Inverse problem for the heat equation and the Schrödinger equation on a tree

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Heat equation on a network

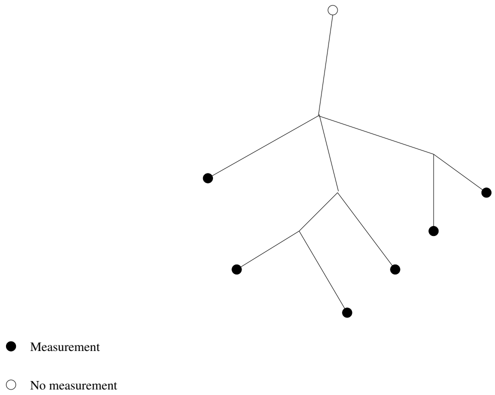
Consider the heat equation on a network Γ

$$\begin{cases} \mathbf{u}_t - \Delta_\Gamma \mathbf{u} + \mathbf{p}(x)\mathbf{u} = 0, & \text{in } \Gamma \times (0,T), \\ \mathbf{u} = \mathbf{h}, & \text{on } \partial\Gamma \times (0,T), \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0, & \text{in } \Gamma, \end{cases}$$

$\mathbf{u} = \{u^j\}$, $\mathbf{p} = \{p^j\}$, etc.

1. At the **internal** nodes N : **continuity** conditions ($u^i = u^j$) and **Kirchhoff's law** ($\sum_j u_x^j(N) = 0$)
2. **Inverse problem**: estimation of $\mathbf{p}$ from Neumann boundary measurements at the external nodes
3. Same framework as in **Julie**'s presentation, but for heat.

Tree with observation on all but one external nodes



Stability estimate

Thm: Under some technical (but classical) assumptions

$$||\mathbf{p}-\mathbf{q}||_{L^2(\Gamma)}$$
$$\leq C\left(||[\mathbf{u}(\mathbf{p})-\mathbf{u}(\mathbf{q})](t_0)||_{H^2(\Gamma)}+\sum_{v\in\mathcal{E}}||\partial_x[u(p)-u(q)](v)||_{H^1(0,T)}\right)$$

$\mathcal{E}$ = set of all the external nodes except one

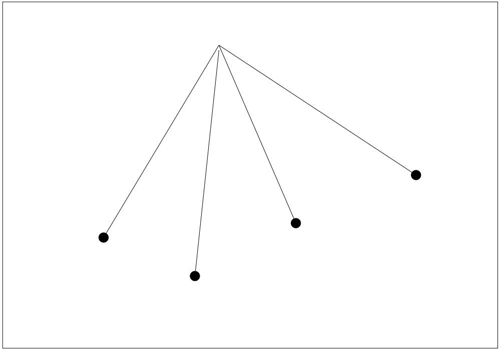
1. **Method**: by Bukhgeim-Klibanov approach, it is "sufficient " to prove some **Carleman estimate** on Γ
2. **Difficulty**: the weight function will fulfill the continuity condition, but not Kirchhoff law at the internal nodes !
3. **Question**: do the same with less boundary measurements ?

Schrödinger equation on a network

Consider the Schrödinger equation on a network Γ

$$\begin{cases} i\mathbf{u}_t + \Delta_\Gamma \mathbf{u} + \mathbf{p}(x)\mathbf{u} = 0, & \text{in } \Gamma \times (0,T), \\ \mathbf{u} = \mathbf{h}, & \text{on } \partial\Gamma \times (0,T), \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0, & \text{in } \Gamma, \end{cases}$$

Star-shaped tree



Stability estimate

Thm:

$$||\mathbf{p} - \mathbf{q}|| \leq C \sum_{v \in \mathcal{E}} ||\partial_x [u(p) - u(q)](v)||_{H^1(0,T)}.$$

1. **Method**: by Bukhgeim-Klibanov approach, it is still "sufficient " to prove some **Carleman estimate** on Γ
2. **Difficulty**: there is some bad term (u_t) at the internal node that we cannot control well
3. **Solution**: we used N weights functions defined on Γ

Carleman estimate for Schrödinger on a tree

$$\sum_{1\le j,k\le N}\int_0^T\int_0^{l_j}[\lambda^2 s\theta_j^k|q_{j,x}|^2+\lambda^4(s\theta_j^k)^3|q_j|^2$$
$$\le C_0\sum_{1\le j,k\le N}\left(\int_0^T\int_0^{l_j}|q_{j,t}+iq_{j,xx}|^2 e^{-2s\varphi_j^k}dxdt\right.$$
$$\left.+\int_0^T\lambda s\theta_j^k(l_j)|q_{j,x}(l_j)|^2 e^{-2s\varphi_j^k}dt\right)$$

Open questions

1. Carleman estimate with one weight function?
2. General tree?
3. Observation at all but one external node?