $l^1 - l^\infty$ property directly from the properties of the operator A?

Remarks: A is not a diagonal operator, so we cannot use the Fourier analysis to obtain a symbol for A and to use oscillatory integrals

A can be decomposed as $A = \Delta_d + B$ where

$$\Delta_d = \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}$$

and

$$B = \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}.$$

The solution of (9) is given by $U(t) = e^{it(\Delta_d + B)}$. How we can use the dispersive properties of $e^{it\Delta_d}$ and some properties of B in order to prove the $l^1 - l^\infty$ estimate for U?

DLSE with "non-constant coefficients"

The model (D. Stan, L.I., JFAA 2011)

$$\begin{cases} iu_t(j) + b_1^{-2}(\Delta_d u)(j) = 0 & j \leq -1, \\ iv_t(j) + b_2^{-2}(\Delta_d v)(j) = 0 & j \geq 1, \\ u(t,0) = v(t,0), & t > 0, \\ b_1^{-2}(u(t,-1) - u(t,0)) = b_2^{-2}(v(t,0) - v(t,1)), & t > 0 \\ u(0,j) = \varphi(j), & j \leq -1, \\ v(0,j) = \varphi(j), & j \geq 1. \end{cases}$$

Question: $\|(u,v)(t)\|_\infty \leq (1+|t|)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z}^*)}$

Matrix formulation

$U=(u(j))_{j\neq 0}$ satisfies $iU_t+AU=0$ where A is given by

$$\begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0\\ 0 & b_1^{-2} & -2b_1^{-2} & b_1^{-2} & 0 & 0 & 0 & 0\\ 0 & 0 & b_1^{-2} & -b_1^{-2}-\frac{1}{b_1^2+b_2^2} & \frac{1}{b_1^2+b_2^2} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{b_1^2+b_2^2} & -\frac{1}{b_1^2+b_2^2}-b_2^{-2} & b_2^{-2} & 0 & 0\\ 0 & 0 & 0 & 0 & b_2^{-2} & -2b_2^{-2} & b_2^{-2} & 0\\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}.$$

No chance to use Fourier transform, sums, etc... unless we answer to the previous open problem.

Use of the resolvent

Theorem

For any b_1 and b_2 positive the spectrum of the operator A satisfies

$$\sigma(A) \subset I = [-4\max\{b_1^{-2}, b_2^{-2}\}, 0]. \tag{10}$$

For any $\omega \in I$ define

$$R^{\pm}(\omega) = \lim_{\epsilon \downarrow 0} R(\omega \pm i\epsilon).$$

We can prove that

$$R^{-}(\omega) = \overline{R}^{+}(\omega), \quad \forall \omega \in I.$$

Then

$$e^{itA} = \frac{1}{2i\pi} \int_I e^{it\omega} [R^{+}(\omega) - R^{-}(\omega)] d\omega$$

Big Problem: computing the resolvent

Lemma

Let $\lambda \in \mathbb{C} \setminus [-4\max\{b_1^{-2}, b_2^{-2}\}, 0]$. Any solution of the equation $(A - \lambda I)f = g$ is given by

$$f(j) = \frac{-r_s^{|j|}}{b_2^{-2}(1-r_2) + b_1^{-2}(1-r_1)}\Big[\sum_{k\in I_2} r_2^{|k|}g(k) + \sum_{k\in I_1} r_1^{|k|}g(k)\Big] \qquad (11)$$

$$+\frac{b_s^2}{r_s - r_s^{-1}}\sum_{k\in I_s}(r_s^{|j-k|} - r_s^{|j|+|k|})g(k), \quad j \in I_s,$$

where r_s, $s \in \{1,2\}$ is the unique solution with $|r_s| < 1$ of the equation

$$r_s^2 - 2r_s + 1 = \lambda b_s^2 r_s.$$

A small part of the proof

Let assume $b_2 < b_1$ and take $I = [-4b_1^{-2}, 0]$. "Essentially" we have to prove that

$$|\int_I e^{it\omega} r_1(\omega)^j r_2(\omega)^k| \leq C|t|^{-1/3}$$

uniformly on j and k, where

$$r_s^2 - 2r_s + 1 = \omega b_s^2 r_s, s \in \{1, 2\}.$$

On I, $r_1 = e^{i\theta_1(\omega)}$ and $r_2 = e^{i\theta_2(\omega)}$ and we have to prove that

$$|\int_I e^{it\omega} e^{ij\theta_1(\omega)} e^{ik\theta_2(\omega)} d\omega| \leq C|t|^{-1/3}$$

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$$|\int_I e^{it\omega} e^{ij\theta_1(\omega)} e^{ik\theta_2(\omega)} d\omega| \leq C|t|^{-1/3}$$

With a change of variables $\omega = 2b_1^{-2}(\cos\theta - 1)$ it remains to prove the following result

Lemma

Let $a \in (0,1]$. There exists a positive constant $C(a)$ such that the following

$$\Big| \int_0^{\pi} e^{it(2\cos\theta + 2z\arcsin(a\sin\frac{\theta}{2}))} e^{ity\theta} \sin\theta d\theta \Big| \leq C(a)(|t|+1)^{-1/3} \qquad (12)$$

holds for any real numbers y, z and t.

Obs: For $z = 0$ the estimate appears in the case of simpler DLSE.

Oscillatory integrals

Lemma (Van der Corput)

Suppose ψ *is real-valued and smooth in* I, *and that* $|\psi^{(k)}(x)| \geq 1$ *for all* $x \in I$. *Then*

$$\Big|\int_I e^{i\lambda\psi(x)}\phi(x)dx\Big| \leq c_k\lambda^{-1/k}(\|\phi\|_{L^\infty(I)} + \int_I |\phi'|).$$

We need to use two or three derivatives of the phase function

$$p_a(\theta) = 2\cos\theta + y\theta + z\arcsin(a\sin\frac{\theta}{2}).$$

But there are cases when the above Lemma is not sufficient

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Refinements of Van der Corput's Lemma

Lemma (Kenig, Ponce, Vega 91)

The following

$$|\int_a^b e^{i(t\psi(\xi)-x\xi)}|\psi''(\xi)|^{1/2}\phi(\xi)d\xi| \leq c_\psi |t|^{-1/2}\{\|\phi\|_{L^\infty(a,b)} + \int_a^b |\phi'(\xi)|d\xi\}.$$

holds for all real numbers x and t.

But there are cases when the above Lemma is still not sufficient

Refinements of Van der Corput's Lemma

Lemma (Kenig, Ponce, Vega 91)

The following

$$| \int_a^b e^{i(t\psi(\xi)-x\xi)} |\psi''(\xi)|^{1/2} \phi(\xi) d\xi | \leq c_\psi |t|^{-1/2} \{ \|\phi\|_{L^\infty(a,b)} + \int_a^b |\phi'(\xi)| d\xi \}.$$

holds for all real numbers x and t.

But there are cases when the above Lemma is still not sufficient

A new Lemma

Lemma (D. Stan, LI, JFAA 2011)

Assuming that at the critical points we have

$$\phi'(\xi) \sim \xi^{\alpha}, \alpha \geq 2$$

then

$$I(x,t) = \Big| \int_{\Omega} e^{i(t\phi(\xi) - x\xi)} |\phi'''(\xi)|^{\frac{1}{3}} d\xi \Big| \leq ct^{-\frac{1}{3}}.$$

Finally apply careful Van der Corput and KpV with $k = 2$ or $k = 3$ and even brute force

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Some Open Problems

I. Give sufficient conditions for a symmetric matrix A with few diagonals such that for the equation $iU_t + AU = 0$ we can prove similar decay properties, even with other type of decay: $t^{-1/4}$, etc..

II. Coupling more than two equations.

Outline

Schrodinger equation on trees (or network trees)

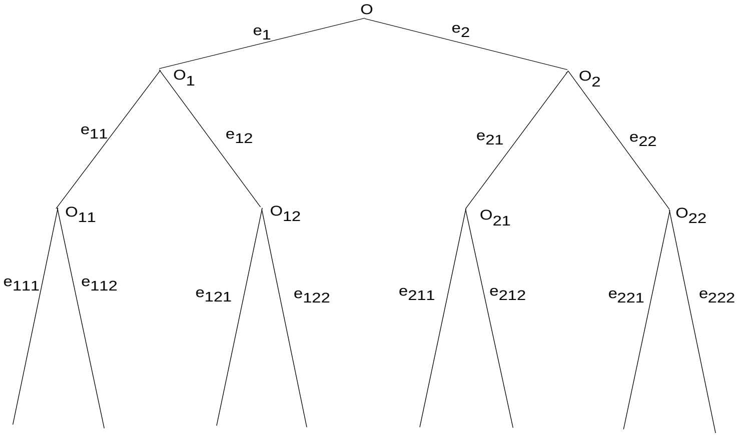
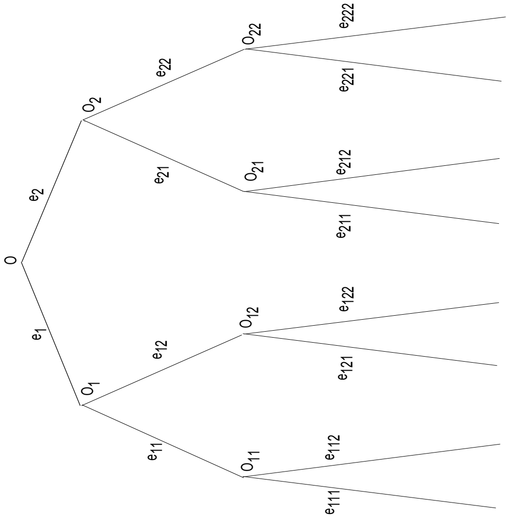


Figure: A tree with the third generation formed by infinite edges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$$\begin{cases} i\mathbf{u}_t(t,x) + \Delta_\Gamma \mathbf{u}(t,x) = 0, & x \in \Gamma, t \neq 0, \\ \mathbf{u}(0) = \mathbf{u}_0, & x \in \Gamma. \end{cases} \tag{13}$$

$$\begin{cases} iu_t^{\overline{\alpha}}(t,x) + u_{xx}^{\overline{\alpha}}(t,x) = 0, & x \in (0,1), 1 \leq |\overline{\alpha}| \leq n, \\ iu_t^{\overline{\alpha}}(t,x) + u_{xx}^{\overline{\alpha}}(t,x) = 0, & x \in (0,\infty), |\overline{\alpha}| = n+1, \\ \begin{cases} u^{\overline{\alpha}}(t,1) = u^{\overline{\alpha\beta}}(t,0), & \beta \in \{1,2\}, 1 \leq |\overline{\alpha}| \leq n, \\ u^1(0,t) = u^2(0,t), \end{cases} \\ \begin{cases} u_x^{\overline{\alpha}}(t,1) = \sum_{\beta=1}^{2} u_x^{\overline{\alpha\beta}}(t,0), & 1 \leq |\overline{\alpha}| \leq n, \\ u_x^1(0,t) + u_x^2(0,t) = 0, \end{cases} \\ u^{\overline{\alpha}}(0,x) = u_0^{\overline{\alpha}}(x). \end{cases} \tag{14}$$

$$\begin{cases} i\mathbf{u}_t(t,x) + \Delta_\Gamma \mathbf{u}(t,x) = 0, & x \in \Gamma, t \neq 0, \\ \mathbf{u}(0) = \mathbf{u}_0, & x \in \Gamma. \end{cases} \tag{13}$$

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For regular trees we have similar dispersive estimates.
Main Tool: A result on LSE with discontinuous coefficients

Theorem (Banica, SIAM JMA 2003)

Consider a partition of the real axis $-\infty = x_0 < x_1 < \cdots < x_{n+1} = \infty$ and a step function $\sigma(x) = \sigma_i$ for $x \in (x_i, x_{i+1})$, where σ_i are positive numbers.
The solution u of the Schrödinger equation

$$\begin{cases} iu_t(t,x) + (\sigma(x)u_x)_x(t,x) = 0, & \text{for } x \in \mathbb{R}, t \neq 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

satisfies the dispersion inequality

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C|t|^{-1/2}\|u_0\|_{L^1(\mathbb{R})}, \quad t \neq 0.$$

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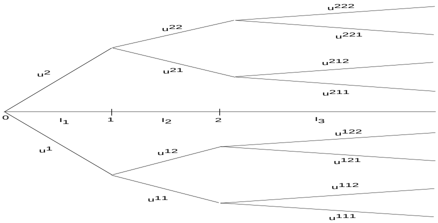
Idea of the proof in the case of regular trees

Look to the tree in a different way

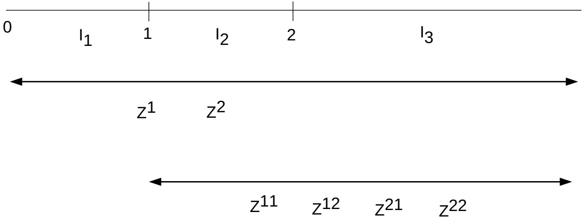


The functions situated above each interval are defined on that interval, for example u^1 and u^2 are defined on I_1, etc... where

$$I_k = \begin{cases} (k-1,k) & \text{if} \quad 1 \leq k \leq n, \\ (n,\infty) & \text{if} \quad k = n+1. \end{cases}$$

In order to obtain $L^1 - L^\infty$ estimates we need to introduce some averages

$$Z^{\overline{\alpha}} = \frac{\sum_\beta u^{\overline{\alpha\beta}}}{2^{|\beta|}} \quad \text{on} \quad I_{|\alpha|+|\beta|}, \quad 0 \le |\beta| \le n+1-|\alpha|$$



The first generation of Z's

$Z(t,x)=(-Z^1(t,-x),Z^2(t,x))$ satisfies

$$\begin{cases} iZ_t+Z_{xx}=0 & x\in\mathbb{R}\backslash\{k, 1\leq |k|\leq n\} \\ Z(t,k-)=Z(t,k+), & 1\leq |k|\leq n \\ Z_x(t,k-)=2Z_x(t,k+), & 1\leq |k|\leq n \\ Z(0,x)=Z_0(x), & x\in\mathbb{R}\backslash\{k, 1\leq |k|\leq n\}. \end{cases} \quad (15)$$

Using that Z satisfies

$$\|Z(t)\|_{L^\infty(\mathbb{R})}\leq |t|^{-1/2}\|Z(0)\|_{L^1(\mathbb{R})}$$

we have the same information about u^1 and u^2:

$$\max\{\|u^1(t)\|_{L^\infty(I_1)},\|u^2(t)\|_{L^\infty(I_1)}\}\leq |t|^{-1/2}\sum_{k=1}^{n+1}\frac{1}{2^{k-1}}\Big\|\sum_{|\alpha|=k}u_0^{\overline{\alpha}}\Big\|_{L^1(I_k)}.$$

Next generations: induction

Question: What about a general tree? another ideas ...

The general case

Theorem (V. Banica, L.I, JMP 2011)

The solution of the linear Schrödinger equation on a tree is of the form

$$e^{it\Delta_\Gamma}u_0(x)=\sum_{\lambda\in\mathbb{R}}\frac{a_\lambda}{\sqrt{|t|}}\int_{I_\lambda}e^{i\frac{\phi_\lambda(x,y)}{t}}\,u_0(y)\,dy. \tag{16}$$

with $\phi_\lambda(x,y)\in\mathbb{R}$, $I_\lambda\in\{I_e\}_{e\in E}$, $\sum_{\lambda\in\mathbb{R}}|a_\lambda|<\infty$, *and it satisfies the dispersion inequality*

$$\|e^{it\Delta_\Gamma}u_0\|_{L^\infty(\Gamma)}\leq\frac{C}{\sqrt{|t|}}\|u_0\|_{L^1(\Gamma)},\ \ t\neq 0. \tag{17}$$

Ingredients for the proof

1. If $R_\omega \mathbf{f} = (-\Delta_\Gamma + \omega^2 I)^{-1}\mathbf{f}$ then $\omega R_\omega \mathbf{f}(x)$ can be analytically continued in a region containing the imaginary axis
2. A spectral calculus argument to write

$$e^{it\Delta_\Gamma}\mathbf{u}_0(x) = \int_{-\infty}^{\infty} e^{it\tau^2}\tau R_{i\tau}\mathbf{u}_0(x)\frac{d\tau}{\pi}.$$

3. The representation of the resolvent

$$\tau R_{i\tau}\mathbf{u}_0(x) = \sum_{\lambda\in\mathbb{R}} b_\lambda e^{i\tau\psi_\lambda(x)} \int_{I_\lambda} \mathbf{u}_0(y) e^{i\tau\beta_\lambda y} dy, \qquad (18)$$

with $\psi_\lambda(x), \beta_\lambda \in \mathbb{R}$, $I_\lambda \in \{I_e\}_{e\in E}$ and $\sum_{\beta\in\mathbb{R}} |b_\lambda| < \infty$.

Main steps

1.On each edge parametrized by I_e,

$$R_\omega \mathbf{f}(x) = ce^{\omega x} + \tilde{c}e^{-\omega x} + \frac{1}{2\omega}\int_{I_e} \mathbf{f}(y)\, e^{-\omega|x-y|} dy, \ x \in I_e.$$

2. The continuity of $R_\omega \mathbf{f}$ and of transmission of $\partial_x R_\omega \mathbf{f}$ at the vertices of the tree give the system of equations on the coefficients c's
3.

$$R_\omega \mathbf{f}(x) = \frac{1}{\omega \ \det D_\Gamma(\omega)} \sum_{\lambda=1}^{N(\Gamma)} c_\lambda e^{\pm\omega\Phi_\lambda(x)} \int_{I_\lambda} \mathbf{f}(y)\, e^{\pm\omega y} dy \tag{19}$$

$$+ \frac{1}{2\omega}\int_{I_e} \mathbf{f}(y)\, e^{-\omega|x-y|} dy, \tag{20}$$

4. Induction on the number of the vertices to prove that

$$\exists c_\Gamma, \epsilon_\Gamma > 0, \ |\det D_\Gamma(\omega)| > c_\Gamma, \forall \omega \in \mathbb{C}, |\Re\omega| < \epsilon_\Gamma.$$

5. Results on almost periodic functions to write

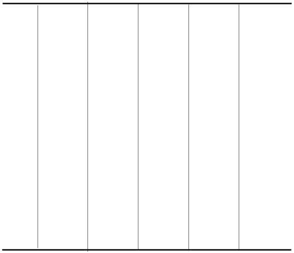
$$\frac{1}{\det D_\Gamma(i\tau)} = \sum_\lambda d_\lambda e^{i\tau\lambda}$$

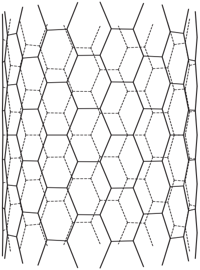
with $\sum_\lambda |d_\lambda| < \infty$

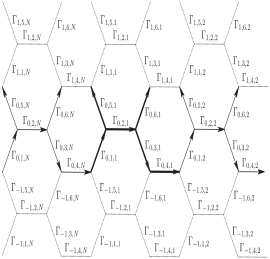
Some Open Problems

1. Other coupling conditions $A(v)\mathbf{f}(v) + B(v)\mathbf{f}'(v) = 0$ where
   1. the joint matrix $(A(v), B(v))$ has maximal rank, i.e. $d(v)$,
   2. $A(v)B(v)^T = B(v)A(v)^T$.
2. clarify if the dispersion is possible only on trees or there are graphs (with some of the edges infinite) with suitable couplings where the dispersion is still true
3. Some applications to control/stabilization on trees/networks
4. Discrete Schrödinger equations on trees, graphs
5. some magnetic operators: in the presence of an external magnetic field the effect of the topology of the graph becomes more pronounced
6. Strichartz estimates for "exotic" graphs

Exotic structures







THANKS for your attention !!!