One example in simultaneous homogenization and dimensional reduction in nonlinear elasticity

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Outline

- Plate models in elasticity
- Homogenization & dimensional reduction
- Periodically wrinkled plate
- Conclusions

Plate models in elasticity

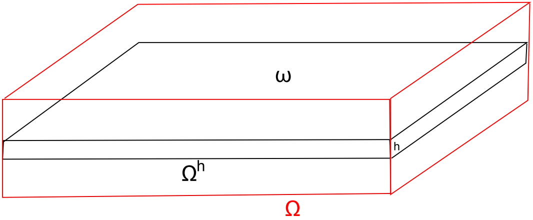
Minimization functional of $3D$ elasticity

$$\int_{\Omega^h} W(\nabla \boldsymbol{y}^h)dx - \int_{\Omega^h} \boldsymbol{f}^h \cdot \boldsymbol{y}^h dx,$$

- $\boldsymbol{y}^h : \Omega^h \to \mathbb{R}^3$ deformation
- $\boldsymbol{f}^h : \Omega^h \to \mathbb{R}^3$, external volume dead loads
- $W : \mathbb{R}^{3\times 3} \to [0, \infty]$, stored energy function with the properties
  1. class C^2 in a neighborhood of $\mathrm{SO}(3)$;
  2. W is frame-indifferent, i.e., $W(\mathbf{F}) = W(\mathbf{RF})$ for every $\mathbf{F} \in \mathbb{R}^{3\times 3}$ and $\mathbf{R} \in \mathrm{SO}(3) \Leftrightarrow W(\mathbf{F}) = \tilde{W}(\mathbf{F}^T\mathbf{F})$.
  3. $W(\mathbf{F}) \geq C_W \operatorname{dist}^2(\mathbf{F}, \mathrm{SO}(3))$, for some $C_W > 0$ and all $\mathbf{F} \in \mathbb{R}^{3\times 3}$, $W(\mathbf{F}) = 0$ iff $\mathbf{F} \in \mathrm{SO}(3)$.

Plate models in elasticity

- $\Omega^h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$, $\omega \subset \mathbb{R}^2$ Lipschitz domain.



What happens as $h \to 0$?
Rescaling: $P^h : \Omega \to \Omega^h$, $P^h(x_1, x_2, x_3) = (x_1, x_2, hx_3)$,
$\Omega = \omega \times [-\frac{1}{2}, \frac{1}{2}]$. Minimization functional:

$$\frac{1}{h}\left(\int_\Omega W(\nabla_h \boldsymbol{y}^h)dx - \int_\Omega \boldsymbol{f}^h \cdot \boldsymbol{y}^h dx\right),$$

$$\nabla_h = \nabla_{\boldsymbol{e}_1, \boldsymbol{e}_2} + \frac{1}{h}\nabla_{\boldsymbol{e}_3}$$

Plate models in elasticity-Γ-convergence

Definition

Let (X, d) be metric space. We say that the sequence $f^n : X \to \overline{\mathbb{R}}$ Γ-converges to $f : X \to \overline{\mathbb{R}}$ if for every $x \in X$ we have

1. *(*$\liminf$ *inequality) for every sequence $(x^n)_n$ which converges to x we have*

$$f(x) \leq \liminf_n f^n(x^n);$$

2. *(*$\limsup$ *inequality) there exist $(x^n)_n$ (recovery sequence) which converges to x such that*

$$f(x) = \lim_n f^n(x^n).$$

- Γ-limit + precompactness ofminimizing sequence $\Longrightarrow$ convergence of global minimizers;
- the non-uniqueness or non-existence of the minimizers does not bother us.

Plate models in elasticity-hierarchy of models

Depending on parameter $\alpha \geq 0$ and boundary conditions (i.e. appropriate space) we want to find (without the term of forces)

$$\frac{1}{h^\alpha}\frac{1}{h}\int_\Omega W(\nabla_h \boldsymbol{y})dx \xrightarrow{\Gamma} ?$$

- $\alpha = 0$ membrane model Le Dret, Raoult (1995).
- $\alpha > 0$ the key fact is the frame-indifference of the stored energy function; the order of the strain energy can be smaller than the order of the volume for non-trivial (close to metric preserving) deformations.
  $\alpha = 2$ (2002) the limit deformations are exact isometries, defined on ω, i.e. $\nabla y \in \mathrm{SO}(3)$ a.e. in ω. The key ingredient is the geometric rigidity theorem proved by Friesecke, James, Müller.

Plate models in elasticity-theorem on geometric rigidity

Theorem (on geometric rigidity)

Let $U \subset \mathbb{R}^m$ be a bounded Lipschitz domain, $m \geq 2$. Then there exists a constant $C(U)$ with the following property: for every $\boldsymbol{v} \in W^{1,2}(U; \mathbb{R}^m)$ there is associated rotation $\mathbf{R} \in \mathrm{SO}(m)$ such that

$$\|\nabla \boldsymbol{v} - \mathbf{R}\|_{L^2(U)} \leq C(U)\| \operatorname{dist}(\nabla \boldsymbol{v}, \mathrm{SO}(m))\|_{L^2(U)}.$$

The constant $C(U)$ can be controlled for the class of Billipschitz domains whose Billipschitz constants we can control.

Plate models in elasticity-bending model

The typical deformation of order $\alpha = 2$. Let us take $\mathbf{R} = (\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3) \in W^{1,2}(\omega; \mathrm{SO}(3))$ such that for $\boldsymbol{y} \in W^{2,2}(\omega; \mathbb{R}^3)$ we have $\partial_\alpha \boldsymbol{y} = \mathbf{R}_\alpha$. Defining the deformation

$$\boldsymbol{y}^h(x', x_3) = \boldsymbol{y}(x') + h x_3 \mathbf{R}_3 + O(h^2)$$

we can conclude

$$\nabla_h \boldsymbol{y}^h = \mathbf{R} + O(h),\ (\nabla_h \boldsymbol{y})^T \nabla_h \boldsymbol{y} = \mathbf{I} + O(h),$$

$$W(\nabla_h \boldsymbol{y}^h) = \tilde{W}\big((\nabla_h \boldsymbol{y}^h)^T \nabla_h \boldsymbol{y}^h\big) = \tilde{W}\big(\mathbf{I} + O(h)\big) = O(h^2).$$

Fortunately we do not need the corrector (for non smooth data) to prove liminf inequality.

Plate models in elasticity-Föppl-von Kármán model

The "typical deformation" of order $\alpha = 4$ looks like

$$\boldsymbol{y}^h = \begin{pmatrix} x' \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2\boldsymbol{u} \\ hv \end{pmatrix} - h^2 x_3 \begin{pmatrix} \partial_1 v \\ \partial_2 v \\ 0 \end{pmatrix} + O(h^2),$$

$x' = (x_1, x_2)$, $\boldsymbol{u} : \omega \to \mathbb{R}^2$, $v : \omega \to \mathbb{R}$.

$$\begin{aligned} \nabla_h \boldsymbol{y}^h &= \mathbf{I} + h \left(\begin{array}{c|c} 0 & -(\nabla' v)^T \\ \hline \nabla' v & 0 \end{array} \right) + O(h^2) \\ (\nabla_h \boldsymbol{y}^h)^T \nabla_h \boldsymbol{y}^h &= \mathbf{I} + O(h^2). \end{aligned}$$

Taylor expansion of $\tilde{W}$ about $\mathbf{I}$ $\Longrightarrow$

$$W(\nabla_h \boldsymbol{y}^h) = \tilde{W}\left((\nabla_h \boldsymbol{y}^h)^T \nabla_h \boldsymbol{y}^h\right) = \tilde{W}\left(\mathbf{I} + O(h^2)\right) = O(h^4).$$

Plate models in elasticity

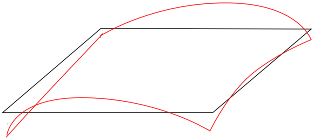
Conclusion: The property of objectivity of energy density influence the process of dimensional reduction implying the hierarchy of lower dimensinal models for plates, rods, shells.

Fox, D.D., Raoult, A., Simo, J.C.: *A justification of nonlinear properly invariant plate theories*, *ARMA* (1993)

P.G. Ciarlet, **Mathematical elasticity, Vol. II: Theory of plates**

G. Frisecke, R.D. James, S. Müller: *A hierarchy of plate models derived from nonlinear elasticity by Gamma-convergence*, *ARMA* (2006)

Plate models in elasticity

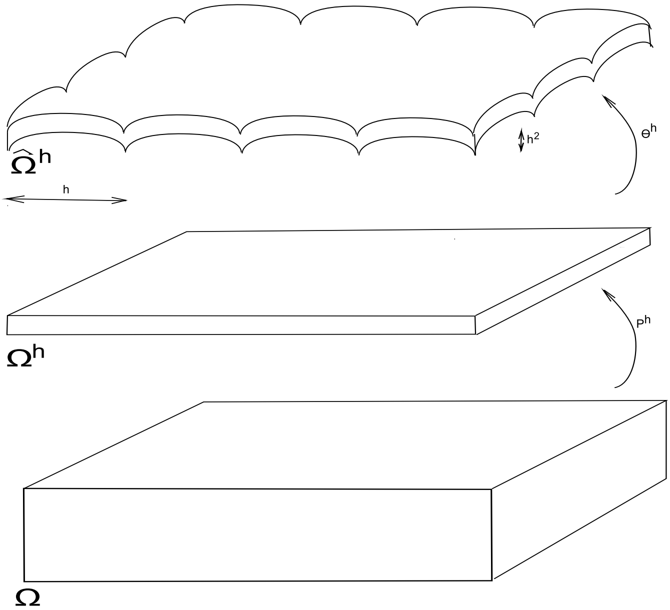


General question: How do the (geometrical) imperfections of the domain influence the model and what is their order (compared to the thickness) to influence it non-trivially.

Shallow shell models: By taking curvature of order h and taking the energy h^4 we derive Marguerre von Kármán shallow shell model.

Ciarlet, P.G., Paumier, J.C.: *A justification of Marguerre-von Kármán equations*, *Computational Mech.* (1986)

Homogenization & dimensional reduction



Homogenization & dimensional reduction

Dimensional reduction + objectivity $\Longrightarrow$ hierarchy of models
Linearization & dimensional reduction do not commute.
Linearization & homogenization commute

Müller, S. and Neukamm, S.: *On the commutability of homogenization and linearization in finite elasticity*, *ARMA* (2010).

Homogenization & dimensional reduction $\Longrightarrow$ there are inter effects depending on the relation between the scales of dimensional reduction and homogenization.

Braides, A., Fonseca, I., Francfort G. : *3D-2D asymptotic analysis for inhomogeneous thin films*, *Indiana Univ. Math. J.* (2000)

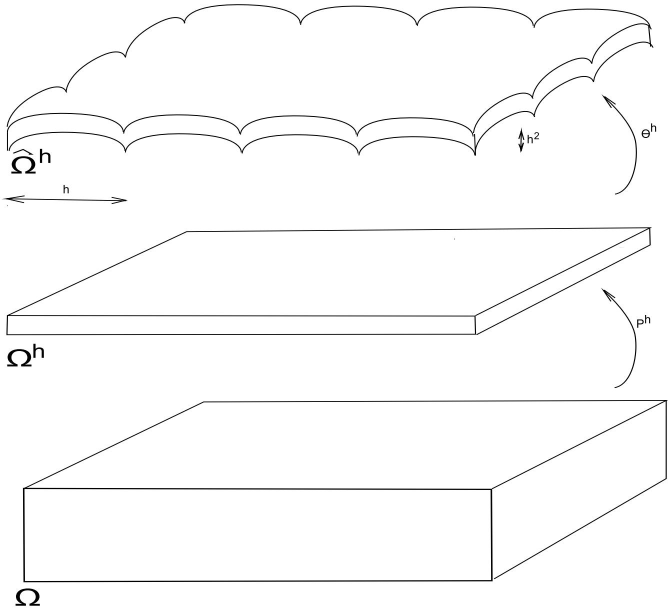
Homogenization & dimensional reduction

Homogenization & dimensional reduction
Bending regime

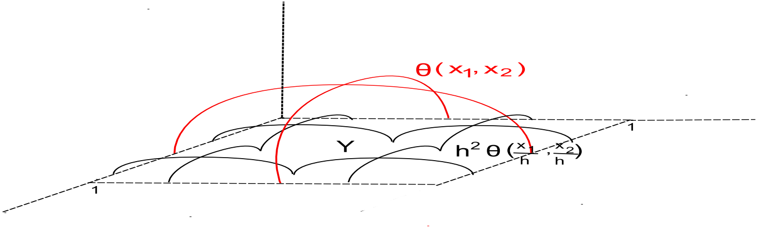
- Case of rods: 3 scenarios depending on relation between period of homogenization and thickness. Techniques include theorem on geometric rigidity, Γ-convergence and two scale convergence. Oscillations are on the same scale as period of material (different than in the membrane case and general 3D) (Müller, S., Neukamm, S.)
- Case of plates: partial results; not able to treat the case when the period of the material is on the same scale or faster than the thickness (geometric difficulties).

Von Kármán ???

Periodically wrinkled plate



Periodically wrinkled plate



$$\Theta^h(x^h) = (x_1, x_2, h^2\theta(\frac{x_1}{h}, \frac{x_2}{h})) + x_3^h \boldsymbol{n}^h(x_1, x_2), \quad \theta \in C^2_\#(Y)$$

$\boldsymbol{n}^h$- unit normal to the mid-surface.
Functional:

$$\frac{1}{(h^2)^4}\frac{1}{h}\int_{\hat{\Omega}^h} W(\nabla \boldsymbol{y})dx = \frac{1}{(h^2)^4}\int_{\Omega} W((\nabla \boldsymbol{y}) \circ \Theta^h \circ P^h)dx \xrightarrow{\Gamma} ?$$

Periodically wrinkled plate

Example of the deformation of the order $(h^2)^4 = h^8$

$$
\begin{aligned}
\boldsymbol{y}^h(\boldsymbol{\Theta}^h(x_1,x_2,h^2x_3)) \quad &= \quad \boldsymbol{\Theta}^h(x_1,x_2,h^2x_3) \\
&+ \begin{pmatrix} h^4\boldsymbol{u}(x_1,x_2) \\ h^2 v(x_1,x_2) \end{pmatrix} \\
&- h^2(h^2x_3) \begin{pmatrix} \partial_1 v(x') \\ \partial_2 v(x') \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&-h^4\partial_1 v(x_1,x_2)\theta_0(\frac{x_1}{h},\frac{x_2}{h})\boldsymbol{e}_1 \\
&-h^4\partial_2 v(x_1,x_2)\theta_0(\frac{x_1}{h},\frac{x_2}{h})\boldsymbol{e}_2 \\
&-h^3(h^2x_3)\Big(\partial_1 v(x_1,x_2)\partial_1\theta(\frac{x_1}{h},\frac{x_2}{h}) + \partial_2 v(x_1,x_2)\partial_2\theta(\frac{x_1}{h},\frac{x_2}{h})\Big)\boldsymbol{e}_3 \\
&+O(h^4)
\end{aligned}
$$

$$
\theta_0 = \theta - \langle\theta\rangle, \quad \langle\theta\rangle := \int_Y \theta dy
$$

$$
\boldsymbol{y}^h : \hat{\Omega}^h \to \mathbb{R}^3
$$

Periodically wrinkled plate

$$
\begin{aligned}
\nabla \boldsymbol{y}^h &= \mathbf{I} + h^2 \mathbf{A} + O(h^4) \\
\mathbf{A} &= \begin{pmatrix} 0 & 0 & -\partial_1 v \\ 0 & 0 & -\partial_2 v \\ \partial_1 v & \partial_2 v & 0 \end{pmatrix} \\
(\nabla \boldsymbol{y}^h)^T \nabla \boldsymbol{y}^h &= \mathbf{I} + O(h^4)
\end{aligned}
$$

Taylor expansion of $\tilde{W}$ about $\mathbf{I}$ $\Longrightarrow$

$$
W(\nabla \boldsymbol{y}^h) = \tilde{W}\big((\nabla \boldsymbol{y}^h)^T \nabla \boldsymbol{y}^h\big) = \tilde{W}\big(\mathbf{I} + O(h^4)\big) = O(h^8).
$$

Periodically wrinkled plate

Proof of Γ-convergence

- Compactness result-tells us how the deformations whose energy is of order h^8 look like.
- Lim-inf inequality (we use two-scale convergence).
- Lim-sup inequality i.e. the construction of the recovery sequence.

Periodically wrinkled plate-Compactness result

$$\frac{1}{h^2}\int_{\hat{\Omega}^h} \mathrm{dist}^2(\nabla \boldsymbol{y}^h, \mathrm{SO}(3))dx \le Ch^8.$$

theorem on geometric rigidity $\implies$ there exists map $\mathbf{R}^h \in W^{1,2}(\omega, \mathrm{SO}(3))$ such that

$$\begin{aligned}\|(\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h - \mathbf{R}^h\|_{L^2(\Omega)} &\le Ch^4,\\ \|\nabla \mathbf{R}^h\|_{L^2(\omega)} &\le Ch^2.\end{aligned}$$

Periodically wrinkled plate-Compactness result

For the corrected in-plane and the out-of-plane displacements

$$
\begin{aligned}
\boldsymbol{u}^h(x_1,x_2) \quad :=& \frac{1}{h^4}\left(\int_{-1/2}^{1/2}\begin{pmatrix} \boldsymbol{y}_1^h\circ\boldsymbol{\Theta}^h\circ P^h \\ \boldsymbol{y}_2^h\circ\boldsymbol{\Theta}^h\circ P^h \end{pmatrix}(x_1,x_2,x_3)dx_3 - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \\
&+\frac{1}{h^2}\begin{pmatrix} \mathbf{R}_{31}^h(x_1,x_2)\theta_0(\frac{x_1}{h},\frac{x_2}{h}) \\ \mathbf{R}_{32}^h(x_1,x_2)\theta_0(\frac{x_1}{h},\frac{x_2}{h}) \end{pmatrix}, \\
v^h(x_1,x_2) \quad :=& \quad \frac{1}{h^2}\int_{-1/2}^{1/2}(\boldsymbol{y}_3^h\circ\boldsymbol{\Theta}^h\circ P^h)(x_1,x_2,x_3)dx_3 - \theta(\frac{x_1}{h},\frac{x_2}{h})
\end{aligned}
$$

$$
\begin{aligned}
v^h \quad &\to \quad v \quad \text{in } W^{1,2}(\omega), \quad v\in W^{2,2}(\omega) \\
\boldsymbol{u}^h \quad &\rightharpoonup \quad \boldsymbol{u} \quad \text{in } W^{1,2}(\omega;\mathbb{R}^2), \\
\frac{1}{h^2}\begin{pmatrix} \mathbf{R}_{31}^h \\ \mathbf{R}_{32}^h \end{pmatrix} \quad &\rightharpoonup \quad \begin{pmatrix} \partial_1 v \\ \partial_2 v \end{pmatrix} \quad \text{in } W^{1,2}(\omega),
\end{aligned}
$$

Periodically wrinkled plate, two-scale convergence

To obtain sharp enough estimate we need more information on the datas, concerning the oscillations of order h.

Definition

$L^2(\Omega) \ni u_h \rightharpoonup\rightharpoonup u_0 \in L^2(\omega \times Y)$ if for every $\psi \in L^2(\omega; C_\#(Y))$,

$$\int_\omega u_h(x)\psi(x, \frac{x}{h}) \to \int_\omega \int_Y u_0(x,y)\psi(x,y) \quad \text{as } h \to 0.$$

Periodically wrinkled plate- identification of the limit strain

On a subsequence we have

$$\nabla \boldsymbol{u}^h \rightharpoonup\rightharpoonup \nabla_x \boldsymbol{u}(x) + \nabla_y \boldsymbol{u}_1(x,y) \text{ in } L^2(\omega \times Y; \mathbb{R}^{2\times 2}),$$
$$\nabla \frac{1}{h^2} \begin{pmatrix} \mathbf{R}^h_{31} \\ \mathbf{R}^h_{32} \end{pmatrix} \rightharpoonup\rightharpoonup \nabla_x^2 v(x) + \nabla_y^2 v_1(x,y) \text{ in } L^2(\omega \times Y; \mathbb{R}^{2\times 2}).$$

Two scale convergence of strain

$$\mathbf{G}^h := \frac{(\mathbf{R}^h)^T((\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h) - \mathbf{I}}{h^4} \rightharpoonup\rightharpoonup \mathbf{G} \quad \text{in } L^2(\Omega \times Y; \mathbb{R}^{3\times 3}),$$

Partially identification of the strain: 2×2 sub-matrix $\mathbf{G}''$ satisfies

$$\begin{aligned}
\mathbf{G}''(x', x_3, y) &= \mathbf{G}_0(x_1, x_2, y) + x_3 \mathbf{G}_1(x_1, x_2, y), \\
\operatorname{sym} \mathbf{G}_0(x_1, x_2, y) &= \operatorname{sym} \nabla_x \boldsymbol{u}(x) + \operatorname{sym} \nabla_y \boldsymbol{u}_1(x,y) \\
&\quad - \nabla_x^2 v(x)\theta_0(y) - \nabla_y^2 v_1(x,y)\theta_0(y) + \frac{1}{2}\nabla_x v(x) \otimes \nabla_x v(x) \\
\mathbf{G}_1(x_1, x_2, y) &= -\nabla_x^2 v(x) - \nabla_y^2 v_1(x,y).
\end{aligned}$$

Periodically wrinkled plate- lim inf inequality

By Taylor expansion and convexity of Q_3 on symetric matrices it can be concluded

$$
\begin{aligned}
& \liminf_{h\to 0} \frac{1}{h^8} \int_\Omega W\big((\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h\big) dx \\
= \ & \liminf_{h\to 0} \frac{1}{h^8} \int_\Omega W\big((\mathbf{R}^h)^T ((\nabla \boldsymbol{y}^h) \circ \boldsymbol{\Theta}^h \circ P^h)\big) dx \\
\geq \ & \frac{1}{2} \int_\Omega \int_Y Q_3\big(\mathbf{G}(x,y)\big) dy dx \geq \frac{1}{2} \int_\Omega \int_Y Q_2\big(\mathbf{G}''(x,y)\big) dy dx .
\end{aligned}
$$

$$
Q_3(\mathbf{F}) = D^2 W(\mathbf{I})(\mathbf{F},\mathbf{F}),\ Q_2(\mathbf{G}) = \min_{\boldsymbol{a}\in\mathbb{R}^3} Q_3(\mathbf{G} + \boldsymbol{a} \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{a}),
$$

quadratic forms.

Periodically wrinkled plate-lim inf inequality

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \int_Y Q_2(\mathbf{G}''(x,y))dydx = \\
&\int_\omega \int_Y \Big(\frac{1}{2}Q_2\big(\operatorname{sym}\nabla_x \boldsymbol{u}(x) + \operatorname{sym}\nabla_y \boldsymbol{u}_1(x,y) - \nabla_x^2 v(x)\theta_0(y) \\
&\qquad\qquad - \nabla_y^2 v_1(x,y)\theta_0(y) + \frac{1}{2}\nabla_x v \otimes \nabla_x v\big)\Big)dydx \\
&+\frac{1}{24}\int_\omega \int_Y Q_2\big(\nabla_y^2 v_1(x,y)\big)dydx + \frac{1}{24}\int_\omega Q_2\big(\nabla_x^2 v(x)\big)dx.
\end{aligned}
$$

Relax the energy by the appropriately chosen oscillations $\boldsymbol{u}_1, v_1$ to obtain the candidate for Γ-limit.

Periodically wrinkled plate- candidate for the Γ limit

$Q_2^H : \mathbb{R}^{2\times 2}_{\mathrm{sym}} \times \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}_0^+$ denote the functional

$$
\begin{aligned}
&Q_2^H\big(\mathbf{G},\mathbf{F}\big) = \\
&\min_{\substack{\boldsymbol{u}_1 \in \dot{H}^1_\#(Y),\\ v_1 \in \dot{H}^2_\#(Y)}} \int_Y \Big(Q_2\big(\mathbf{G} + \mathbf{F}\theta_0 + \mathrm{sym}\,\nabla \boldsymbol{u}_1(y) - \nabla^2 v_1(y)\theta_0(y)\big) \\
&\qquad + \frac{1}{12} Q_2(\nabla^2 v_1(y))\Big) dy,
\end{aligned}
$$

$$
\begin{aligned}
I_0^{WvK}(\boldsymbol{u}, v) &= \frac{1}{2}\int_\omega Q_2^H(\mathrm{sym}\,\nabla \boldsymbol{u} + \frac{1}{2}\nabla v \otimes \nabla v, -\nabla^2 v)dx \\
&\quad + \frac{1}{24}\int_\omega Q_2(\nabla^2 v)dx
\end{aligned}
$$

Periodically wrinkled plate-properties of the Γ-limit

Properties:

- Q_2, Q_2^H quadratic forms, positive definite on symetric matrices.
- For $\mathbf{G}, \mathbf{F}$ symetric, $Q_2(\mathbf{F}) \geq C\|\mathbf{F}\|^2$, $Q_2^H(\mathbf{G}, \mathbf{F}) \geq C(\|\mathbf{G}\|^2 + \|\mathbf{F}^\perp\|^2)$, $\mathbf{F}^\perp$ projection of $\mathbf{F}$ on $V^\perp$ where

$$\begin{aligned} V &= \Big\{ \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}; \forall y \in Y, \\ &\qquad a_{11}\partial_{22}\theta(y) + a_{22}\partial_{11}\theta(y) - 2a_{12}\partial_{12}\theta(y) = 0 \Big\}. \end{aligned}$$

Periodically wrinkled plate-lim sup inequality

Recovery sequence can be constructed from the mentioned example, by adding terms of higher order and by relaxing by them.

Periodically wrinkled plate-lim sup inequality

$$
\begin{aligned}
\boldsymbol{y}^h(\boldsymbol{\Theta}^h(x_1,x_2,h^2x_3)) \quad &= \quad \boldsymbol{\Theta}^h(x_1,x_2,h^2x_3) \\
&\quad + \begin{pmatrix} h^4\boldsymbol{u}(x_1,x_2) + h^5\boldsymbol{u}_1(x_1,x_2,\frac{x_1}{h},\frac{x_2}{h}) \\ h^2v(x_1,x_2) + h^4v_1(x_1,x_2,\frac{x_1}{h},\frac{x_2}{h}) \end{pmatrix} \\
&\quad -h^2(h^2x_3)\begin{pmatrix} \partial_1 v(x') + h\partial_{y_1}v_1(x_1,x_2,\frac{x_1}{h},\frac{x_2}{h}) \\ \partial_2 v(x') + h\partial_{y_2}v_1(x_1,x_2,\frac{x_1}{h},\frac{x_2}{h}) \\ 0 \end{pmatrix} \\
&-h^4\Big(\partial_1 v(x_1,x_2) + h\partial_{y_1}v_1(x_1,x_2,\frac{x_1}{h},\frac{x_2}{h})\Big)\theta_0(\frac{x_1}{h},\frac{x_2}{h})\boldsymbol{e}_1 \\
&-h^4\Big(\partial_2 v(x_1,x_2) + h\partial_{y_2}v_1(x_1,x_2,\frac{x_1}{h},\frac{x_2}{h})\Big)\theta_0(\frac{x_1}{h},\frac{x_2}{h})\boldsymbol{e}_2 \\
&-h^3(h^2x_3)\Big(\partial_1 v(x_1,x_2)\partial_1\theta(\frac{x_1}{h},\frac{x_2}{h}) + \partial_2 v(x_1,x_2)\partial_2\theta(\frac{x_1}{h},\frac{x_2}{h})\Big)\boldsymbol{e}_3 \\
&+h^4(h^2x_3)\boldsymbol{d}_0(x_1,x_2,\frac{x_1}{h},\frac{x_2}{h}) + \frac{1}{2}h^2(h^2x_3)^2\boldsymbol{d}_1(x_1,x_2,\frac{x_1}{h},\frac{x_2}{h}),
\end{aligned}
$$

Relax by $\boldsymbol{u}_1 \in \mathcal{D}(\omega; C^\infty_\#(Y))^2$, $v_1 \in \mathcal{D}(\omega; C^\infty_\#(Y))$, $\boldsymbol{d}_0, \boldsymbol{d}_1 \in \mathcal{D}(\omega; C^\infty_\#(Y))^3$.

Periodically wrinkled plate-properties of the Γ-limit

$$
\begin{aligned}
I_0^{vK}(\boldsymbol{u}, v) &= \underbrace{\frac{1}{2}\int_\omega Q_2(\operatorname{sym}\nabla \boldsymbol{u} + \frac{1}{2}\nabla v \otimes \nabla v)dx}_{\text{streching part of the energy}} \\
&\quad + \underbrace{\frac{1}{24}\int_\omega Q_2(\nabla^2 v)dx}_{\text{bending part of the energy}} \\
I_0^{WvK}(\boldsymbol{u}, v) &= \frac{1}{2}\int_\omega Q_2^H(\operatorname{sym}\nabla \boldsymbol{u} + \frac{1}{2}\nabla v \otimes \nabla v, -\nabla^2 v)dx \\
&\quad + \frac{1}{24}\int_\omega Q_2(\nabla^2 v)dx
\end{aligned}
$$

Periodically wrinkled plate-Conclusion

- we derived the model of periodically wrinkled plate of Föppl-von Kármán type from $3D$ minimization formulation of nonlinear elasticity by means of Γ-convergence. We also used two scale convergence techniques.
- periodic imperfections of the plate can cause mixing of the bending and the streching part of the energy! This does not appear for the non periodic imperfections like shallow shell.
- the linearization of the obtained model is different from the previously obtained models of wrinkled plate by Aganović, I., Marušić-Paloka, E., Tutek, Z. from linear Koiter shell (2D) model. This model can not be seen from (2D) theories!

Periodically wrinkled plate-Conclusion

- What happens for

  $$\mathbf{\Theta}^h(x^h) = (x_1, x_2, h^2\theta(\frac{x_1}{h}, \frac{x_2}{h})) + x_3^h \boldsymbol{n}^h(x_1, x_2),\ x_3^h = h^\alpha x_3,\ \alpha > 2.$$

  The same. Dimensional reduction has already taken its part! $1 < \alpha < 2$? Other combinations?
- Rod model?

  The metric part of strain:
  $u' + \frac{1}{2}(v_1')^2 - v_1''\theta_0 + u_o'(y) - v_o''(y)\theta_0,$
  $u_o \in \dot{H}^1_\#([0,1]),\ v_0 \in \dot{H}^2_\#([0,1])$. Take
  $u_o = v_1'' \int_0^y \theta_0(s)ds, v_o = 0$. We obtain the usual von Kármán model. Other combinations?

Periodically wrinkled plate-Conclusion

- General approach: Take energy density function of the form $\hat{W}(\mathbf{F}) = W\big(\mathbf{F}(\mathbf{I} + h^{\alpha}\mathbf{A}_h(x))\big)$, where W satisfies the frame indiference and $\mathbf{A}_h \rightharpoonup\rightharpoonup \mathbf{A}_0$. α ? Constraints on $\mathbf{A}_h$? Non-local effects on $\mathbf{A}_h$?