Diffraction by random dielectric structures.

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joint work with:

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Physical context

Maxwell System in harmonic regime: $\exp(-i\omega t)$
ω : angular frequency: $\frac{\omega}{c} = \frac{2\pi}{\lambda} = k$ (wave number)

Finite scatter $\Omega \subset \mathbb{R}^3$ filled (periodically or not) with high permittivity inclusions:

$$\varepsilon = \varepsilon' + i\,\varepsilon'' \text{ with } |\varepsilon| \gg 1 \text{ (permittivity)} \quad , \quad \mu \sim \mu_0 \text{ (permeability)}$$

GOAL: **Find geometries and good scalings for**
d (period) , ε permittivity , θ (volume fraction)

$\rightsquigarrow$ Negative effective permittivity tensor $\varepsilon^{\text{eff}}(\omega)$?

$\rightsquigarrow$ Negative permeability tensors $\mu^{\text{eff}}(\omega)$?

Singular limit of 3D- Maxwell system

- The distance between inclusions d is viewed as an infinitesimal parameter η (although in practice $d \sim \frac{\text{wavelength}}{10}$)
- The relative permittivity $\varepsilon_\eta(x)$ is very large.
- The metallic inclusions have a filling ratio θ_η which may vanish in the limit process.
- The electromagnetic field (E_η, H_η) satisfies **on all** $\mathbb{R}^3$

$$\begin{cases} \operatorname{curl} E_\eta = i\omega\mu_0 H_\eta \\ \operatorname{curl} H_\eta = -i\omega\varepsilon_0\, \varepsilon_\eta\, E_\eta \end{cases} \tag{1}$$

$$(E_\eta - E^i, H_\eta - H^i) \quad \text{satisfies the O.W.C.} \tag{2}$$

where ε_η is the stiff parameter and O.W.C. means 'outgoing radiation condition at infinity' (Silver Müller)

Arrays of metallic nanorods

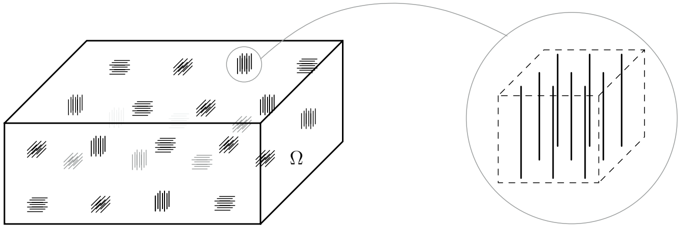


Figure: metallic fibers $||$, lenght $\sim$ period, diameter $\ll$ period

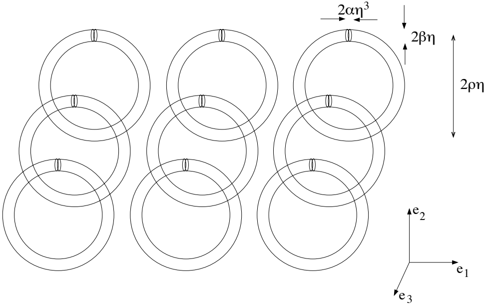
Finite obstacle in $\mathbb{R}^3$, volume fraction of fibers $\theta_\eta \ll 1$.

Each block of fibers acts as an electrostatic resonator

$\rightsquigarrow$ $\varepsilon^{\mathit{eff}}(\omega)$ negative (all symmetric tensor are realizable)

joined work with C. Bourel , to appear CICP

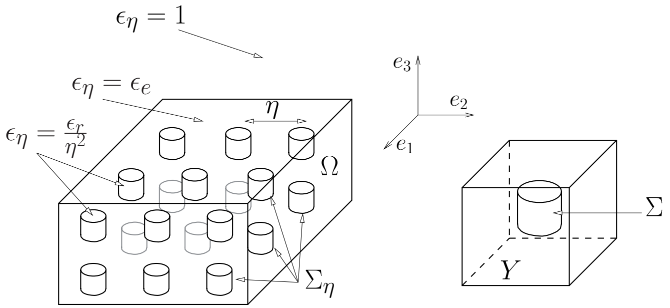
Pendry split rings structure



- The scatter $\Omega \subset \mathbb{R}^3$ contains $O(\eta^{-3})$ split-rings of size $O(\eta)$.
- $\theta_\eta = \theta$ is positive.

with Ben Schweizer (Siam MMS 2010) $\rightsquigarrow$ negative $\mu^{eff}(\omega)$

AN ALTERNATIVE TO METALLIC PENDRY SPLIT RINGS ?



- η is a small parameter (period)
- Finite domain $\Omega \subset \mathbb{R}^3$ contains $O(\eta^{-3})$ periodic inclusions of diameter $O(\eta)$ filled with high permittivity $\frac{\varepsilon_r}{\eta^2}$.
- Volume fraction (of dielectric) constant as $\eta \to 0$

Oscillations of the magnetic field

The zero order term in the expansions in

$$\begin{aligned} H_\eta(x) &= H_0(x, x/\eta) + \eta\, H_1(x, x/\eta) + \eta^2\, H_2(x, x/\eta) \\ J_\eta := \eta\varepsilon_\eta E_\eta &= J_0(x, x/\eta) + \eta\, J_1(x, x/\eta) + \eta^2\, J_2(x, x/\eta) \end{aligned}$$

saisfies a cell problem

$$\mathrm{curl}_y\ H_0 + i\omega\varepsilon_0 J_0 = 0 \quad \text{in } Y \quad , \quad \mathrm{div}_y\, H_0 = 0 \quad \text{in } Y \qquad (3)$$

$$\mathrm{curl}_y\ J_0 + i\varepsilon_r\omega\mu_0 H_0 = 0 \quad \text{in } \Sigma \quad , \quad J_0 = 0 \quad \text{in } Y \setminus \Sigma \qquad (4)$$

Observations :

• By (3), $H_0(x, \cdot)$ belongs to the Sobolev space $H^1_\sharp(Y; \mathbb{C}^3)$.

• In contrast $J_0(x, \cdot)$ (supported in Σ) which may have a tangential jump across $\partial\Sigma$.

• Exploiting (3)(4), on subset Σ: $\Delta_y H_0 + k_0^2\, \varepsilon^r H_0 = 0$

Micro-resonator problem on $Q = (0,1)^3$

$$b_0(\varphi_n, \psi) \;=\; \lambda_n \int \varphi_n \cdot \overline{\psi}\, dy \quad , \quad \forall \psi \in X_0 \;, \tag{5}$$

where b_0 and Hilbert space X_0 are given by

$$b_0(u,v) \;:=\; \int_Q (\mathrm{curl}\, u \cdot \overline{\mathrm{curl}\, v} + \mathrm{div}\, u \cdot \overline{\mathrm{div}\, v})\, dy \;,$$

$$X_0 \;=\; \left\{ u \in W^{1,2}_{\sharp}(Q;\mathbb{C}^3) \,:\, \mathrm{curl}\ u = 0 \text{ on } Q \setminus \Sigma \,,\; \oint u = 0 \right\}$$

Remark

- non zero constant functions are ruled out in previous definition.
- Contributing eigenvectors in expansion (6) are all divergence-free.

Effective permeability law

$$\mu_{ij}^{\text{eff}}(k) = \delta_{ij} + \sum_{n\in\mathbb{N}} \frac{\varepsilon_r k^2}{\lambda_n - \varepsilon_r k^2} \left(e_j . \int_Y \varphi_n \right) \left(e_i . \int_Y \varphi_n \right) \ , \qquad (6)$$

where the λ_n, φ_n's are related to a spectral problem on the unit cell satisfied by the microscopic magnetic field (Micro resonator problem)

Remarks:

- The real positive λ_n and periodic eigenfunctions φ_n depend only on the geometry.
- The validity of homogenized law (6) requires that

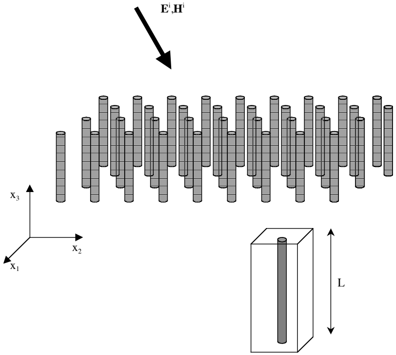
$$\mathrm{dist}(\varepsilon_r k^2, \mathcal{S}) > 0 \quad , \quad \text{where} \quad \mathcal{S} := \{\lambda_n \ , \ n \in \mathbb{N}\} \ .$$

(for instance $\Im(\varepsilon_r) > 0$)

- The effective permittivity law ε^{eff} is the same as for perfect metallic inclusions (Electric field vanishes)

TM-Polarized case

Assume the obstacle consists of e_3 parallel cylindrical rods with length $L = \infty$. The magnetic $H = u(x_1, x_2)e_3$ field is assumed to be e_3-parallel $\rightsquigarrow$ 2D-analysis



Let $u = u(x_1, x_2)\, e_3$ such that $u \in X_0$ and $\Sigma = D \times [-1/2, 1/2]$. Then

$$\operatorname{curl} u = 0 \quad \text{in } Q \setminus \Sigma \text{ and } \quad \oint u = 0 \quad \Rightarrow u(x_1, x_2) = 0 \ \text{ on } Y \setminus D$$

Thus solving solving cell system (3), (4) reduces to a 2D Laplace spectral problem

$$-\Delta \varphi_n = \lambda_n \varphi_n \quad , \quad \varphi_n = 0 \quad \text{on } \partial D$$

Comment: The micro-resonances are localized on each inclusion and no interactions between inclusions is expected is the limit as $\eta \to 0$.

- D.Felbacq, GB *Theory of mesoscopic magnetism in photonic crystals,* **Phys. Rev. Lett. 94**, *183902 (2005)* , Phys. Rev. Lett. 94, (2005)
- C.Bourel,D. Felbacq, GB: *Homogenization of the 3D Maxwell system near resonances and artificial magnetism*, C. R. Math. Acad. Sci. Paris I, Volume 347, 2009, 571–576

Aim and outline

GOAL: Study the stability of the dielectric resonator model under random perturbations

Method: Stochastic homogenization

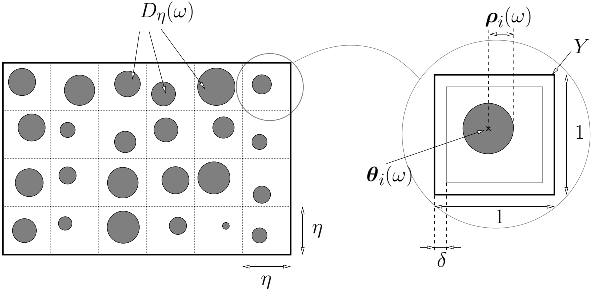
NB: we only consider the 2D case and TM polarization

Outline of the talk

- Stochastic framework
- Homogenization result
- About the proof
- Limit of validity and vanishing dissipation

Randomly perturbed geometry in finite $\mathcal{B} \subset \mathbb{R}^2$

Note: ω denotes now the random event !



$$D_\eta(\omega) := \bigcup_{i \in J_\eta(\omega)} D_\eta^i(\omega) \quad , \quad D_\eta^i(\omega) := \eta \left[i - y(\omega) + B\big(\boldsymbol{\theta}_i(\omega), \boldsymbol{\rho}_i(\omega)\big) \right] \tag{7}$$

where $J_\eta(\omega) = \{ i \in \mathbb{Z}^2 \mid \eta(i - y(\omega) + Y) \subset \mathcal{B} \}$, $y(\omega)$ random lattice translation.

Diffraction problem

The diffracting obstacle $D_\eta(\omega) \subset \mathcal{B}$ is illuminated by a monochromatic incident wave traveling in the $H||$ mode. The magnetic field takes the form $u_\eta(x_1, x_2, \omega)\, e_3$ and is characterized by

$$\begin{cases} \operatorname{div}\left(a_\eta(x,\omega)\nabla u_\eta(x,\omega)\right) + k^2 u_\eta(x,\omega) = 0 \quad x \in \mathbb{R}^2, \\ u_\eta - u^i \quad \text{verifies the outgoing Sommerfeld radiating condition,} \end{cases} \tag{8}$$

The scalar random function $a_\eta(x,\omega) \in \mathbb{C}$ represents the inverse of the permittivity at the point x and is given by

$$a_\eta(x,\omega) = 1_{\mathcal{B}\setminus D_\eta(\omega)}(x) + \sum_{i \in J_\eta(\omega)} \frac{\eta^2}{\varepsilon_i(\omega)} 1_{D^i_\eta(\omega)}(x) \tag{9}$$

Probability space

The random variables $\{m_i := (\theta_i, \rho_i, \varepsilon_i) \ : \ i \in \mathbb{N}^2\}$ are independent and identically distributed with a given probability law p on

$$M := \{(\theta, \rho, \varepsilon) \in Y \times [0, 1/2] \times \mathbb{C}^+ : \mathrm{dist}(\theta, \partial Y) \geq \rho + \delta\}$$

and the translation parameter $y(\omega)$ follows a uniform density law on Y. The probability space $(\Omega, \mathcal{B}, \mathbb{P})$ is therefore

$$\Omega := \prod_{\mathbb{Z}^2} M \times Y \ , \ \mathbb{P} := \bigotimes_{\mathbb{Z}^2} \mathrm{p}(dm) \otimes dy,$$

being $\mathcal{B}$ the Borel tribe.

Dynamical system (A.Piatnitski, S.Kozlov, V. Zhikov)

For $x \in \mathbb{R}^2$, $[x] = ([x_1], [x_2])$, we denote the element of $\mathbb{Z}^2$ made of integer parts. We introduce the group of transformations in Ω

$$T_x : \omega = \Big((m_i)_{i\in\mathbb{Z}^2}, y\Big) \longrightarrow T_x(\omega) = \Big((m_{i+[x+y]})_{i\in\mathbb{Z}^2}, x+y-[x+y]\Big)$$

One checks that T_x is a group preserving the measure $\mathbb{P}$ and ergodic (i.e. $\mathbb{P}(T_x A \Delta A) = 0 \Rightarrow \mathbb{P}(A) \in \{0,1\}$). Now let

$$\Sigma = \{\omega \in \Omega : |y - \theta_0| < \rho_0\}, \qquad \Sigma^* = \Omega \setminus \Sigma.$$

$\mathbb{P}(\Sigma)$ represents the volume fraction of inclusions

We may rewrite (9) as

$$a_\eta(x,\omega) = 1_{\mathcal{B}}(x)\left(\frac{\eta^2}{\varepsilon(T_{\frac{x}{\eta}}\omega)} 1_\Sigma\left(T_{\frac{x}{\eta}}\omega\right) + 1_{\Sigma^*}\left(T_{\frac{x}{\eta}}\omega\right)\right) + 1_{\mathbb{R}^2\setminus\mathcal{B}}(x) \qquad (10)$$

2- The homogenization result

Let $\mathcal{S}_0 = \{\lambda_n, n \in \mathbb{N}\}$ be the eigenvalues and $\{\varphi_n, n \in \mathbb{N}\}$ the normalized eigenvectors of the 2D-Dirichlet Laplace operator on the **unit disk** $(\int \varphi_n^2 = 1)$. Let $[\varphi_n] := \int \varphi_n$.

- **Frequency dependent effective permeability**

$$\mu^{\text{eff}}(k) = 1 + \sum_n \mathbb{E}\left[\frac{\varepsilon\rho^4 k^2}{\lambda_n - \varepsilon\rho^2 k^2}\right][\varphi_n]^2 \qquad (*)$$

- **Real positive effective permittivity:** $\varepsilon^{\text{eff}} = \mathbb{E}\left[\frac{1}{A(\rho)}\right]$

where, for θ, e arbitrary (e unit vector)

$$A(\rho) = \inf\left\{\int_{Y\setminus B(\theta,\rho)} |e + \nabla w|^2 \ , \ w \ \ Y\text{-periodic}\right\}.$$

(Neumann problem on perforated domain)

Stability condition and Main result

To avoid a blow-up of solutions $u_\eta(\omega, x)$, we need the folllowing

$$\mathrm{p}(\{\Im(\varepsilon) > 0\}) > 0 \quad , \quad \int_M \left(\frac{\varepsilon\rho}{\mathrm{dist}\left(\varepsilon\rho^2 k^2, \mathcal{S}_0\right)} \right)^{2+h} d\mathrm{p} < \infty \qquad (**)$$

for a suitable $h > 0$ (higher integrability)

Theorem

Under (**), for almost all event ω, the sequence u_η does converge in $L^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \mathcal{B})$ to the unique (deterministic) solution of the diffraction problem where the scatterer $\mathcal{B}$ is filled with an homogeneous material of permittivity and permeability $\varepsilon^{\mathrm{eff}}, \mu^{\mathrm{eff}}(k)$ given in (**).

The pointwise convergence holds only outside of the scatter.
Q: What happens inside ?

What happens inside ?

Recalling that $\omega = ((m_i)_{i\in\mathbb{N}}, y)$, we define the random function

$$\Lambda(\omega, k) := 1 + 1_\Sigma(\omega) \sum_{n\in\mathbb{N}} \frac{k^2\varepsilon(\omega)\boldsymbol{\rho}_0^2(\omega)[\varphi_n]}{\lambda_n - k^2\varepsilon(\omega)\boldsymbol{\rho}_0^2(\omega)}\ \varphi_n\left(\frac{y - \boldsymbol{\theta}_0(\omega)}{\boldsymbol{\rho}_0(\omega)}\right) \qquad (11)$$

As expected the field u_η oscillates inside $\mathcal{B}$. Precisely, for $\mathbb{P}$ a.a. ω

$$\lim_{\eta\to 0} \int_{\mathcal{B}} |u_\eta(x,\omega) - u(x)\,\Lambda(T_{x/\eta}(\omega), k)|^2\, dx \;=\; 0\ , \qquad (12)$$

being u the unique solution of the limit diffration problem

Comments

- **Under (**), the effective medium is dissipative**
  For every k, $\Im(\mu^{\mathrm{eff}}(k)) > 0 \;\Rightarrow\;$ well posed limit Pb

- **Our result includes the determinist case** (GB, Felbacq , PRL(2005)) Let p be a Dirac mass at some $(\theta_0, \rho_0, \varepsilon_r)$, then:
  - the probality space Ω reduces to $(\theta_0, \rho_0, \varepsilon_r) \times Y$
  - $T_{x/\eta}(\omega) \equiv x/\eta$
  - $\Lambda(\omega, k)$ becomes the periodic solution $w_k(y)$ of

$$\Delta w + k^2 w = 0 \quad , \quad w = 1 \text{ on } B(\theta_0, \rho_0)$$

  - From (12) follows the strong two-scale convergence

$$\lim_{\eta \to 0} \int_{\mathcal{B}} |u_\eta(x) - u(x)\, w_k(x/\eta)|^2 \, dx \;=\; 0 \; .$$

3- About the proof

We use a variant of the stochastic two scale convergence introduced by Bourgeat , Kozlov and Wright (1994).

NB: The realization $\tilde\omega$ is fixed (following Piatnitski)

Definition: $u_\eta(x,\tilde\omega) \rightharpoonup\rightharpoonup u_0(x,\omega,\tilde\omega)$ if for every Ψ continuous on Ω,

$$u_\eta(\cdot,\tilde\omega)\,\Psi(T_{x/\eta}(\tilde\omega) \rightharpoonup \mathbb{E}[(u_0(x,\cdot)\Psi(\cdot)]$$

Remark: By Birkhoff's Thm, for every $\tilde\omega \in \tilde\Omega$ of full measure

$$\Psi(T_{x/\eta}(\tilde\omega) \rightharpoonup \mathbb{E}(\Psi(\omega))$$

In our case the two scale limit of $u_\eta(x,\omega)$ we find is independant of $\tilde\omega$:

$$u_0(x,\omega) = u(x)(1_{\mathbb{R}^2\setminus\mathcal{B}}(x) + \Lambda(\omega)1_{\mathcal{B}}(x)) \ .$$

Ergodicity and stochastic derivative

The constancy of $u_0(x,\cdot)$ outside $\mathcal{B}$ is deduced from the ergodicity of the dynamical system $(\Omega, \mathbb{P}, T_x)$ thanks to

Lemma

Let $\mathcal{U} \subset \Omega$, $Q(\omega) := \{x : T_x\omega \in \mathcal{U}\} \subset \mathbb{R}^2$, and $f \in L^1(\mathcal{U}; \mathbb{P})$. If $f(T_x\omega) = f(\omega)$ for almost all $\omega \in \mathcal{U}$, $x \in Q(\omega)$ then f is constant on $\mathcal{U}$.

Definition The map $f \mapsto (U_x f)(\omega) = f(T_x\omega)$ defines a continuous group in $L^2(\Omega, \mathbb{P})$ with infinitesimal generators

$$D(\partial_i^s) = \left\{ f \in L^2(\Omega, \mathbb{P}) \quad : \quad \exists \lim_{t\to 0} \frac{U_{te_i} f - f}{t} \in L^2(\Omega, \mathbb{P}) \right\}$$

Accordingly we define Sobolev spaces $H^1_s(\Omega), H^2_s(\Omega)$. Then

$f \in H^1_s(\Omega) \ \Rightarrow$ for a.a.ω $\quad f(T_x\omega) \in H^1_{loc}(\mathbb{R}^2)\ ,\ \partial_i(f(T_x\omega) = (\partial_i^s f)(T_x\omega)\ .$

Identification of u_0

Lemma

The two-scale limit u_0 belongs to $L^2(\mathcal{B}, H^1_s(\Omega))$ and for a.a. $x \in \mathcal{B}$, $u_0, \cdot)$ satisfies

$$\nabla^s u_0(x,\cdot) = 0 \quad \text{in } \Omega \setminus \Sigma \quad , \quad \Delta_s u_0(x,\omega) + \varepsilon_0(\omega) k^2 u_0(x,\omega) = 0 \quad \text{in } \Sigma \ .$$

L^2 estimate

The L^2- bound for $\{u_\eta\}$ requires an estimate involving the distance of k to the resonance frequencies k_n in the rods where $k_n^2 = \frac{\lambda_n}{\varepsilon\,\rho^2}$

Lemma

Fix $\delta \in (0, 1/2)$ and let $\mathcal{S}_0 = \{\lambda_n\}$. Then there exists $c_\delta > 0$ such that, for any $\alpha \in \mathbb{C}$ and $u \in H^1(Y)$ such that $\Delta u \in L^2(B(\theta,\rho))$ where $\mathrm{dist}(\theta, \partial Y) \geq \rho + \delta$, it holds for every $\alpha \in \mathbb{C}$

$$\int_{B(\theta,\rho)} |u|^2 \;\leq\; \frac{2}{\mathrm{dist}^2\left(\alpha, \frac{\mathcal{S}_0}{\rho^2}\right)} \int_{B(\theta,\rho)} |\Delta u + \alpha u|^2 + 2\,c_\delta \left(1 + \frac{|\alpha|}{\mathrm{dist}\left(\alpha, \frac{\mathcal{S}_0}{\rho^2}\right)}\right)^2 \int_{Y\setminus B(\theta,\rho)} (|u|^2 + |\nabla u|^2)$$

We apply this to $v_\eta(x) = u_\eta(x, \tilde{\omega}) - u_0(x, T_{\frac{x}{\eta}}\tilde{\omega})$.

Lemma

Let $\{X_i\ ,\ i = 1, 2, \dots, n, \dots\}$ *be a sequence of independent and identically distributed non negative random variables in* $L^1(\Omega, \mathcal{A}, \mathcal{P})$. *Define*

$$Z_n := \sup\{X_i, 1 \leq i \leq n\}$$

Then we have

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}(Z_n) = 0 \quad , \quad \frac{Z_n}{n} \xrightarrow{a.s.} 0$$

.

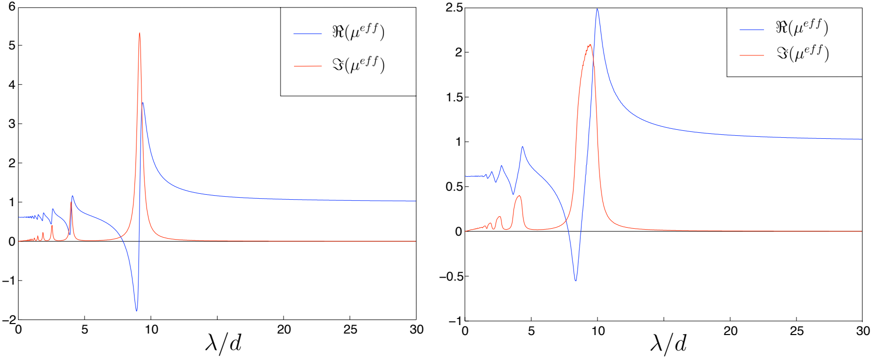
We apply the Lemma above to show that

$$b_\eta \ := \ \sup_{j\in J_\eta} \frac{\eta^2}{\mathrm{dist}(\varepsilon_j \rho_j^2 k^2, \mathcal{S}_0)} \quad \xrightarrow{a.s.} 0 \ .$$

4- Limit of validity of the model

- Random fluctuations on the radius or permittivity should reduce the amplitude of large terms in the series expansion (*)
- Are negative permittivities stable ?
- Can we start with permittivity laws on the real axis ? (condition (**))

Influence of the law p



On the left, the radius of inclusions are fixed to 0.35.

On the right radius follows an uniform law between 0.32 and 0.38. In both the law of permittivity is a Dirac mass in $100 + 5i$.

Larger fluctuations

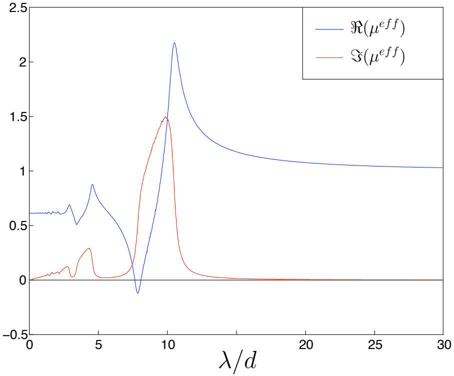


Figure: The radius follows an uniform law between 0.3 and 0.4. The permittivity law is a Dirac mass in $100 + 5i$.

Small vanishing dissipation

Q: What happens if ε is randomly distributed on the reals, for instance

$$p(\theta, \rho, a, b) := \delta(\theta - \theta_0, \rho - \rho_0) \otimes g(a)\, da \otimes \delta(b) \quad ??$$

It seems natural to approximate $\tilde{p}$ introducing some small loss

$$\mathrm{p}_\delta(\theta, \rho, a + ib) = \delta(\theta - \theta_0, \rho - \rho_0) \otimes g(a)\, da \otimes \frac{1}{\delta}\zeta(\frac{b}{\delta})$$

being ζ a probability on $]0, +\infty[$ compatible with (**).
Owing to expansion (*), we expect the effective permeability to be the limit $\delta \to 0$ of

$$\mu_\delta^{\mathrm{eff}}(k) := 1 + \sum_n \int_M \frac{\varepsilon \rho^4}{\nu_n - \rho^2 \varepsilon}\, p_\delta(d\theta\, d\rho\, d\varepsilon), \qquad \nu_n := \frac{\lambda_n}{k^2}.$$

The vanishing loss limit

Assume that the density $g(a)$ is smooth (Lipschitz) in the vicinity of the λ_n's. Then, as $\delta \to 0$:

$$\mu_\delta^{\text{eff}}(k) \to \mu^{\text{eff}}(k) = 1 + \sum_n [\varphi_n]^2 \, I_n(k) \ ,$$

where (PV refers to the Cauchy principal value)

$$\Re(I_n(k)) = \mathrm{PV}\left(\int \frac{a\rho_0^4}{\lambda_n - a\rho_0^2} g(a)\, da\right) \ ,$$

$$\Im(I_n(k)) = \frac{\pi \lambda_n}{k^2}\, g\left(\frac{\lambda_n}{k_0^2 \rho_0^2}\right) \ .$$

We take ζ to be Dirac masses at $b = 1$ and $b = 5$
$g(a)$ is smooth supported in $[90, 110]$ and $\rho_0 = 0.35$.
The black line repesents the limit as $\delta \to 0$ (ζ Dirac at $b = 0$).

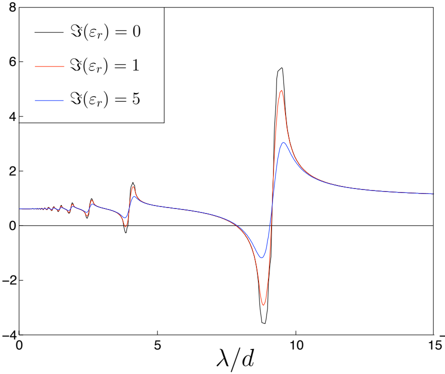


Figure: Dependence of $\Re(\mu^{eff})$.

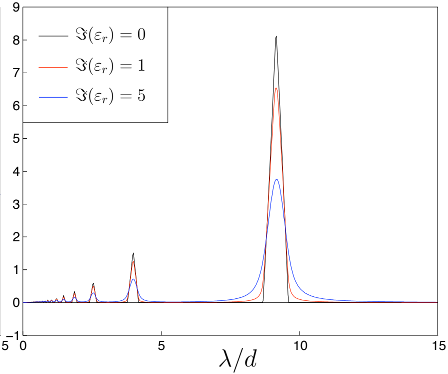


Figure: Dependence of $\Im(\mu^{eff})$

5- Conlusions and open issues

• **Conclusion:** The limit medium has **positive loss** whenever the density law for permittivity $\varepsilon(\omega)$ is positive close to the resonance frequencies.
The classical homogenization fails when starting with lossless dielectric
Due to asymptotic analysis in harmonic regime ??

• **Open problems:**
- Full 3D case ? (random resonators are coupled to each other)
- How to manage more general random perturbations ?