Controllability of coupled hyperbolic and parabolic systems by a reduced number of controls

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Outline

The control of scalar wave type equations, either by a locally distributed control or a boundary control is by now quite well-understood.

What happens for non scalar equations, that is in case for instance of coupled systems?

These appear naturally in mechanics as for instance in case of Timoshenko beams, or in reaction-diffusion equations, insensitizing controls...

Let us recall how it "works" for a scalar wave equation:

Let Ω be a bounded open set of $\mathbb{R}^N$ with a sufficiently smooth boundary Γ, and Γ_1 a subset of Γ.

The boundary control problem reads as follows for the wave equation:

For initial data (y_0, y_1) in a space to be determined, does there there exist a control $v \in L^2(\Gamma_1 \times (0,T))$ such that the solution of

$$\begin{cases} y_{tt} - \Delta y = 0 \text{ in } \Omega \times (0,T) \\ y = v \text{ on } \Sigma_1 = \Gamma_1 \times (0,T)\,, y = 0 \text{ on } \Sigma_0 = (\Gamma - \Gamma_1) \times (0,T) \\ y(.,0) = y_0(.)\,, y_t(.,0) = y_1(.) \text{ in } \Omega\,, \end{cases}$$

satisfies in addition $y(T,\cdot) = y_t(T,\cdot) = 0$ in Ω?

i.e. the control v drives back the system to equilibrium at time T.

In the case of locally distributed control problem, one looks for controls v such that the solution of

$$\begin{cases} y_{tt} - \Delta y = \chi_\omega v \text{ in } \Omega \times (0, T) \\ y = 0 \text{ on } \Sigma = \Gamma \times (0, T)\,, \\ y(.,0) = y_0(.)\,, y_t(.,0) = y_1(.) \text{ in } \Omega\,, \end{cases}$$

satisfies $y(T,\cdot) = y_t(T,\cdot) = 0$ in Ω.

Thanks to the Hilbert Uniqueness Method (HUM) J.-L Lions 1988 the above controllability problems " are equivalent" to:

proving suitable observability (and also a direct) inequality for the dual problem

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \Omega \times (0, T) \\ u(.,0) = u_0(.)\,, u_t(.,0) = u_1(.) \text{ in } \Omega\,, \end{cases}$$

in which no control source term appears (homogeneous problem).

Well-posedness for the dual problem in the natural energy space is well-known, the energy of a solution being defined as

$$E(t) = \frac{1}{2} \int_\Omega (|u_t|^2 + |\nabla u|^2) \, dx \, .$$

The dual problem is conservative, that is

$$E(t) = E(0) \quad \forall \, t \geq 0 \, .$$

In the case of boundary control, one can show that the weak solutions of the dual problem satisfy the direct inequality

$$\int_0^T \int_\Gamma |\frac{\partial u}{\partial \nu}|^2 \, d\gamma \, dt \leq c_2 \, E(u(0)) \, ,$$

This allows to define rigorously by the transposition method (by duality) the solutions of the boundary control problem.

If Γ_1 is a part of the boundary which satisfies certain geometric conditions, either the Geometric Condition of Bardos Lebeau Rauch 1992, or (stronger) multiplier geometric conditions, and if T is sufficiently large, then there exists $c_1 > 0$ such that the following inverse inequality

$$\int_0^T \int_{\Gamma_1} |\frac{\partial u}{\partial \nu}|^2 \, d\gamma \, dt \geq c_1 \, E(0) \, ,$$

holds.

References: Ho, JL Lions, Zuazua, Bardos Lebeau Rauch, Komornik and many others . . .

It is also well-known that the geometric conditions

as well as the fact that T has to be sufficiently large

are due to the finite speed of propagation for the wave equation.

Such conditions do not occur for the corresponding heat equation.

One can also show suitable direct and observability inequalities for the locally distributed case.

We turn now to coupled systems and start by a model problem.

Consider the following weakly coupled system of two wave equations

$$\begin{cases} u_{1\,,tt} - \Delta u_1 + \alpha u_2 = 0 \text{ in } \Omega \times (0, T)\,, \\ u_{2\,,tt} - \Delta u_2 + \alpha u_1 = 0 \text{ in } \Omega \times (0, T)\,, \\ u_1 = 0 \text{ on } \Sigma = \Gamma \times (0, T)\,, u_2 = 0 \text{ on } \Sigma\,, \\ u_i(0) = u_i^0\,, u_{i,t}(0) = u_i^1\,. \end{cases}$$

where α is a coupling parameter.

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Several different notions of observability/controllability can be considered for coupled systems:

- *complete* : observe all components $\longmapsto$ recover all initial data (Kim Renardy for Timoshenko beams 1987 ...)
- *partial* : observe only one component with initial vanishing data for the other $\longmapsto$ initial data of the observed component (Lions 1988, Komornik-Loreti 2000, ...)
- *indirect* : observe only one component $\longmapsto$ recover all initial data (Russell 1993, A. 2001, ...)
- simultaneous : observe simultaneously both components of the system $\longmapsto$ recover all initial data (Lions 1988, Loreti Komornik 2003, Tucsnak and Weiss ...)

Here we consider indirect observability.

We would like to observe this system by a reduced number of observations.

This means, we want to get information on the full initial state, observing only one component of the unknown state.

This is more demanding, since information is missing on one component.

More precisely, he question is:

Is it possible to get, for sufficiently large time T, the following type of observability inequality

$$\int_0^T \int_{\Gamma_1} |\frac{\partial u_1}{\partial \nu}|^2 \, d\gamma \, dt \geq c \left(e_1(u_1(0)) + e_2(u_2(0)) \right),$$

where $e_i(u_i(t))$ stands for some energy of the corresponding component of the unknown.

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where $e_i(u_i(t))$ stands for some energy of the corresponding component of the unknown.

If $\alpha = 0$, the two waves are uncoupled, so that we cannot hope to get such a result.

What can be said for $\alpha \neq 0$?

How to compensate the lack of information on the unobserved component of the dual coupled system?

Which type of observability inequality can hold true?

Due to results on the lack of exponential stabilization for the corresponding indirect stabilization problems A.-B. Cannarsa and Komornik 2002 (globally distributed case) and A.-B. 2002 (boundary case),

one cannot expect an observability inequality in the natural spaces.

We assume the following (multiplier geometric conditions)

Ω is a non-empty bounded open set in $\mathbb{R}^N$ having a boundary Γ of class C^2.

Moreover, $\{\Gamma_0, \Gamma_1\}$ is a partition of Γ such that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ and x_0 is a point in $\mathbb{R}^N$ such that $m \cdot \nu \leq 0$ on Γ_0 and $m \cdot \nu \geq 0$ on Γ_1, where $m(x) = x - x_0$.

We denote by $|\ |$ the L^2-norm on Ω. Then, we prove

Theorem (A.-B. 2001, 2003)

There exists $\alpha^\star > 0$ *such that for all* $0 < |\alpha| < \alpha^\star$, *there exists* $T_0 = T_0(\alpha) > 0$ *such that for all* $T > T_0$ *and all* $U^0 = (u_1^0, u_1^1, u_2^0, u_2^1) \in \mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2$ *the solution* (u_1, u_2) *satisfies*

$$\int_0^T \int_{\Gamma_1} |\frac{\partial u_1}{\partial \nu}|^2 \geq c_1 \left(|u_1^1|^2 + |\nabla u_1^0|^2\right) + c_2 \left(|u_2^1|^2_{H^{-1}(\Omega)} + |u_2^0|^2\right) ,$$

where the constants $c_1 = c_1(\alpha, T)$, $c_2 = c_2(\alpha, T)$ *are given by*

Theorem (continued)

$$c_1 = \frac{a_1(T - T_3)}{(1+\alpha T)(1+\alpha T_3)}\,,\ c_2\frac{\alpha a_2(T - T_2)(T - T_2^-)}{1+\alpha T}\,,$$

where a_1 , a_2 are constants independent on α and T.

T_0, $|T_2|$, $|T_2^-|$ behave as $C\alpha^{-1}$ and T_3 behaves as C as α goes to zero.

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Theorem (continued)

Moreover, if in addition the solution satisfies

$$\frac{\partial u_1}{\partial \nu} = 0 \textit{ on } \Gamma_1 \times (0, T)\,,$$

then, unique continuation holds that is: $u_1 = u_2 = 0$ *in* $\Omega \times [0, T]$.

By duality, using the HUM method and thanks to the proof of a suitable direct inequality, the above result can be translated into an exact *indirect controllability result* for the following control problem:

For given initial data, determine a L^2 control v such that the solution of

$$\begin{cases} y_{1\,,tt} - \Delta y_1 + \alpha y_2 = 0 \quad \text{in} \quad \Omega \times (0,T)\,, \\ y_{2\,,tt} - \Delta y_2 + \alpha y_1 = 0 \quad \text{in} \quad \Omega \times (0,T)\,, \\ y_1 = v \quad \text{on} \quad \Sigma_1 = \Gamma_1 \times (0,T)\,, y_1 = 0 \quad \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,T)\,, \\ y_2 = 0 \quad \text{on} \quad \Sigma = \Gamma \times (0,T)\,, \\ (y_1, y_{1\,,t})(0) = (y_1^0, y_1^1), (y_2, y_{2\,,t})(0) = (y_2^0, y_2^1) \quad \text{on} \quad \Omega\,. \end{cases}$$

satisfies

$$(y_1, y_2, y_{1,t}, y_{2,t})(T) = 0 \text{ on }]\Omega\,.$$

Theorem (A.-B. SICON 2003)

Under the multiplier geometric condition: there exists $\alpha^\star > 0$ such that for all $0 < |\alpha| < \alpha^\star$, there exists $T_0 = T_0(\alpha) > 0$ such that for all $T > T_0$ and all $Y^0 = (y_1^0, y_1^1, y_2^0, y_2^1) \in L^2(\Omega) \times H^{-1}(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ there exists a control $v \in L^2([0,T]; L^2(\Gamma_1))$ such that the solution $Y(t) = (y_1, y_1', y_2, y_2')$ satisfies

$$y_i(.,T) = \partial_t y_i(.,T) = 0 \text{ in } \Omega\,, \text{ for } i = 1\,,2\,.$$

Note that the initial data for the uncontrolled component have to be taken in a smaller space than the space of the controlled one.

These results are obtained through a two-level energy method (A.-B. SICON 2003).

Roughly speaking, it consists in compensating the lack of observation of the second component by a balance effect between the natural energy of the observed component and the weakened energy of the unobserved one.

This means that we have to work with the $H_0^1 \times L^2$ norm of the observed component, whereas we have to consider the $L^2 \times H^{-1}$ norm of the unobserved component.

This balance can be adjusted through fine estimates. A key point is the conservation of the total natural energy but also of the total weakened energies of the solutions.

These results extend to indirect observability (resp. controllability) for adjoint (resp. direct) abstract coupled systems, with applications to coupled waves, plates, that is for instance for observability for

$$\begin{cases} u_1'' + A_1 u_1 + \alpha C u_2 = 0 \quad \text{in } V_1' \,, \\ u_2'' + A_2 u_2 + \alpha C^\star u_1 = 0 \quad \text{in } V_2' \,, \\ (u_1, u_1')(0) = (u_1^0, u_1^1) = U_1^0 \in V_1 \times H \,, \\ (u_2, u_2')(0) = (u_2^0, u_2^1) = U_2^0 \in V_2 \times H \,, \end{cases}$$

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Here H, $V_1 \subset H$ and $V_2 \subset H$ are separable Hilbert spaces, with $V_i \subset H$ with dense, compact and continuous embedding for $i = 1, 2$.

A_1, A_2 are coercive self adjoint unbounded operators in H,

C is a coupling operator assumed to be bounded in H, $C^\star$ is the adjoint operator of C and α is a coupling parameter.

Also one can consider different coupling parameters α_1 and α_2 in each equation.

The total energy of a solution (u_1, u_2) is defined by

$$E(u_1(t), u_2(t)) = \frac{1}{2}\Big(|u_1'(t)|^2 + |u_2'(t)|^2 + |A_1^{1/2}u_1(t)|^2 + |A_2^{1/2}u_2(t)|^2\Big) + \alpha(u_1, Cu_2)\,,$$

where

$$e_i(u(t)) = \frac{1}{2}(|u'|_H^2 + |u|_i^2)\,, i = 1\,, 2\,,$$

stand for the partial energies, $|\cdot|_i$ is the norm in V_i.

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One can prove well-posedness and generalize the above indirect observability/controllability results for an unbounded observation/control operator $B^\star$ resp. B.

If $A_1 \neq A_2$, one has to add compatibility conditions between A_1 and A_2 and have restrictive geometric conditions, together with restriction on the coupling operator.

The above results are valid only for bounded coercive coupling operators C.

That is under the assumption

$$\exists \eta > 0 \text{ such that } \langle Cu, u \rangle \geq \eta |u|_H^2 , \quad \forall\, u \in H .$$

What can be said in the situation of noncoercive coupling operators

or equivalently

for systems of coupled PDE's when the coupling coefficient is localized on some part of the domain and vanishes outside a a subset of Ω?

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As a model example, let us consider the following system

$$\begin{cases} y_{1\,,tt} - \Delta y_1 + p(.)y_2 = b\,v \quad \text{in} \quad \Omega \times (0,T)\,, \\ y_{2\,,tt} - \Delta y_2 + p(.)\,y_1 = 0 \quad \text{in} \quad \Omega \times (0,T)\,, \\ y_1 = y_2 = 0 \quad \text{on} \quad \Sigma = \Gamma \times (0,T)\,, \\ (y_1, y_{1\,,t})(0) = (y_1^0, y_1^1), (y_2, y_{2\,,t})(0) = (y_2^0, y_2^1) \quad \text{on} \quad \Omega\,. \end{cases}$$

where v is the control, and b and p are functions which are resp. localized on ω_b (resp. on Γ_b in the boundary control case) and ω_p, and satisfying

$$b \geq 0\,, p \geq 0 \text{ on } \Omega\,,$$

$$b > 0 \text{ on } \overline{\omega}_b\,, (\text{ resp. on } \overline{\Gamma}_b)$$

$$p > 0 \text{ on } \overline{\omega}_p\,.$$

The above results can be extended to the case of noncoercive couplings.

This is a work in collaboration with Léautaud.

Using the two-level energy method we prove indirect observability estimates and exact controllability results in the same spirit than above. A key point is to replace the use of the multiplier method as in A.-B. 2001 2003 by GCC so that we have results for two coupled waves under geometric conditions on ω_b and ω_p with empty intersection in multi-dimensions, as well as in abstract form and for localized as well as boundary observation/control.

This extension requires new ideas to compensate the loss of coercivity of the coupling operator.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary (or a smooth connected compact riemannian manifold with or without boundary).

Moreover $p = p(x)$ and $b = b(x)$ are smooth real-valued functions on Ω, $\delta > 0$ is a constant parameter and f is the control function, that can act on the system.

We set $p^+ = ||p||_{L^\infty(\Omega)}$.

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We recall the classical Geometric Control Conditions GCC due to Bardos Lebeau Rauch 1992, itis a necessary and sufficient condition for the internal observability and controllability of a single wave equation.

We say that $\omega \subset \Omega$ satisfies GCC if every generalized geodesic traveling at speed one in Ω meets ω (resp. meets Γ on a non-diffractive point) in finite time.

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Theorem (A.-B.-Léautaud 2011)

For all b satisfying $b \geq 0$ on Ω (resp. Γ), $\{b > 0\} \supset \overline{\omega_b}$ (resp. $\{b > 0\} \supset \overline{\Gamma_b}$) for some open subset $\omega_b \subset \Omega$ (resp. $\Gamma_b \subset \Gamma$) satisfying GCC*, there exists a constant $p_* > 0$ such that for all $p^+ < p_*$, there exists a time $T_* > 0$ such that for all $T > T_*$, all p satisfying $p \geq 0$ on Ω, $\{p > 0\} \supset \overline{\omega_p}$ for some open subset $\omega_p \subset \Omega$ satisfying* GCC*, , and all initial data $(y_1^0, y_2^0, y_1^1, y_2^1) \in H_0^1(\Omega) \times H^2 \cap H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$ (resp. $(y_1^0, y_2^0, y_1^1, y_2^1) \in L^2(\Omega) \times H_0^1(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$), there exists a control $v \in L^2((0, T) \times \Omega)$ (resp. $v \in L^2((0, T) \times \Gamma)$), such that the solution satisfies $(y_1, y_2, y_1', y_2')|_{t=T} = 0$.*

Moreover, we show that the reachable set is exactly

$H_0^1(\Omega) \times H^2 \cap H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$. in the internal control case and $L^2(\Omega) \times H_0^1(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ in the boundary control case.

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Using the transmutation method, based on the Kannai transform Phung 2001, Miller 2005, Ervedoza and Zuazua 2010 (also Russell in 1973).

controllability properties for the heat equation can be deduced from controllability properties for the wave equation

Thus our results for wave coupled systems lead to indirect exact controllability results for coupled heat equations, and coupled Schrödinger equations,

with empty intersection between the localization of the coupling and of the control regions in the multi-dimensional case,

and for boundary dampings as well.

They also hold when one replaces the operator $-\Delta$ by a second order uniformly elliptic operator or by replacing the homogeneous Dirichlet boundary conditions by Neumann or Fourier boundary conditions, so that the method is flexible.

Consider the corresponding null controllability problem for parabolic, diffusive or Schrödinger systems. That is for each initial data in a suitable space, determine a L^2 control v such the solution of

$$\begin{cases} e^{i\theta}y_1' - \Delta y_1 + p(.)y_2 = bv & \text{in } (0,T)\times\Omega, \\ e^{i\theta}y_2' - \Delta y_2 + p(.)y_1 = 0 & \text{in } (0,T)\times\Omega, \\ y_1 = y_2 = 0 & \text{on } (0,T)\times\Omega, \\ (y_1, y_2)|_{t=0} = (y_1^0, y_2^0) & \text{in } \Omega, \end{cases}$$

satisfies $(y_1, y_2)(T) = 0$ on Ω.

Here $\theta = 0$ (heat case), $\theta \in (0, \pi/2)$ (diffusive case) or $\theta = \pi/2$ (Schrödinger case).

Combining the above controllability result for the coupled wave system, we can deduce a controllability result for heat/diffusive coupled systems

Corollary (Heat-type systems)

Suppose that ω_p *satisfies GCC and that* ω_b *(resp* Γ_b*) satisfies GCC (resp. GCC for boundary case). Then, there exists a constant* $p_* > 0$ *such that for all* $p^+ < p_*$*, for all* $T > 0$*,* $\theta \in (-\pi/2, \pi/2)$*, for all initial data* $(y_1^0, y_2^0) \in \left(L^2(\Omega)\right)^2$ *(resp* $(y_1^0, y_2^0) \in \left(H^{-1}(\Omega)\right)^2$*), there exists a control* $v \in L^2((0,T) \times \Omega)$ *(resp* $v \in L^2((0,T) \times \Gamma)$*) such that the solution of heat coupled type systems satisfies* $(y_1, y_2)|_{t=T} = 0$*.*

We also obtain the following result for Schrödinger coupled systems

Corollary (Schrödinger-type systems)

Assume $\theta = \pm\pi/2$. Under the above conditions, the same null-controllability result holds for any $T > 0$, taking initial data $(y_1^0, y_2^0) \in L^2(\Omega) \times H_0^1(\Omega)$ (resp. $(y_1^0, y_2^0) \in H^{-1}(\Omega) \times L^2(\Omega)$) for a suitable L^2 control v

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Rosier and De Teresa obtained at the same time, a null controllability result for heat coupled cascade systems, that is

$$\begin{cases} e^{i\theta} y_1' - \Delta y_1 + p(.) y_2 = bv & \text{in } (0,T) \times \Omega, \\ e^{i\theta} y_2' - \Delta y_2 = 0 & \text{in } (0,T) \times \Omega, \\ y_1 = y_2 = 0 & \text{on } (0,T) \times \Omega, \\ (y_1, y_2)|_{t=0} = (y_1^0, y_2^0) & \text{in } \Omega, \end{cases}$$

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Their method is based on Däger's approach and a controllability result for the corresponding cascade wave system.

It requires that the semigroup generated by the free wave equation is periodic, so that it is valid only in 1-D domains.

They also have a positive null controllability result for Schrödinger cascade coupled systems in the torus and for sufficiently large time T.

There exists a rich literature on the controllability of parabolic couples systems by a reduced number of controls whch started around 2000, motivated by studying insensitizing control De Teresa 2000.

Some references:

De Teresa Kavian 2010 have proven unique continuation results for cascade parabolic type problems, for control/coupling regions with empty intersections.

There are several results by Ammar-Khodja Benabdallah Fernàndez-Cara Gonzalez-Burgos and de Teresa with Kalman type conditions for coupled parabolic PDE's ranging from 2006 to now.

The methods are based direct methods for the parabolic system via Carleman estimates, or the methods of moments.

The limitation is that such results require that the coupling and control region have a nonempty intersection.

One of the important question on these problems is to know if whether it is possible to derive positive null controllability results in case of localized coupling and localized control regions with empty intersections?

Also, the situation is well-known to be more complicated in the boundary control case

For instance negative results for the case of boundary control for coupled parabolic equations with different diffusion coefficients (de Teresa et al.)

Thus, the results we saw before:

based on the transmutation method relying on controllability/observability results for the corresponding wave equations

are important, since they provide a positive (not yet complete in the parabolic case as we will see in the open problems) answer to the above problem.

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Both results are based on a controllability result for wave coupled systems of the form

$$\begin{cases} y_{tt} - \Delta y + \mathcal{C} y = Bv \,, \text{in } Q_T = \Omega \times (0, T) \,, \\ y = 0 \,, \text{on } \Sigma_T = \partial\Omega \times (0, T) \,, \\ (y(0, .), y_t(0, .)) = (y_0(.), y_1(.)) \,, \text{in } \Omega \,, \end{cases}$$

where $y = (y_1, y_2)$, $Bv = (0, v\mathbf{1}_\omega)^t$ and where

$$\mathcal{C} = \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$$

in A.-B. (2003) and A.-B. and Léautaud (2011) case which is multi-dimensional,
and

$$\mathcal{C} = \begin{pmatrix} 0 & \mathbf{1}_{O} \\ 0 & 0 \end{pmatrix}$$

in Rosier and de Teresa case, under restrictive geometric assumptions (one-dimensional or in the torus).

Recent extensions:

We prove A.-B. 2011 positive indirect controllability results for coupled systems of N equations by a reduced number of controls.

Open problems

- (Gcc) is not natural for coupled heat systems. How to derive positive results without geometric conditions?
- Our controllability/observabilty results hold for sufficiently small couplings. What about extension to large couplings? Work in this direction with De Teresa.
- What about couplings with no coercivity type properties?
- Higher order couplings. Semilinear problems.

Open problems

- What is the minimal time of control for coupled wave systems? Some positive results by Dehman Léautaud and Le Rousseau 2011
- Estimates on the cost of control?
- Optimization of the localization and shape of the control and localized regions to minimize the cost.

Open problems

- Have more general results characterizing sharp conditions on the structure of the coupled systems that can be indirectly controlled.
- General spectral analysis of such problems
- Numerical discretization of such problems and convergence results

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Part of the work in collaboration (for the stabilization aspects) with Léautaud has been done at the Centro de Ciencias de Benasque at this conference in 2009. So

Thanks to the organizers and for your attention

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