Asymptotic stability of dissipative systems with on/off damping

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Stabilization & intermittent control

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$$\dot{x} = Ax + Bu$$

and a feedback $u = Kx$ stabilizing at 0.

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$$\dot{x} = Ax + \alpha BKx.$$

α may model

- Unfaithful transmission of the control law ($\alpha(t) \in \{0, 1\}$)
- Cyclic parameter affecting the control efficiency
- Allocation of control resources

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PB: under which conditions is the switched system asymptotically stable at 0?

A nonlinear multiD-signal example

Model studied by Astolfi and Lovera [2004]: attitude control of a spacecraft by means of magnetic actuators

$$\begin{aligned} \dot{R} &= RS(\omega) \\ J\dot{\omega} &= J\omega \times \omega + u(t) \times b(t) \end{aligned}$$

(R attitude, ω angular velocity, J inertia matrix, b Earth's magnetic field) by applying a feedback transformation of the control which changes the second equation in

$$J\dot{\omega} = J\omega \times \omega - S(b(t))S(b(t))^T v(t)$$

The stability is obtained from the inequality

$$\int_t^{t+T} S(b(\tau))S(b(\tau))^T d\tau \geq \mu I_3$$

due to the cyclical rotation of the satellite around the earth

Rationale of the talk

- Finite-dimensional behavior
- New phenomena for PDE evolution
- Exponential/strong/weak stability for PDE evolution

Finite dimension: stabilizable linear control system

A linear control system

$$\dot{x} = Ax + Bu, \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}^m \qquad (A, B)$$

is stabilizable at the origin if there exists a feedback $u = Kx$ such that $A + BK$ is Hurwitz.

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If (A, B) is controllable, than the system can be stabilized with an arbitrary rate of convergence, ie, for every $\lambda > 0$ the exist K and $C > 0$ such that

$$\|x(t)\| \leq C\|x(0)\|e^{-\lambda t}$$

for every trajectory x of $\dot{x} = Ax + BKx$.

Persistent excitation

Definition

Let $0 < \mu \leq T$. A (T, μ)-signal is a function $\alpha \in L^\infty(\mathbf{R}, [0, 1])$ satisfying

$$\int_t^{t+T} \alpha(s) ds \geq \mu \,, \quad \forall t \in \mathbf{R} \,.$$

Definition ((T, μ)-stabilizer)

Let $0 < \mu \leq T$. The feedback $u = Kx$ is said to be a (T, μ)-stabilizer if there exist $C, \gamma > 0$ such that, for every (T, μ)-signal α, and every $x_0 \in \mathbf{R}^n$, the solution x of $\dot{x} = (A + \alpha BK)x$, $x(0) = x_0$, satisfies

$$\|x(t)\| \leq Ce^{-\gamma t}\|x_0\| \,, \quad \forall t \geq 0 \,.$$

A neutrally stable

Lemma

Let (A, B) be stabilizable and A neutrally stable ($Re(\sigma(A)) \leq 0$ and the eigenvalues with real-part equal to zero have trivial Jordan blocks). Then there exists K which is a (T, μ)-stabilizer for every $0 < \mu \leq T$.

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Without loss of generality A skew-symmetric and $K = -B^T$ (independent on (T, μ)). $V(t) = \|x(t)\|^2$ Lyapunov function.

$$\dot{V} = -2\alpha(t)\|B^T x\|^2 \,.$$

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The α-uniform exponential decay of V follows by compactness: if

$$\int_{t_0}^{t_0+T} \alpha_j(t)\|B^T x_j(t)\|^2 dt \to 0, \quad \|x_j(t_0)\| = 1,$$

then, $\alpha_j \overset{*}{\rightharpoonup} \alpha_\infty$ and $x_j \to x_\infty$ in $C^0([t_0, t_0 + T])$. Hence, $0 \equiv \alpha_\infty(t)\|B^T x_\infty(t)\|^2 \equiv \alpha_\infty(t)\|B^T e^{(t-t_0)A} x_\infty(t_0)\|^2$.

Contradiction

Spectra with non-positive real part

Theorem (Y. Chitour, M. S., SICON, 2010)

Let $(A, b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$ *be a controllable pair and assume that* $Re(\sigma(A)) \leq 0$*. Then, for every* $0 < \mu \leq T$ *there exists a* (T, μ)*-stabilizer.*

The uncontrolled system $\dot{x} = Ax$ can have trajectories such that $\|x(t)\| \to \infty$ as $t \to +\infty$.

The proof is based on a compactness argument and a time-contraction procedure, transforming asymptotically the integral constraint in a pointwise one.

On the maximal rate of convergence

Proposition (Y. Chitour, M. S., SICON, 2010)

There exists $\rho_* \in (0,1)$ *such that for every controllable pair* $(A,b) \in M_2(\mathbf{R}) \times \mathbf{R}^2$ *and every* $T > 0$, *if* $\mu/T < \rho_*$ *then the maximal rate of exponential convergence is finite.*

In particular, there exist controllable pairs (A,b) that are not (T,μ)-stabilizable for some $T > \mu > 0$.

$A = J_2 + \lambda \mathrm{Id}_2$, λ large, $T/\mu < \rho_*$

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Proposition (Y. Chitour, M. S., SICON, 2010)

There exists $\rho^* \in (0,1)$ (only depending on n) such that for every controllable pair $(A,b) \in M_n(\mathbf{R}) \times \mathbf{R}^n$ and every $T > 0$, the system $\dot{x} = Ax + \alpha bu$ can be (T,μ)-stabilized with an arbitrarily large rate of convergence if $\mu/T > \rho^*$.

Persistent excitation in the infinite-dimensional case

Let us go back to:

Lemma

Let A be skew-symmetric and (A, B) stabilizable. Then $K = -B^T$ is a (T, μ)-stabilizer for every $0 < \mu \leq T$.

Such result do not generalize to infinite-dimensional systems.

Persistent excitation in the infinite-dimensional case

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Consider the wave equation on a string of finite length L, fixed at both ends and damped on a subset $(a, b) \subsetneq (0, L)$,

$$v_{tt}(t,x) = v_{xx}(t,x) - \alpha(t)1_{(a,b)}(x)v_t(t,x)$$
$$v(t,0) = v(t,L) = 0$$

Given $T \geq \mu > 0$, it suffices to take a traveling wave with sufficiently small support in order to design α that satisfies the persistent excitation condition and switches off the actuator when the wave passes through (a, b).

A positive stability result

[Martinez-Vancostenoble, 2002] and [Haraux-Martinez-Vancostenoble, 2005] studied (a class of second-order systems generalizing) the damped wave equation

$$v_{tt}(t,x) = v_{xx}(t,x) - \alpha(t)v_t(t,x)$$
$$v(t,0) = v(t,L) = 0.$$

They proved that if

$$\{t \mid \alpha(t) = 1\} = \cup_{n\in\mathbf{N}}(a_n, b_n)$$

with $b_n \leq a_{n+1}$ and

$$\sum_{n\in\mathbf{N}} (b_n - a_n)^3 = \infty$$

then the solution converges exponentially to zero in $H_0^1(0,L) \times L^2(0,L)$.

Infinite-dimensional framework

H Hilbert space

$$\begin{cases} \dot{z}(t) = Az(t) + \alpha(t)Bu(t) \\ u(t) = -B^*z(t) \\ z(0) = z_0 \end{cases}$$

with

- $A : H \supset D(A) \to H$ a (possibly unbounded) linear operator generating a strongly continuous contraction semigroup $\{e^{tA}\}_{t\geq 0}$
- $B : U \to H$ bounded linear operator
- $\alpha : [0,\infty) \to [0,1]$ measurable
- mild solutions: $z(t) = e^{tA}z_0 - \int_0^t e^{(t-s)A}\alpha(s)BB^*z(s)\,ds$

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Let $V(z) = \frac{1}{2}\|z\|_H^2$. Then

$$V(z(t+\tau)) - V(z(t)) \leq -\int_t^{t+\tau} \alpha(s)\|B^*z(s)\|_H^2\,ds \quad \text{for all } \tau \geq 0$$

Exponential stability

Theorem (F. Hante, M. S., M. Tucsnak)

Let $\vartheta, c > 0$ be such that

$$\int_0^{\vartheta} \alpha(t) \|B^* e^{tA} z_0\|_H^2 \, dt \geq c \|z_0\|_H^2, \quad \textit{for each } (T,\mu)\textit{-signal } \alpha(\cdot).$$

Then $-B^*$ is a (T,μ)-stabilizer, i.e., there exist $C, \gamma > 0$ such that all solutions $z(\cdot)$ of $\dot{z} = Az - \alpha BB^* z$ satisfy

$$\|z(t)\|_H \leq C e^{-\gamma t} \|z(0)\|_H$$

uniformly with respect to the (T,μ)-signal $\alpha(\cdot)$.

Stability is guaranteed by a generalized observability inequality. Inequalities of this type were obtained for the heat equation studying bang-bang properties for optimal control [Mizel-Seidman,1997],[Fattorini,2005],[Wang,2008],[Phung,2011]

Idea of the proof

- for a trajectory $z(\cdot)$ of $\dot{z} = Az - \alpha BB^* z$ and (T, μ)-signal $\alpha(\cdot)$,

$$\int_0^{\vartheta} \alpha(t) \|B^* z(t)\|_H^2 \, dt \geq c' \int_0^{\vartheta} \alpha(t) \|B^* e^{tA} z(0)\|_H^2 \, dt$$

- we conclude by standard considerations on the real-valued map $t \mapsto V(z(t)) = \|z(t)\|^2/2$

Example: wave equation

Ω bounded domain of $\mathbf{R}^N$

$$\begin{aligned} v_{tt}(t,x) &= \Delta v(t,x) - \alpha(t) d(x)^2 v_t(t,x), && (t,x) \in (0,\infty)\times\Omega, \\ v(0,x) &= y_0(x), && x \in \Omega, \\ v_t(0,x) &= y_1(x), && x\in\Omega, \\ v(t,x) &= 0, && (t,x)\in(0,\infty)\times\partial\Omega, \end{aligned}$$

with $d \in L^\infty(\Omega)$, $|d(x)| \geq d_0 > 0$.

The generalized observability inequality is satisfied with $\vartheta = T$, $H = H_0^1(\Omega)\times L^2(\Omega)$ and $\|(z_1,z_2)\| = \|\nabla z_1\|_{L^2(\Omega)} + \|z_2\|_{L^2(\Omega)}$.

$$\int_0^T \alpha(t)\|B^* z(t)\|^2\,dt = \int_0^T\int_\Omega \alpha(t) d(x)^2 |v_t(x,t)|^2 dx dt$$

Weak stability

Theorem (F. Hante, M. S., M. Tucsnak)

Let $\vartheta > 0$ *be such that*

$$\int_0^{\vartheta} \alpha(s) \|B^* e^{sA} z_0\|_H^2 \, ds = 0 \quad \Rightarrow \quad z_0 = 0$$

for every (T, μ)*-signal* $\alpha(\cdot)$*.*
Then each solution $t \mapsto z(t)$ *of* $\dot{z} = Az - \alpha BB^* z$ *converges weakly to* 0 *in* H *as* $t \to \infty$ *for any initial data* $z_0 \in H$ *and any* (T, μ)*-signal* $\alpha(\cdot)$*.*

The sufficient condition for weak stability can be seen as a generalized unique continuation principle.

Idea of the proof

Let $z(t_n) \rightharpoonup z_{\infty,0}$ and consider the translations

$$z_n(t) = z(t + t_n; z_0) \quad \alpha_n(t) = \alpha(t + t_n)$$

We have the energy estimates

$$V(z_n(t)) - V(z(t_n; z_0)) \leq -\int_0^t \alpha_n(s) \|B^* z_n(s)\|_H^2 \, ds \quad \text{for all } t \geq 0.$$

One proves by compactness that

$$z_n(t) \rightharpoonup z_\infty(t) \quad \text{for all } t \in [0, \vartheta]$$

where $z_\infty(\cdot)$ is the solution of the undamped equation

$$\begin{cases} \dot{z}(t) = Az(t) \\ z(0) = z_{\infty,0} \end{cases}$$

and $\int_0^t \alpha_\infty(s) \|B^* z_\infty(s)\|_H^2 \, ds = 0$. Then $z_{\infty,0} = 0$.

Example: Schrödinger equation

Ω bounded domain of $\mathbf{R}^N$.

$$\begin{aligned} y_t(t,x) &= i\Delta y(t,x) - \alpha(t)1_\omega(x)y(t,x), && (t,x)\in(0,\infty)\times\Omega,\\ y(t,x) &= 0, && t\in(0,\infty)\times\partial\Omega,\\ y(0,x) &= y_0(x), && t\in\Omega, \end{aligned}$$

with $\alpha(\cdot)$ a (T,μ)-signal and $\omega\subset\Omega$ open nonempty.

Take $\vartheta > T-\mu$ and then

$$(s,x)\mapsto (e^{sA}z_0)(x)\equiv 0 \quad \text{on } \Xi\times\omega$$

with $\Xi\subset(0,\theta)$ and $\text{meas}(\Xi)>0$. By Privalov's theorem

$$(s,x)\mapsto (e^{sA}z_0)(x)\equiv 0 \quad \text{on } (0,\theta)\times\omega$$

and we can conclude by Holmgren's uniqueness theorem.

Strong stability

Theorem (F. Hante, M. S., M. Tucsnak)

Let $\rho, T_0 > 0$ *and* $c : (0,\infty) \to (0,\infty)$ *be a continuous function satisfying for all* $T \in (0, T_0]$ *and* $\tilde{\alpha} \in L^\infty([0,T],[0,1])$

$$\int_0^T \tilde{\alpha}(t)\,dt \geq \rho T \Rightarrow \int_0^T \tilde{\alpha}(t)\|B^* e^{tA} z_0\|_H^2\,dt \geq c(T)\|z_0\|_H^2, \quad \forall z_0.$$

Let (a_n, b_n), $n \in \mathbf{N}$, *a sequence of disjoint intervals in* $[0,\infty)$ *and* $\alpha \in L^\infty([0,\infty),[0,1])$ *be such that* $\int_{a_n}^{b_n} \alpha(t)\,dt \geq \rho(b_n - a_n)$ *and* $\sum_{n=1}^\infty c(b_n - a_n) = \infty$. *Then each solution of* $\dot{z} = Az - \alpha B B^* z$ *satisfies* $\|z(t)\|_H \to 0$ *as* $t \to \infty$.

Stability results are then obtained by estimating the asymptotic behavior of $c(T)$ for T small.

Special case: $\rho = 1 \to \alpha \equiv 1$ on each (a_n, b_n)

Examples

- $\rho = 1$, 1D Schrödinger with internal control

$$c(T) \sim T^{-\frac{1}{2}} e^{-\frac{\pi}{2T}}$$ [Tenenbaum–Tucsnak, 2007]

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- Wave damped everywhere: $c(T) \sim T^3$ (same behavior as in case $\rho = 1$ studied in [Haraux-Martinez-Vancostenoble]). Then conditions in [H-M-V] not necessary (question raised in [Fragnelli–Mugnai, 2008, 2010]).

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- Finite-dimensional control systems

Proposition

$H = \mathbf{R}^n$, A skew-symmetric, (A, B) controllable, r minimal such that

$$\operatorname{rank}[B, AB, \dots, A^r B] = n.$$

Then for every $\rho > 0$ there exists $\kappa > 0$ such that, for every $T \in (0, 1]$ and every $\alpha \in L^\infty([0, T], [0, 1])$, if $\int_0^T \alpha(s)ds \geq \rho T$ then $\int_0^T \alpha(s)\|B^\top e^{sA} z_0\|^2 ds \geq \kappa T^{2r+1}\|z_0\|^2$.

$c(T) \sim T^{2r+1}$ as proved in [Seidman, 1988] for $\rho = 1$.

Open problems

- Semilinear extensions
- Unbounded damping operators
- Strong stability (generalized observability inequality) for Schrödinger?
- Relax neutral stability (nontrivial Jordan blocs)
- Generalized inequalities/uniqueness principles for other systems (e.g. wave equation damped almost everywhere not uniformly)

Intermittent damping for a star-shaped networks of strings

N strings of length $L_1, \ldots, L_N > 0$ joined at a common point.
A damping actuator at the other end of each string.

$$
\begin{aligned}
v^{(i)}_{tt}(t,x) &= v^{(i)}_{xx}(t,x)\\
v^{(i)}(t,0) &= v^{(j)}(t,0)\\
0 &= v^{(1)}_x(t,0) + v^{(2)}_x(t,0) + \cdots + v^{(N)}_x(t,0)\\
v^{(i)}_x(t,L_i) &= -\alpha_i(t)\kappa_i v^{(i)}_t(t,L_i)\\
v^{(i)}(0,x) &= y^{(i)}_0(x),\ v^{(i)}_t(0,x) = y^{(i)}_1(x),\ x \in (0,L_i)
\end{aligned}
$$

for $i, j \in \{1, ..., N\}$, $\kappa_i > 0$, $t > 0$.

PB: which stability if $\sum_{i=1}^{N} \alpha_i \geq \hat{N} > 0$?

Finite dimensional case

Lemma

Let A be neutrally stable and assume that, for every $1 \leq i_1 < \cdots < i_{\hat{N}} \leq N$,

$$\dot{x} = Ax + \sum_{j=1}^{\hat{N}} u_{i_j} b_{i_j}$$

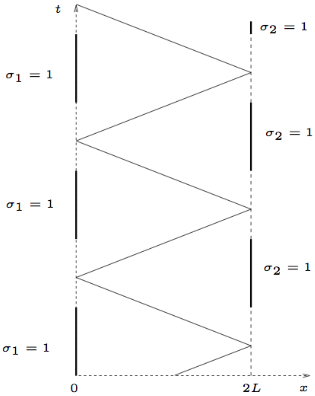
is stabilizable. Then, taking $K = -B^T$,

$$\dot{x} = Ax - \sum_{i=1}^{N} \alpha_i(t) b_i b_i^T x, \quad \alpha_i \in \{0, 1\}$$

is globally uniformly exponentially stable with respect to α such that $\sum_{i=1}^{N} \alpha_i(t) \geq \hat{N}$.

The finite-dimensional result do not extend to infinite dimension

In particular, taking the string (i.e., the string network with $N = 2$):



The question is open for $N \geq 3$ and $\hat{N} = N - 1$.

First stabilization result: forward condition

Theorem (M. Gugat, M. S., NHM, 2010)

If $N \geq 3$ *and*

$$\sum_{i=1}^{N} \alpha_i \left(t + L_i\right) \geq N - 1 \quad \text{for almost every } \; t \qquad \text{(FwdC)}$$

then $E(t) = \frac{1}{2} \sum_{i=1}^{N} \int_0^{L_i} \left(v_t^{(i)}(t,x)^2 + v_x^{(i)}(t,x)^2 \right) dx$ *satisfies*

$$E(t) \leq C_1 \exp\left(-C_2 t\right) E(0),$$

for some $C_1, C_2 > 0$ *independent of* $(y_0^{(i)}, y_1^{(i)})_{i=1}^{N}$ *and of* α *verifying* (FwdC).

Second stabilization result: backward condition

Theorem (M. Gugat, M. S., NHM, 2010)

Let $N \geq 3$ *and* $\lambda = \max\{L_1, \ldots, L_N\}$ *and* $f = \max\left\{\frac{2}{N},\ \frac{N-2}{N}\right\}$.
If

$$F := \sqrt{N} \max_{i=1,\ldots,N} \frac{|\kappa_i - 1|}{\kappa_i + 1} + f < 1$$

and

$$\sum_{i=1}^{N} \alpha_i \left(t - L_i\right) \geq N - 1 \quad \text{for almost every } \; t \qquad \text{(BwdC)}$$

then for almost every $t > 0$

$$\|v_x(t,\cdot)\|_\infty + \|v_t(t,\cdot)\|_\infty \leq C F^{\frac{t}{2\lambda}}$$

with C *independent of* α *verifying* (BwdC).

Open problems

- Conditions of the type $\sum_{i=1}^{N} \sigma_i(t) \geq N-1$ (without time shifts)
- More general networks (possibly with internal, localized damping)
- Strings of rationally independent length
- Persistent excitation conditions