Open session on networks

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Benasque, September, 2011

Outline

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Schrodinger equation on trees (or network trees)

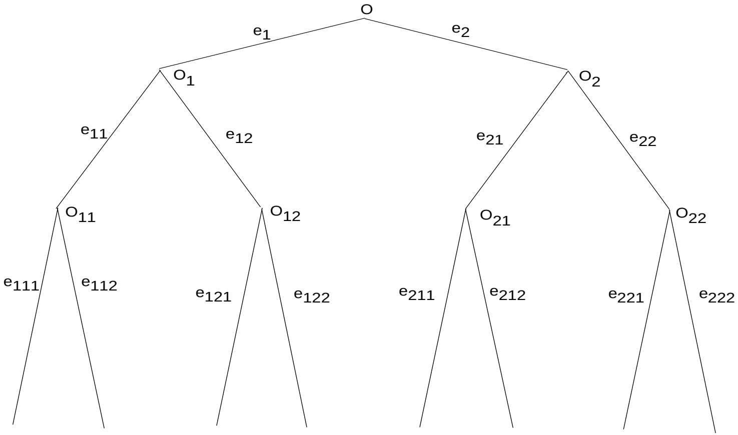


Figure: A tree with the third generation formed by infinite edges

$$\begin{cases} i\mathbf{u}_t(t,x)+\Delta_\Gamma \mathbf{u}(t,x)=0, & x\in\Gamma, t\neq 0,\\ \mathbf{u}(0)=\mathbf{u}_0, & x\in\Gamma. \end{cases} \tag{1}$$

$$\begin{cases} iu_t^{\overline{\alpha}}(t,x)+u_{xx}^{\overline{\alpha}}(t,x)=0, & x\in(0,1), 1\leq|\overline{\alpha}|\leq n,\\ iu_t^{\overline{\alpha}}(t,x)+u_{xx}^{\overline{\alpha}}(t,x)=0, & x\in(0,\infty), |\overline{\alpha}|=n+1,\\ \begin{cases} u^{\overline{\alpha}}(t,1)=u^{\overline{\alpha\beta}}(t,0), & \beta\in\{1,2\}, 1\leq|\overline{\alpha}|\leq n,\\ u^1(0,t)=u^2(0,t), \end{cases} \\ \begin{cases} u_x^{\overline{\alpha}}(t,1)=\sum_{\beta=1}^{2} u_x^{\overline{\alpha\beta}}(t,0), & 1\leq|\overline{\alpha}|\leq n,\\ u_x^1(0,t)+u_x^2(0,t)=0, \end{cases} \\ u^{\overline{\alpha}}(0,x)=u_0^{\overline{\alpha}}(x). \end{cases} \tag{2}$$

$$\begin{cases} i\mathbf{u}_t(t,x) + \Delta_\Gamma \mathbf{u}(t,x) = 0, & x \in \Gamma, t \neq 0, \\ \mathbf{u}(0) = \mathbf{u}_0, & x \in \Gamma. \end{cases} \tag{1}$$

$$\begin{cases} iu_t^{\overline{\alpha}}(t,x) + u_{xx}^{\overline{\alpha}}(t,x) = 0, & x \in (0,1), 1 \leq |\overline{\alpha}| \leq n, \\ iu_t^{\overline{\alpha}}(t,x) + u_{xx}^{\overline{\alpha}}(t,x) = 0, & x \in (0,\infty), |\overline{\alpha}| = n+1, \\ \begin{cases} u^{\overline{\alpha}}(t,1) = u^{\overline{\alpha\beta}}(t,0), & \beta \in \{1,2\}, 1 \leq |\overline{\alpha}| \leq n, \\ u^1(0,t) = u^2(0,t), \end{cases} \\ \begin{cases} u_x^{\overline{\alpha}}(t,1) = \displaystyle\sum_{\beta=1}^{2} u_x^{\overline{\alpha\beta}}(t,0), & 1 \leq |\overline{\alpha}| \leq n, \\ u_x^1(0,t) + u_x^2(0,t) = 0, \end{cases} \\ u^{\overline{\alpha}}(0,x) = u_0^{\overline{\alpha}}(x). \end{cases} \tag{2}$$

Main goal:

$$\|\mathbf{u}(t,x)\|_{L^\infty} \le t^{-1/2}\|\mathbf{u}_0\|_{L^1}$$

More general coupling conditions

$$\Delta(A, B)$$

• it acts on functions on the graph Γ as the second derivative $\frac{d^2}{dx^2}$
• its domain consists in all f that belong to $H^2(e)$ and satisfy boundary condition at the vertices:

$$A(v)\mathbf{f}(v) + B(v)\mathbf{f}'(v) = 0 \quad \text{for each vertex } v. \tag{3}$$

For each vertex v of the tree we assume that matrices $A(v)$ and $B(v)$ are of size $d(v)$ and satisfy the following two conditions

1. the joint matrix $(A(v), B(v))$ has maximal rank, i.e. $d(v)$,
2. $A(v)B(v)^T = B(v)A(v)^T$.

Under those assumptions it has been proved that the considered operator, denoted by $\Delta(A, B)$, is self-adjoint.

V. Kostrykin and R. Schrader.
Kirchhoff's rule for quantum wires.
J. Phys. A, 32(4):595–630, 1999.

V. Kostrykin and R. Schrader.
Laplacians on metric graphs: eigenvalues, resolvents and semigroups.
In *Quantum graphs and their applications*, volume 415 of *Contemp. Math.*, pages 201–225. Amer. Math. Soc., Providence, RI, 2006.

The Kirchhoff coupling corresponds to the matrices

$$A(v)=\begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0\\ 0 & 1 & -1 & \dots & 0 & 0\\ 0 & 0 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \vdots & 1 & -1\\ 0 & 0 & 0 & \vdots & 0 & 0 \end{pmatrix},\ B(v)=\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 0 & 0\\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

Open problem

• Under which assumptions on A and B we can guarantee a dispersion property?

Outline

We recall the following result of H. Takaoka and N. Tzvetkov (JFA 2001)

Theorem

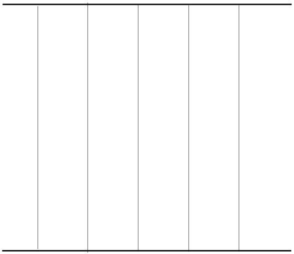
Let $U(t) = \exp(-it(\partial_x^2 + \partial_y^2))$, $x \in \mathbb{R}$, $y \in \mathbb{T}$. *Then*

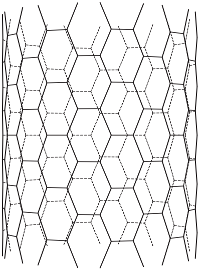
$$\|U(t)\varphi\|_{L^4(t\in I, x\in\mathbb{R}, y\in\mathbb{T})} \leq C(I)\|\varphi\|_{L^2(\mathbb{R}\times\mathbb{T})}.$$

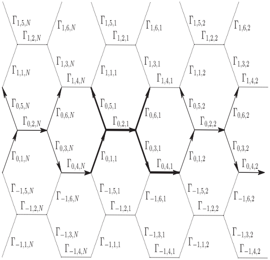
It implies some results for nonlinear models on cylinders.

Exotic structures

Question: it is possible to obtain similar results on infinite X periodic structures?







Outline

Discrete models

$$\begin{cases} i\mathbf{u}_t(t,x) + \Delta_{d,\Gamma}\mathbf{u}(t,x) = 0, & x \in \Gamma, t \neq 0, \\ \mathbf{u}(0) = \mathbf{u}_0, & x \in \Gamma. \end{cases} \tag{4}$$

The basic idea of making averages reduces the problem to some discrete equations on $\mathbb{Z}$ with "non-constant coefficients".

DLSE with discontinuous coefficients

$$\begin{cases} iu_t(j) + b_1^{-2}(\Delta_d u)(j) = 0 & j \leq -1, \\ iv_t(j) + b_2^{-2}(\Delta_d v)(j) = 0 & j \geq 1, \\ u(t,0) = v(t,0), & t > 0, \\ b_1^{-2}(u(t,-1) - u(t,0)) = b_2^{-2}(v(t,0) - v(t,1)), & t > 0 \\ u(0,j) = \varphi(j), & j \leq -1, \\ v(0,j) = \varphi(j), & j \geq 1. \end{cases}$$

D. Stan, L.I., JFAA 2011:

$$\|(u,v)(t)\|_\infty \leq (1+|t|)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z}^*)}$$

Matrix formulation

$U=(u(j))_{j\neq 0}$ satisfies $iU_t+AU=0$ where A is given by

$$\begin{pmatrix}
\dots & \dots & \dots & 0 & 0 & 0 & 0 & 0\\
0 & b_1^{-2} & -2b_1^{-2} & b_1^{-2} & 0 & 0 & 0 & 0\\
0 & 0 & b_1^{-2} & -b_1^{-2}-\frac{1}{b_1^2+b_2^2} & \frac{1}{b_1^2+b_2^2} & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{b_1^2+b_2^2} & -\frac{1}{b_1^2+b_2^2}-b_2^{-2} & b_2^{-2} & 0 & 0\\
0 & 0 & 0 & 0 & b_2^{-2} & -2b_2^{-2} & b_2^{-2} & 0\\
0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots
\end{pmatrix}.$$

No chance to use Fourier transform

An easier problem: $b_1 = b_2 = 1$

$$\begin{cases} iU_t + AU = 0, \quad t > 0, \\ U(0) = \varphi, \end{cases} \tag{5}$$

where the operator A is given by

$$A = \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}.$$

Open Problem

How we can obtain the $l^1 - l^\infty$ property directly from the properties of the operator A?

Remarks: A is not a diagonal operator, so we cannot use the Fourier analysis to obtain a symbol for A and to use oscillatory integrals

A can be decomposed as $A = \Delta_d + B$ where

$$\Delta_d = \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}$$

and

$$B = \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}.$$

How we can use the dispersive properties of $e^{it\Delta_d}$ and some properties of B in order to prove the $l^1 - l^\infty$ estimate for $U(t) = e^{it(\Delta_d + B)}$?

Some Open Problems

I. Give sufficient conditions for a symmetric matrix A with few diagonals such that for the equation $iU_t + AU = 0$ we can prove similar decay properties, even with other type of decay: $t^{-1/4}$, etc..
II. Coupling more than two equations.

Outline

Wave propagation in multi-layer structures

Consider the wave equation $u_{tt} - (\sigma(x)u_x)_x = 0$ and prove

$$\int_{\mathbb{R}} |u(t,x)|dt \leq C(\|\sigma\|_{BV})\|(u(0,\cdot),0)\|_{L^1(\mathbb{R})\times L^1(\mathbb{R})}$$

or

$$\int_{\mathbb{R}} |u(t,x)|dx \leq C(\|\sigma\|_{BV})\|(u(0,\cdot),0)\|_{L^1(\mathbb{R})\times L^1(\mathbb{R})}$$

L. Demanet and G. Peyre, Compressive Wave Computation, Foundations of Computational Mathematics, 2011 proved that the second one holds assuming that

- $\log \sigma \in BV([0,1])$
- $Var(\log \sigma) < 1$

the first estimate

- Reflexion coefficients $c_j = \frac{\sigma_{j+1}-\sigma_j}{\sigma_{j+1}+\sigma_j}$
- Define sequence $R_j(\omega)$ as follows $R_0 \equiv 0$,

$$R_{j+1}(\omega) = \frac{c_{j+1} + R_j(\omega)}{1 + c_{j+1}R_j(\omega)} e^{i\omega}$$

- For each j write

$$R_j(\omega) = \sum_{k\geq 1} a_{jk} e^{ik\omega}$$

  and define

$$\|R_j\| = \sum_{k\geq 1} |a_{jk}|.$$

- Estimate $\|R_j\|$ in a useful way in terms of $\{c_j\}$
- **Our result**: for $\sum |c_k|$ small, $\|R_j\| \leq C(\sum_{k=1}^{j} |c_k|)$ for all $j \geq 1$