Stabilization for the semilinear wave equation with geometric control condition

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The damped nonlinear wave equation

$$\begin{cases} \Box u = \partial_t^2 u - \Delta u &= -\gamma(x)\partial_t u + f(u) \\ (u(0), \partial_t u(0)) &= (u_0, u_1) \in X = H^1 \times L^2. \end{cases} \tag{1}$$

Ω is a connected bounded open set with boundary in dimension 3 (for simplicity)
$f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies

$f(0) = 0$, 0 is an equilibrium solution

$sf(s) \geq 0$, f is defocusing

$|f(s)| \leq C(1+|s|)^p$, $|f'(s)| \leq C(1+|s|)^{p-1}$ with $1 \leq p < 5$ f is subcritical.

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$$E(t) = \frac{1}{2}\left(\int_\Omega |\partial_t u|^2 + \int_\Omega |\nabla u|^2\right) + \int_\Omega V(u)$$

where $V(u) = \int_0^u f(s)ds$.

local theory by Strichartz estimates (Burq-Lebeau-Planchon 2006)

Bibliography

Linear results with Geometric Control Condition : Rauch-Taylor (75), Bardos-Lebeau-Rauch (92)

Assumption (Geometric Control Condition)

There exists $T_0 > 0$ *such that every ray of geometric optic travelling at speed* 1 *meets* ω *in a time* $t < T_0$*.*

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Nonlinear stabilization results : If ω is the exterior of a ball of $\mathbb{R}^d$:

- Dehman-Lebeau-Zuazua (03) (subcritical case, controllability in large time)
- Dehman-Gérard (02) (critical case on $\mathbb{R}^3$ using profile decomposition)
- Aloui-Ibrahim-Nakanishi (09) (any nonlinearity for weak solutions, uses Morawetz-type estimates)

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Other type of nonlinear result, controllability at high frequency :
Dehman-Lebeau (09) : in subcritical case, (same time and geometrical assumption as linear case),
C.L. (10) in critical case with non-focusing assumptions

Stabilization theorem

Theorem (R.J.,C.L.)

Let $R_0 > 0$, ω satisfying assumption Geometric Control Condition and $\gamma \in C^\infty(\Omega, \mathbb{R}^+)$ satisfying $\gamma(x) > \eta > 0$ for all $x \in \omega$. Assume moreover that f satisfies the previous assumptions and is analytic. Then, there exist $C, \lambda > 0$ such that for any (u_0, u_1) in $H^1 \times L^2$, with

$$\|(u_0, u_1)\|_{H^1 \times L^2} \leq R_0;$$

the unique strong solution of (1) satisfies $E(u)(t) \leq Ce^{-\lambda t} E(u)(0)$ for $t \geq 0$.

Idea of proof in the subcritical case (Dehman-Lebeau-Zuazua)

We have

$$E(T) = E(0) - \int_0^T \int_\Omega \gamma(x) |\partial_t u|^2.$$

So to get exponential decay, we need to prove an observability estimate

$$\int_0^T \int_\Omega \gamma(x) |\partial_t u|^2 \geq C E(0)$$

for solutions of the damped wave equation bounded in energy by R_0.

Idea of proof in the subcritical case (DLZ)

By contradiction : let u_n be a bounded sequence of solutions with :

$$\int_0^T \int_\Omega \gamma(x)|\partial_t u_n|^2 \leq \frac{1}{n} E(u_n)(0). \quad (2)$$

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Contradiction to $\alpha > 0$.

Main assumptions

Their method of proof could allow to prove the stabilization in more general domains under the more general assumptions

- **Geometric Control Condition**
- **Unique Continuation** $u \equiv 0$ *is the unique strong solution in the energy space of*

$$\left\{\begin{array}{rcccl} \Box u + f(u) & = & 0 & \text{on} & [0,T]\times\Omega \\ \partial_t u & = & 0 & \text{on} & [0,T]\times\omega. \end{array}\right.$$

The problem of unique continuation

Classical technique : use Carleman estimate for $w = \partial_t u$ solution of

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There are some other available unique continuation results using partial analyticity.

Unique continuation under partial analyticity

Theorem (Zuily-Robbiano, a particular case)

Let v be a solution on an open set $\mathcal{U}$ of

$$\Box v + d(x,t)v = 0$$

where d is smooth, analytic in time.
Let $\varphi \in C^2(\mathcal{U},\mathbb{R})$ such that $\varphi(x_0,t_0) = 0$ and $(\nabla\varphi,\partial_t\varphi)(x,t) \neq 0$ for all $(x,t) \in \mathcal{U}$.
Moreover, assume

- $v \equiv 0$ *in* $\{(x,t) \in \mathcal{U}\ ,\ \varphi(x,t) \leq 0\}$.
- φ *not characteristic at* (x_0,t_0).

Then, $v \equiv 0$ in a neighbourhood of (x_0,t_0).

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Pbm : in our case, $V = f'(u)$ has no reason to be analytic in time...

A result of analyticity

Theorem – J.K. Hale and G. Raugel (2003)

Let $U(t)$ be a global solution of

$$\partial_t U(t) = AU(t) + F(U(t)) \quad \forall t \in \mathbb{R} .$$

We assume that

(i) $\|e^{At}\|_{\mathcal{L}(X)} \leq Me^{-\lambda t}$.
(ii) $\{U(t), t \in \mathbb{R}\}$ *is contained in a compact set* K *of* X.
(iii) F *is a compact and lipschitzian map and is analytic in a neighbourhood of* K.
(iv) there exist projectors (P_n) *converging to the identity and commuting with the unbounded part of* A.

Then, the solution $U(t)$ *is analytic from* $t \in \mathbb{R}$ *into* X.

Idea of the proof

Goal : prove that $t \mapsto U(t)$ is $\mathcal{C}^1$ with t in a complex strip $\mathbb{R} + i(-\varepsilon, \varepsilon)$.

Idea : use the fixed point theorem for contracting maps as in the proof of Cauchy-Lipschitz theorem.

$$U(t) \longmapsto e^{A(t-t_0)} U_0 + \int_{t_0}^{t} e^{As} F(U(t-s))\, ds \,.$$

Proof of our main result

Let (u_n) solutions with $E(u_n(0)) \leq E_0$ and $T_n \to +\infty$ such that

$$\int_0^{T_n} \int_\Omega \gamma(x) \left|\partial_t u_n\right|^2 \, dtdx \leq \frac{1}{n} E(u_n(0)) \leq \frac{1}{n} E_0.$$

Assume that $E(u_n(0)) \to \alpha > 0$ and set $u_n^*(\cdot) = u_n(\cdot + T_n/2)$.
It remains to :

- show that (u_n^*) converges strongly to a global solution u^* which does not dissipate energy.
- apply previous theorem to show that u^* is analytic in time and smooth in space.
- use the unique continuation property of Robbiano and Zuily to show that u^* is constant in time and so $u^* \equiv 0$.

Asymptotic compactness

$$U_n^*(0) = e^{AT_n/2}U_n(0) + \int_0^{T_n/2} e^{A(T_n/2-\tau)} \begin{pmatrix} 0 \\ f(u_n(\tau)) \end{pmatrix} d\tau$$

We use that :

- $\|e^{At}\| \leq Ce^{-\lambda t}$
- $U_n(t)$ is bounded in $H^1(\Omega) \times L^2(\Omega)$
- for $f(u) = o(|u|^3)$, f maps a bounded set of $H^1(\Omega)$ into a compact set of $L^2(\Omega)$.

Compactness for $f(u) = o(|u|^5)$

For $f(u) \sim |u|^p$ with $p < 5$, we use

Theorem – B. Dehman, G. Lebeau and E. Zuazua (2003)

Let $s \in [0,1)$, $R > 0$ and $T > 0$.
There exist $\varepsilon > 0$ and (q,r) satisfying $\frac{1}{q} + \frac{3}{r} = \frac{1}{2}$, $q \in [7/2, +\infty]$ and $C > 0$ such that,
if $v \in L^\infty([0,T], H^{1+s}(\Omega))$ has a Strichartz $L^q([0,T], L^r(\Omega))$ norm bounded by R, then

$$\|f(v)\|_{L^1([0,T],H^{s+\varepsilon}(\Omega))} \leq C\|v\|_{L^\infty([0,T],H^{1+s}(\Omega))} \ .$$

(proof by using Meyer's multipliers)

Regularity of u^*

$$U^*(t) = \int_{-\infty}^{t} e^{A(t-\tau)} F(U^*(\tau))\, d\tau\,.$$

We use several times the result of Dehman, Lebeau and Zuazua until u^* belongs to $H^2(\Omega)$ and then the usual Sobolev imbeddings are sufficient for the bootstrap argument.

$\Longrightarrow$ u^* is $\mathcal{C}^\infty$
$\Longrightarrow$ we can apply the analyticity result of Hale and Raugel ($F(U^*)$ well defined)

Conclusion of the proof

We know that u^* is analytic in time, smooth in space and does not dissipate energy. Set $v = u_t^*$, we have

$$v \equiv 0 \text{ on the support of } \gamma$$

$$v_{tt} = \Delta v + f'(u(t))v \; .$$

A global version of the unique continuation result of Robbiano and Zuily *Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients* shows that $v \equiv 0$.

Control/Dynamical point of view

- **Stabilisation / existence of a compact global attractor**
- **Propagation of compactness / asymptotic compactness**
- **Propagation of space regularity / asymptotic smoothness** (regularity of the trajectories of the attractor)
- **? ? ? / asymptotic analyticity** (the solutions of the attractor are analytic in time if the nonlinearity is analytic)
- **Unique continuation properties / gradient structure** (equilibria are the only trajectories which do not dissipate the energy)

Further results

- The stabilisation also holds for **unbounded manifolds** with bounded $\mathcal{C}^\infty-$geometry if $\gamma \geq \alpha > 0$ outside a bounded set.
- The stabilisation also holds for **almost all the nonlinearities** f, even non-analytic ones (generic result).
- We get **control of the wave equation** by using the stabilisation and a local control near 0.
- Same kind of technics can be used to show **existence of a compact global attractor** for a more complex nonlinearity $f(x,u)$.

THANK YOU FOR YOUR ATTENTION ! ! ! ! !