Dynamics of a rigid body in a perfect incompressible fluid

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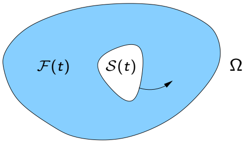
Outline

1. Presentation of the model: a solid immersed in a perfect incompressible fluid
2. Representation of the equation as a geodesic flow
3. The problem of a small body

I. Presentation of the model

A rigid body immersed in an incompressible perfect fluid:

- We consider the motion of a rigid body immersed in an incompressible perfect fluid in a regular domain $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3. (Let us say $n = 3$ to fix the ideas.)



Typically, $\Omega = \mathbb{R}^n$ or is a bounded domain.

- The solid occupies at each instant $t \geq 0$ a closed connected regular subset $\mathcal{S}(t) \subset \Omega$, and the fluid occupies $\mathcal{F}(t) := \Omega \setminus \mathcal{S}(t)$.

Fluid equation

- In $\mathcal{F}(t)$, the fluid satisfies the Euler equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0 \quad \text{in } [0, T] \times \mathcal{F}(t), \\ \\ \operatorname{div} u = 0 \quad \text{in } [0, T] \times \mathcal{F}(t), \end{cases}$$

where

  - $u = u(t, x) : \mathcal{F}(t) \to \mathbb{R}^n$ is the fluid velocity,
  - $p = p(t, x) : \mathcal{F}(t) \to \mathbb{R}$ denotes the pressure.

Boundary conditions

- At the boundaries, the fluid satisfies the slip condition :

$$u(t,x)\cdot n(x) = 0 \text{ for } x \in \partial\Omega,$$
$$u(t,x)\cdot n(t,x) = V_{\mathcal{S}}(t,x)\cdot n(t,x) \text{ for } x \in \partial\mathcal{S}(t),$$

  where n is the normal to the boundaries $\partial\Omega$ and $\partial\mathcal{S}(t)$, and

$$V_{\mathcal{S}}(t,x) = h'(t) + r(t)\times(x - h(t))$$

  is the body velocity, where:

  - $h(t)$ is the position of its center of mass,
  - $h'(t)$ is the linear velocity,
  - $r(t)$ denotes the angular speed.

Dynamics of the solid

- The dynamics of the solid is driven by the action of the pressure on its surface:

$$mh''(t) = \int_{\partial\mathcal{S}(t)} p(t,x)n(t,x)\ d\Gamma(x),$$

$$(\mathcal{J}r)'(t) = \int_{\partial\mathcal{S}(t)} (x - h(t)) \times [p(t,x)n(t,x)]\ d\Gamma(x),$$

  where
  - m is the mass of the body, $\mathcal{J}$ denotes the moment of inertia.
  - Let $Q(t) \in SO(3)$ the rotation matrix defined by:

$$Q'(t) = r(t) \times Q(t) \text{ and } Q(0) = \mathrm{Id}\,.$$

    Then

$$\mathcal{S}(t) = h(t) + Q(t)[\mathcal{S}(0) - h(0)],$$

    and

$$\mathcal{J}(t) = Q(t)\mathcal{J}(0)Q^*(t).$$

Initial data

We prescribe as initial data:

- $\mathcal{S}(0) = \mathcal{S}_0$, with $\mathcal{S}_0 \subset \Omega$ a smooth closed subset of Ω,
- $\mathcal{J}(0) = \mathcal{J}_0$,
- $u|_{t=0} = u_0$, for $x \in \mathcal{F}_0 := \Omega \setminus \mathcal{S}_0$,
- $(h'(0), r(0)) = (h'_0, r_0)$, with (h'_0, r_0, u_0) satisfying

$$\operatorname{div}(u_0) = 0 \text{ in } \mathcal{F}_0,\ \ u_0 \cdot n = 0 \text{ on } \partial\Omega,$$
$$u_0 \cdot n = (h'_0 + r_0 \times (x - x_0)) \cdot n \text{ on } \partial\mathcal{S}_0.$$

References for the Cauchy problem

- This fluid-structure system has been studied by different authors in the context of classical (say C^1) solutions with finite energy:
  - Ortega-Rosier-Takahashi (2005, 2007), in the full plane.
  - Rosier-Rosier (2009), in the full space.
  - Houot-San Martin-Tucsnak (2010) in a bounded domain in Sobolev spaces.
  - See also G.-Sueur-Takahashi

Remark. D'Alembert's paradox does not apply here, because it concerns fluids which are potential in Ω, stationary and constant at infinity. In that case (only), D'Alembert's paradox states that the fluid does not influence the dynamics of the solid.

II. Representation of the system as a geodesic flow

Let us recall Arnold's interpretation of perfect incompressible fluid flows.

- Given $u \in C^1([0,T] \times \overline{\Omega}; \mathbb{R}^n)$ with

  $$\operatorname{div} u = 0 \text{ in } [0,T] \times \Omega \ \text{ and } \ u \cdot n = 0 \text{ on } [0,T] \times \partial\Omega,$$

  the flow is associated by

  $$\partial_t \Phi(t,x) = u(t, \Phi(t,x)) \text{ and } \Phi(0,x) = x, \text{ for } (t,x) \in [0,T] \times \Omega.$$

- For any t, $\Phi(t,\cdot)$ is a diffeomorphism preserving the volume and the orientation. We will write

  $$\Phi(t,\cdot) \in \mathrm{SDiff}^+(\Omega).$$

Arnold's interpretation, 2

- Now one can see $SDiff^+(\Omega)$ as an infinite-dimensional Lie group.
- One can describe its tangent space as follows

$$T_\eta \operatorname{Sdiff}^+(\Omega) := \Big\{ u \circ \eta \text{ with } u \in C^1(\overline{\Omega}; \mathbb{R}^3) \\ \text{such that } \operatorname{div}(u) = 0 \text{ in } \Omega \text{ and } u \cdot n = 0 \text{ on } \partial\Omega \Big\}.$$

- One can endow this Lie group with the right-invariant metric inherited from $L^2(\Omega)$

$$\langle u \circ \eta, v \circ \eta \rangle_\eta := \int_\Omega u(\eta(x)) v(\eta(x))\, dx = \int_\Omega u(x) v(x)\, dx.$$

Arnold's interpretation, 3

Now Arnold's interpretation of perfect incompressible fluid flows states the following.

- Consider a curve $t \mapsto \Phi(t, \cdot)$ from $[0, T]$ to $\mathrm{SDiff}^+(\Omega)$.
- Then Φ is a geodesic on $\mathrm{SDiff}^+(\Omega)$ with respect to the L^2 metric, that is, a critical point of the action functional

  $$\int_0^T \|\partial_t \Phi(t, \cdot)\|^2_{L^2(\Omega)} \, dt,$$

  if and only if

  $$u(t, x) := \partial_t \Phi(t, \Phi^{-1}(x)) \text{ satisfies the Euler equation.}$$

  (See also Ebin-Marsden.)

Equivalent for the fluid/solid problem

The associated flows:

- the fluid flow is given by

$$\partial_t \Phi^{\mathcal{F}}(t,x) = u(t,\Phi^{\mathcal{F}}(t,x)) \text{ and } \Phi^{\mathcal{F}}(0,x) = x, \text{ for } (t,x) \in I \times \mathcal{F}_0,$$

  $\Phi^{\mathcal{F}}(t,\cdot) : \mathcal{F}_0 \to \mathcal{F}(t)$ is a volume and orientation preserving diffeomorphism.

- the solid flow is given by

$$\partial_t \Phi^{\mathcal{S}}(t,x) = v_{\mathcal{S}}(t,\Phi^{\mathcal{S}}(t,x)) \text{ and } \Phi^{\mathcal{S}}(0,x) = x \text{ for } (t,x) \in I \times \mathcal{S}_0.$$

  $\Phi^{\mathcal{S}}(t,\cdot)$ belongs to $SE(3)$, the group of the rigid motions:

$$SE(3) \simeq \mathbb{R}^3 \times SO_3(\mathbb{R}).$$

The set of the possible configurations

- We first describe the set of the possible configurations of the system at a fixed time by setting

$$\mathcal{C} := \Big\{ (\tau, \eta) \in SE(3) \times C^{1,\alpha}(\mathcal{F}_0; \mathbb{R}^3) \text{ such that } \tau(\mathcal{S}_0) \subset \Omega,$$
$$\eta \text{ is a diffeomorphism } \mathcal{F}_0 \to \Omega \setminus [\tau(\mathcal{S}_0)]$$
$$\text{preserving volume and orientation} \Big\}.$$

- One can describe its tangent space at (τ, η) as the set of $(\sigma \circ \tau, u \circ \eta)$ with $\sigma \in \mathfrak{se}(3)$ i.e.

$$\sigma(x) = L + R \times (x - \tau(x_B^0)),$$

and $u \circ \eta \in C^{1,\alpha}(\mathcal{F}_0; \mathbb{R}^3)$ with

$$\text{div } u = 0 \text{ in } \mathcal{F}_0, \;\; u \cdot n = 0 \text{ on } \partial\Omega \;\; \text{ and } \;\; u \cdot n = \sigma \cdot n \text{ on } \partial[\tau(\mathcal{S}_0)].$$

Curves on $\mathcal{C}$

- Given $T > 0$ and two points (τ_1, η_1) and (τ_2, η_2) in $\mathcal{C}$, we define a set of curves of $\mathcal{C}$

$$\mathcal{L} := \Big\{ \, (\tau, \eta) \in C^1([0, T]; \mathcal{C}), \text{ such that:} \\ i. \;\; \tau(0) = \tau_0, \quad \eta(0) = \eta_0, \\ ii. \;\; \tau(T) = \tau_1, \quad \eta(T) = \eta_1 \, \Big\}.$$

- Then we can define the action

$$\mathcal{A}(\tau, \eta) = \frac{1}{2} \int_{[0,T]} \Big(m|L(t)|^2 + \mathcal{J}[\tau(t)] R(t) \cdot R(t) \\ + \int_{\eta_t(\mathcal{F}_0)} |u(t,x)|^2 \, dx \Big) \, dt,$$

where u, L and R are associated to the tangent vector $\partial_t(\tau, \eta)$ as previously; $\mathcal{J}[\tau] := Q \mathcal{J}_0 Q^*$ with Q the linear part of τ.

Interpretation of the fluid/body system

One can show that $\mathcal{A}$ is differentiable on $\mathcal{L}$, and prove the following.

Theorem (G.-Sueur)

If (u, x_B, r) is a classical solution of the PDE system on $[0, T]$ then $(\Phi^{\mathcal{S}}, \Phi^{\mathcal{F}})$ is a geodesic on $\mathcal{L}$, i.e. it satisfies

$$D\mathcal{A}(\Phi^{\mathcal{S}}, \Phi^{\mathcal{F}}) = 0.$$

Conversely, let $(\tau, \eta) \in \mathcal{L}$ be a geodesic. Then (v, L, R) associated as before to the tangent vector $(\partial_t \tau, \partial_t \eta)$, gives a solution of the PDEs formulation on $[0, T]$.

III. The problem of a small body (2D)

- Here, $\Omega = \mathbb{R}^2$.
- Let us be given $\mathcal{S}_0$ a smooth, simply connected, bounded domain as above.
- Given $u_0 \in C^0(\overline{\mathcal{F}}_0; \mathbb{R}^2)$, $(h_0', r_0) \in \mathbb{R}^3$, with corresponding vorticity $w_0 \in L^p_c(\overline{\mathcal{F}}_0)$ as above, one can associate a global solution (h', r, u).
- Question. What can be said if the size ε of the solid goes to zero, so that $\mathcal{S}_0$ shrinks to a point?
- We will be interested in the following particular regime of a massive point in the limit:

$$m_\varepsilon = m \text{ and } \mathcal{J}_\varepsilon = \varepsilon^2 \mathcal{J},$$

where m and $\mathcal{J}$ are fixed constants.

Weak solutions for the Cauchy problem (2D)

Theorem (G.-Sueur)

Let $\mathcal{S}_0$ be a smooth, bounded, simply connected domain in $\Omega \subset \mathbb{R}^2$. Let $p > 2$. For any $u_0 \in C^0(\overline{\mathcal{F}}_0; \mathbb{R}^2)$, $(h_0', r_0) \in \mathbb{R}^3$ such that

$$div\ u_0 = 0,\ curl\ u_0 = w_0 \in L^p_c(\mathcal{F}_0),\ u_0 \cdot n = (h_0' + r(x - h_0)^{\perp}) \cdot n \text{ on } \partial\mathcal{S}_0,$$

there exists a solution $(h, r, u) \in C^2(\mathbb{R}^+) \times C^1(\mathbb{R}^+) \times L^\infty(\mathbb{R}^+; W^{1,p}(\mathcal{F}(t)))$ (resp. $L^\infty(\mathbb{R}^+; \mathcal{LL}(\mathcal{F}(t)))$ for $p = +\infty$). This solution is unique for $p = +\infty$.

Here $$\mathcal{LL}(U) := \Big\{ f \in C^0(U) \ / \ \exists C > 0,\ \forall x, y \in U, \\ |f(x) - f(y)| \leq C|x - y|(1 + \ln^- |x - y|) \Big\}.$$

Remark

In general $u(t, \cdot) \notin L^2(\mathcal{F}(t); \mathbb{R}^2)$. Finite energy solutions would be too particular in the sequel...

References for the Euler equation alone:

- Yudovich (1963) for $p = +\infty$
- DiPerna-Majda (1987) for $p < +\infty$

Vorticity formulation.

In 2D, the fluid part of the system can also be written

$$\begin{cases} \partial_t w + (u \cdot \nabla) w = 0 \ \text{ in } \ \mathcal{F}(t), \\ w_{|t=0} = w_0, \end{cases}$$

and

$$\begin{cases} \text{curl } u = w \ \text{ in } \ \mathcal{F}(t), \\ \text{div } u = 0 \ \text{ in } \ \mathcal{F}(t), \\ u \cdot n = (h' + r(x - h(t))^{\perp}) \cdot n \ \text{ on } \ \partial\mathcal{S}(t), \\ \lim_{|x| \to +\infty} u(t,x) = 0, \\ \oint_{\partial\mathcal{S}(t)} u(t,x) \cdot \tau \, d\sigma = \oint_{\partial\mathcal{S}_0} u_0(x) \cdot \tau \, d\sigma \ \text{ (Kelvin's law).} \end{cases}$$

The problem of a small body, continued

- Let us be given $w_0 \in L^p_c(\mathbb{R}^2)$, $\gamma \in \mathbb{R}$, $(h_0', r_0) \in \mathbb{R}^3$. For $\varepsilon \in (0,1)$, we define
$$\mathcal{S}_0^\varepsilon := h_0 + \varepsilon(\mathcal{S}_0 - h_0), \quad \mathcal{F}_0^\varepsilon := \mathbb{R}^2 \setminus \mathcal{S}_0^\varepsilon.$$
and u_0^ε as to satisfy
$$\begin{cases} \text{curl } u_0^\varepsilon = w_0 \ \text{ in } \ \mathcal{F}_0^\varepsilon, \\ \text{div } u_0^\varepsilon = 0 \ \text{ in } \ \mathcal{F}_0^\varepsilon, \\ u_0^\varepsilon \cdot n = (h_0' + r_0(x - h_0)^\perp) \cdot n \ \text{ on } \ \partial\mathcal{S}_0^\varepsilon, \\ \lim_{|x| \to +\infty} u_0^\varepsilon = 0, \\ \oint_{\partial\mathcal{S}_0^\varepsilon} u_0^\varepsilon \cdot \tau \, d\sigma(x) = \gamma. \end{cases}$$
- What can be said about the sequence of solutions $(h^\varepsilon, r^\varepsilon, u^\varepsilon)$ associated to the data $\mathcal{S}_0^\varepsilon$, h_0', r_0, u_0^ε?
- Call $\theta^\varepsilon := \int_0^t r^\varepsilon$.

Main result

Theorem (G.-Lacave-Sueur). Up to a subsequence, one has:

- $h^\varepsilon \xrightarrow{w*} h$, $\varepsilon\theta^\varepsilon \xrightarrow{w*} 0$ weakly-$*$ in $W^{2,\infty}(0,T;\mathbb{R}^2)$,
- $w^\varepsilon \xrightarrow{w} w$ in $C^0([0,T];L^p(\mathbb{R}^2)w)$ (in $C^0([0,T];L^\infty(\mathbb{R}^2)w*)$ if $p=+\infty$),
- $u^\varepsilon \longrightarrow \tilde{u} + \dfrac{\gamma}{2\pi}\dfrac{(x-h(t))^\perp}{|x-h(t)|^2}$ in $C^0([0,T];L^q_{loc}(\mathbb{R}^2))$, $q<2$,
- one has

$$\frac{\partial w}{\partial t} + \operatorname{div}\left(\left[\tilde{u} + \frac{\gamma}{2\pi}\frac{(x-h(t))^\perp}{|x-h(t)|^2}\right]w\right) = 0 \quad \text{in} \quad [0,T]\times\mathbb{R}^2,$$

$$\tilde{u}(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}w(t,y)\,dy,$$

$$mh''(t) = \gamma\Big(h'(t) - \tilde{u}(t,h(t))\Big)^\perp,$$

$$w|_{t=0} = w_0,\ h(0) = h_0,\ h'(0) = h_0'.$$

Comparison of the limit system, 1

Our limit system:

$$\frac{\partial w}{\partial t} + \operatorname{div}\left(\left[\tilde{u} + \frac{\gamma}{2\pi}\frac{(x-h(t))^\perp}{|x-h(t)|^2}\right] w\right) = 0 \text{ in } [0,T]\times\mathbb{R}^2,$$

$$\tilde{u}(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\,w(t,y)\,dy,$$

$$mh''(t) = \gamma\Big(h'(t) - \tilde{u}(t,h(t))\Big)^\perp.$$

The Euler equation in $\mathbb{R}^2$:

$$\frac{\partial w}{\partial t} + \operatorname{div}\left(\tilde{u}w\right) = 0 \text{ in } [0,T]\times\mathbb{R}^2,$$

$$\tilde{u}(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\,w(t,y)\,dy \quad \text{(Biot-Savart law).}$$

Comparison of the limit system, 2

Our limit system:

$$\frac{\partial w}{\partial t} + \operatorname{div}\left(\left[\tilde{u} + \frac{\gamma}{2\pi} \frac{(x - h(t))^\perp}{|x - h(t)|^2} \right] w \right) = 0 \quad \text{in} \quad [0, T] \times \mathbb{R}^2,$$

$$\tilde{u}(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^2} w(t, y)\, dy,$$

$$m h''(t) = \gamma \Big(h'(t) - \tilde{u}(t, h(t)) \Big)^\perp.$$

The limit of a shrinking obstacle, see Iftimie, Lopes-Filho, Nussenzveig-Lopes (2003):

$$\frac{\partial w}{\partial t} + \operatorname{div}\left(\left[\tilde{u} + \frac{\gamma}{2\pi} \frac{(x - h_0)^\perp}{|x - h_0|^2} \right] w \right) = 0 \quad \text{in} \quad [0, T] \times \mathbb{R}^2,$$

$$\tilde{u}(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^2} w(t, y)\, dy.$$

Comparison of the limit system, 3

Our limit system:

$$\frac{\partial w}{\partial t} + \mathrm{div}\,\left(\left[\tilde{u} + \frac{\gamma}{2\pi} \frac{(x-h(t))^{\perp}}{|x-h(t)|^2} \right] w \right) = 0 \;\text{ in }\; [0,T] \times \mathbb{R}^2,$$
$$\tilde{u}(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2} w(t,y)\, dy,$$
$$mh''(t) = \gamma \Big(h'(t) - \tilde{u}(t,h(t)) \Big)^{\perp}.$$

The wave/vortex system, see Marchioro-Pulvirenti:

$$\frac{\partial w}{\partial t} + \mathrm{div}\,\left(\left[\tilde{u} + \frac{\gamma}{2\pi} \frac{(x-h(t))^{\perp}}{|x-h(t)|^2} \right] w \right) = 0 \;\text{ in }\; [0,T] \times \mathbb{R}^2,$$
$$\tilde{u}(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2} w(t,y)\, dy,$$
$$h'(t) = \tilde{u}(t,h(t)).$$

(See also more recently Lacave-Miot.)

Conjecture

One may conjecture the following. In the natural regime

$$m = m_0 \varepsilon^2 \;\text{ and } \mathcal{J} = \mathcal{J}_0 \varepsilon^4,$$

the system converges toward the wave/vortex system.

Formally if one takes $m = 0$ in

$$m h''(t) = \gamma \Big(h'(t) - \tilde{u}(t, h(t)) \Big)^{\perp},$$

one recovers

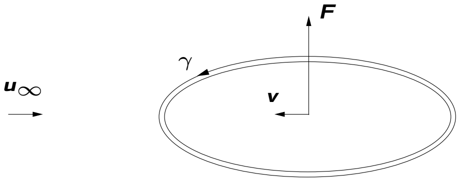
$$h'(t) = \tilde{u}(t, h(t)) \ldots$$

Kutta-Joukowski force

The force appearing in the equation of the point in the limit

$$mh''(t) = \gamma\Big(h'(t) - \tilde{u}(t, h(t))\Big)^{\perp},$$

is similar to the lift force similar to the Kutta-Joukowski force of the irrotational theory:



the force applied to the body at speed v, with fluid velocity u_∞ at infinity and circulation γ around the body is

$$F = \gamma(v - u_\infty)^{\perp}.$$

Ideas of proof for the existence theorem

1. Change of variables. We consider equations in the body frame. Consider:

$$\begin{cases} v^\varepsilon(t,x) = Q^\varepsilon(t)^* \; u^\varepsilon(t, Q^\varepsilon(t)x + h^\varepsilon(t)), \\ q^\varepsilon(t,x) = p^\varepsilon(t, Q^\varepsilon(t)x + h^\varepsilon(t)), \\ \ell^\varepsilon(t) = Q^\varepsilon(t)^* \; (h^\varepsilon)'(t). \end{cases}$$

The equations of the fluid/body system become

$$\partial_t v^\varepsilon + \left[(v^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp) \cdot \nabla \right] v^\varepsilon + r^\varepsilon (v^\varepsilon)^\perp + \nabla q^\varepsilon = 0 \text{ for } x \in \mathcal{F}_0^\varepsilon,$$

$$\text{div } v^\varepsilon = 0 \text{ for } x \in \mathcal{F}_0^\varepsilon,$$

$$v^\varepsilon \cdot n = \left(\ell^\varepsilon + r^\varepsilon x^\perp \right) \cdot n \text{ for } x \in \partial \mathcal{S}_0^\varepsilon,$$

$$m(\ell^\varepsilon)'(t) = \int_{\partial \mathcal{S}_{\mathbf{0}}^\varepsilon} q^\varepsilon n \; ds - m r^\varepsilon (\ell^\varepsilon)^\perp$$

$$\mathcal{J}_\varepsilon (r^\varepsilon)'(t) = \int_{\partial \mathcal{S}_{\mathbf{0}}^\varepsilon} x^\perp \cdot q^\varepsilon n \; ds$$

$$v^\varepsilon(0,x) = v_0^\varepsilon(x) \text{ for } x \in \mathcal{F}_0^\varepsilon, \;\; \ell^\varepsilon(0) = \ell_0, \;\; r^\varepsilon(0) = r_0.$$

2. Kirchoff's potentials.

One introduces Kirchoff's potentials Φ_1, Φ_2, Φ_3:

$$\Delta \Phi_i = 0 \;\text{ in }\; \mathcal{F}_0,$$
$$\partial_n \Phi_i = \begin{cases} n_i \;\; (i = 1, 2), \\ x^\perp \cdot n \;\; (i = 3), \end{cases} \quad \text{on } \; \partial \mathcal{S}_0.$$

The solid equations become

$$\begin{aligned} \begin{bmatrix} m \,\mathrm{Id}_2 & 0 \\ 0 & \mathcal{J} \end{bmatrix} \begin{bmatrix} \ell \\ r \end{bmatrix}' &= \left[\int_{\partial \mathcal{S}_\mathbf{0}} q \partial_n \Phi_i \, dx \right]_{i=1,2,3} - \begin{bmatrix} mr\ell^\perp \\ 0 \end{bmatrix} \\ &= \left[\int_{\mathcal{F}_\mathbf{0}} \nabla q \cdot \nabla \Phi_i \, dx \right]_{i=1,2,3} - \begin{bmatrix} mr\ell^\perp \\ 0 \end{bmatrix} \end{aligned}$$

3. Decomposition of the pressure.

Let P the Leray projector in $L^p(\mathcal{F}_0; \mathbb{R}^2)$, $1 < p < +\infty$, that is, the projection on tangent divergence-free vector fields, parallel to gradient fields. It is continuous in L^p.

The pressure decomposes as follows:

$$\nabla q = \underbrace{(I - P)(\partial_t v)}_{=:\nabla\varphi} + \underbrace{(I - P)(-(v - \ell - rx^{\perp}) \cdot \nabla v - rv^{\perp})}_{=:\nabla\mu}$$

Using that $\partial_t v$ is already divergence-free, one easily deduces that

$$\nabla\varphi = - \begin{pmatrix} \ell \\ r \end{pmatrix}' \cdot \left(\nabla\Phi_i\right)_{i=1,2,3}.$$

We end up with this new equation for the solid:

$$\mathcal{M}\begin{bmatrix}\ell\\ r\end{bmatrix}' = \begin{bmatrix}mr\ell^{\perp}\\ 0\end{bmatrix} + \left[\int_{\mathcal{F}_0} \nabla\mu\cdot\nabla\Phi_i\,dx\right]_{i=1,2,3\}},$$

where

$$\mathcal{M} := \begin{bmatrix}m\,\mathrm{Id}_2 & 0\\ 0 & \mathcal{J}\end{bmatrix} + \underbrace{\left[\int_{\mathcal{F}(t)} \nabla\Phi_i\cdot\nabla\Phi_j\ dx\right]_{i,j=1,2,3}}_{=:\mathcal{M}_2}.$$

The matrix $\mathcal{M}_2$ is a matrix of added inertia, expressing how the fluid opposes the movement of the solid. It is positive as a Gram matrix.

4. Fixed point scheme.

Then one can (as is usual), use a fixed point scheme, for instance relying on the vorticity:

$$\omega := \text{curl } v,$$

which satisfies

$$\partial_t \omega + \left[(v - \ell - r x^{\perp}) \cdot \nabla \right] \omega = 0 \text{ for } x \in \mathcal{F}_0.$$

Hence, knowing (ℓ, r, v), one can

- Transport the initial vorticity and obtain ω,
- Compute $\nabla \mu$ and deduce a new (ℓ, r)
- Define a new v.

A fixed point gives a solution. The solution is defined as long as (ℓ, r) is bounded.

The difficulty here is that we work with infinite energy solutions.

5. Pseudo-energy estimates.

To describe the singular part of the velocity, one introduces $H = \mathcal{F}_0 \to \mathbb{R}^2$ as follows:

$$\begin{gathered} \text{curl } H = \text{div } H = 0 \quad \text{in} \quad \mathcal{F}_0, \\ H \cdot n = 0 \quad \text{on} \quad \partial\mathcal{S}_0, \\ \int_{\partial\mathcal{S}_0} H \cdot \tau \, d\sigma = 1. \end{gathered}$$

Then

$$\hat{v} := v - (\alpha + \gamma) H \in L^2(\mathcal{F}_0),$$

where

$$\alpha := \int_{\mathcal{F}_0} \omega(t, x) \, dx = \int_{\mathcal{F}_0} \omega(0, x) \, dx.$$

Proposition

Let

$$\mathcal{H} := \frac{1}{2}\left[m|\ell|^2 + \mathcal{J} r^2 + \int_{\mathcal{F}_0} \hat{v}^2 + 2(\gamma+\alpha)\hat{v}\cdot H\right].$$

Then $\mathcal{H}$ is conserved.

The "standard" energy would be

$$\mathcal{E} := \frac{1}{2}\left[m|\ell|^2 + \mathcal{J} r^2 + \int_{\mathcal{F}_0} |\hat{v} + (\gamma+\alpha)H|^2\right],$$

but this is infinite in general. The difference $\frac{(\gamma+\alpha)^2}{2}\int_{\mathcal{F}_0}|H|^2$, is "infinite but constant".

Problem. $\mathcal{H}$ is no longer positive...

But mixing with the conservations of $\|\omega\|_{L^p}$, one gets global solutions.

Ideas of proof for the main statement

1. A priori estimates.

Using the pseudo-energy, and tracking the dependence on ε, one obtains:

Proposition

Let $T > 0$. The quantities $|\ell^\varepsilon|$, $\varepsilon|r^\varepsilon|$, $\|v^\varepsilon - \gamma H^\varepsilon\|_\infty$, diam$(Supp(\omega^\varepsilon))$ are bounded on $[0, T]$ independently of ε.

Remark

H^ε is of order $\mathcal{O}(1/\varepsilon)$ on $\partial\mathcal{S}_0^\varepsilon$ ....

2. Added inertia.

It is elementary to check that in the regime under view ($m =$ cst and $\mathcal{J} = \mathcal{J}_0\varepsilon^2$), the added inertia is negligible with respect to the original one.

3. Study of the pressure $\nabla\mu$.

- The part of the pressure that is not included in the inertia is

$$\nabla\mu = -(I - P)[(v - \ell - rx^{\perp}) \cdot \nabla v + rv^{\perp}].$$

  The main part consists in studying the behaviour of $\nabla\mu$ near the boundary $\partial\mathcal{S}_0^{\varepsilon}$.

- Here the non-singular part of the velocity is

$$\check{v} := v - \gamma H.$$

- Using the a priori estimates, one can show that the terms which do not contain γH are negligible as $\varepsilon \to 0^+$.

- The most singular term:

$$\gamma^2(I - P)(H \cdot \nabla H) = \frac{\gamma^2}{2}\nabla|H|^2,$$

  is too singular. However by using a simple computation relying on Cauchy's residue theorem, one can show that it gives no contribution.

- The important term is

$$\begin{aligned}\gamma(I - P)\Big[(\check{v} - \ell - rx^{\perp}) \cdot \nabla H + (H \cdot \nabla)(\check{v} - \ell - rx^{\perp})\Big]\\ = \gamma\nabla[(\check{v} - \ell - rx^{\perp}) \cdot H] + \text{ negligible terms.}\end{aligned}$$

4. Description of the shrinking body's behaviour.

After computation and approximation of the pressure, one arrives to

$$\left(\begin{bmatrix} m\,\mathrm{Id}_2 & 0 \\ 0 & \mathcal{J}_0 \varepsilon^2 \end{bmatrix} + \left[\int_{\mathcal{F}(t)} \nabla \Phi_i^\varepsilon \cdot \nabla \Phi_j^\varepsilon \; dx \right]_{i,j=1,2,3} \right) \begin{bmatrix} \ell \\ r \end{bmatrix}'$$
$$= \gamma \begin{bmatrix} (\ell^\varepsilon - V^\varepsilon(t,0))^\perp + \varepsilon r^\varepsilon \alpha \\ \varepsilon \beta \cdot (\ell^\varepsilon - V^\varepsilon(t,0))^\perp \end{bmatrix} + o(1) \begin{bmatrix} 1 \\ 1 \\ \varepsilon \end{bmatrix},$$

where α and β are constant vectors depending on the geometry of $\mathcal{S}_0$ (and independent of ε), and V^ε is the velocity obtained by Biot-Savart law:

$$V^\varepsilon = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega^\varepsilon(t,y)\, dy,$$

where the vorticity is extended by 0 inside $\mathcal{S}_0^\varepsilon$.

One can deduce $W^{1,\infty}$ weak-$*$ compactness for ℓ^ε and $\varepsilon r^\varepsilon$.

In the result stated above, there is no term containing α, and one should have $\varepsilon r^\varepsilon \xrightarrow{w*} 0$...

But we have to go back to the original frame, and hence to multiply by $Q^\varepsilon(t)$, so that we deduce

$$
\begin{gathered}
m(h^\varepsilon)'' = \gamma((h^\varepsilon)' - U^\varepsilon(t,h^\varepsilon))^\perp - \gamma(\varepsilon r^\varepsilon)Q^\varepsilon(t)\alpha + o(1), \\
\mathcal{J}_0(\varepsilon r^\varepsilon)' = \gamma\beta \cdot Q^\varepsilon(t)^*((h^\varepsilon)' - U^\varepsilon) + o(\varepsilon),
\end{gathered}
$$

with

$$
U^\varepsilon = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} w^\varepsilon(t,y)\,dy,
$$

(w^ε is the vorticity in the original frame.)

The main point is that by an argument close to the unstationary phase, one has:

$$
(\varepsilon r^\varepsilon)Q^\varepsilon(t)\alpha \xrightarrow{w*} 0 \quad \text{and} \quad Q^\varepsilon(t)^*((h^\varepsilon)' - U^\varepsilon) \xrightarrow{w*} 0.
$$

The result follows.