Convergence Proofs of Domain Decomposition Algorithms

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Outline

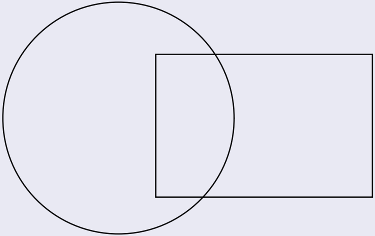
1. Introduction
2. Classical Schwarz
3. Schwarz with Robin
4. Convergence for classical Schwarz
5. Convergence for optimized Schwarz

Classical Schwarz method

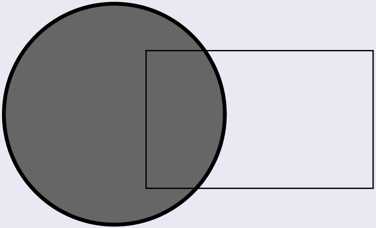
H. Schwarz

In 1860, H. Schwarz proved the existence of a solution for the Poisson equation in a domain being a combination of a circle and a rectangle.

$$\begin{aligned} -\triangle u &= f \quad && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega. \end{aligned}$$

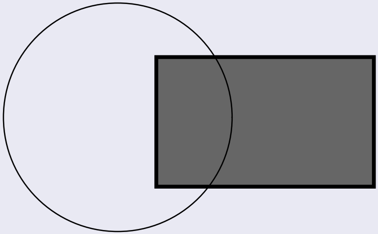


Schwarz's technique



$$\begin{cases} -\triangle u_1^1 = f & \text{in } \Omega_1, \\ v^1 = g & \text{on } \partial\Omega_1 \cap \partial\Omega, \\ v^1 = w^0 & \text{on } \Gamma_1 = \partial\Omega_1 \cap \Omega_2, \end{cases}$$

Schwarz's technique



$$\begin{cases} -\triangle w^1 = f & \text{in } \Omega_2, \\ w^1 = g & \text{on } \partial\Omega_2 \cap \partial\Omega, \\ w^1 = v^1 & \text{on } \Gamma_2 = \partial\Omega_2 \cap \Omega_1. \end{cases}$$

Schwarz's technique

Start with a value w^0, we introduce the two sequence $\{v^k\}_{k=1}^{\infty}$, $\{w^k\}_{k=0}^{\infty}$

$$\begin{cases} -\triangle u_1^k = f & \text{in } \Omega_1, \\ v^k = g & \text{on } \partial\Omega_1 \cap \partial\Omega, \\ v^k = w^{k-1} & \text{on } \Gamma_1 = \partial\Omega_1 \cap \Omega_2, \end{cases}$$

$$\begin{cases} -\triangle w^k = f & \text{in } \Omega_2, \\ w^k = g & \text{on } \partial\Omega_2 \cap \partial\Omega, \\ w^k = v^k & \text{on } \Gamma_2 = \partial\Omega_2 \cap \Omega_1. \end{cases}$$

The parallel algorithm of P. L. Lions, 1987

Start with the initial guess w^0, v^0 we introduce the two sequence $\{v^k\}_{k=0}^{\infty}$, $\{w^k\}_{k=0}^{\infty}$

$$\begin{cases} -\triangle u_1^k = f & \text{in } \Omega_1, \\ v^k = g & \text{on } \partial\Omega_1 \cap \partial\Omega, \\ v^k = w^{k-1} & \text{on } \Gamma_1 = \partial\Omega_1 \cap \Omega_2, \end{cases}$$

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Schwarz methods with Robin

Optimized Schwarz methods

$$\begin{cases} -\Delta u_1^k = f & \text{in } \Omega_1, \\ u_1^k = g & \text{on } \partial\Omega_1 \cap \partial\Omega, \\ (\partial_x + p_1)u_1^k = (\partial_x + p_1)u_2^{k-1} & \text{on } \Gamma_1 = \partial\Omega_1 \cap \Omega_2, \end{cases}$$

$$\begin{cases} -\Delta u_2^k = f & \text{in } \Omega_2, \\ u_2^k = g & \text{on } \partial\Omega_2 \cap \partial\Omega, \\ (\partial_x + p_2)u_2^k = (\partial_x + p_2)u_1^{k-1} & \text{on } \Gamma_2 = \partial\Omega_2 \cap \Omega_1. \end{cases}$$

Optimized Schwarz methods converge faster than classical Schwarz methods

Example

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathbb{R} \times (0, T), \\ u(x, 0) = 0 & \text{on } \mathbb{R}. \end{cases}$$

The classical algorithm

$$\begin{cases} (\partial_t - \Delta)u_1^k = 0 & \text{in } (-\infty, L) \times (0,T), \\ u_1^k(L,.) = u_2^{k-1}(L,.) & \text{in } (0,T), \end{cases}$$

$$\begin{cases} (\partial_t - \Delta)u_2^k = 0 & \text{in } (0, \infty) \times (0,T), \\ u_2^k(0,.) = u_1^{k-1}(0,.) & \text{in } (0,T), \end{cases}$$

$$\begin{aligned} \mathfrak{F}u_1^k(x,\omega) &= \mathfrak{F}u_2^{k-1}(L,\omega)\exp(\sqrt{\frac{i\omega}{\nu}}(x-L)), \\ \mathfrak{F}u_2^k(x,\omega) &= \mathfrak{F}u_1^{k-1}(0,\omega)\exp(-\sqrt{\frac{i\omega}{\nu}}x), \\ \rho &= \left|\exp(-\sqrt{\frac{i\omega}{\nu}}L)\right|. \end{aligned}$$

The optimized algorithm

$$\begin{cases} (\partial_t - \Delta)u_1^k = 0 & \text{in } (-\infty, L) \times (0, T), \\ (\partial_x + \frac{p}{2\nu})u_1^k(L,.) = (\partial_x + \frac{p}{2\nu})u_2^{k-1}(L,.) & \text{in } (0, T), \end{cases}$$

$$\begin{cases} (\partial_t - \Delta)u_2^k = 0 & \text{in } (0, \infty) \times (0, T), \\ (\partial_x - \frac{p}{2\nu})u_2^k(0,.) = (\partial_x - \frac{p}{2\nu})u_1^{k-1}(0,.) & \text{in } (0, T), \end{cases}$$

$$\begin{aligned} \mathfrak{F}u_1^k(x,\omega) &= \left(\frac{2\nu}{\sqrt{4i\omega\nu} + p}\right)\mathfrak{F}u_2^{k-1}(L,\omega)\exp\left(\sqrt{\frac{i\omega}{\nu}}(x-L)\right), \\ \mathfrak{F}u_2^k(x,\omega) &= \left(-\frac{2\nu}{\sqrt{4i\omega\nu} + p}\right)\mathfrak{F}u_1^{k-1}(0,\omega)\exp\left(-\sqrt{\frac{i\omega}{\nu}}x\right), \\ \rho &= \left|\frac{2\sqrt{i\omega\nu} - p}{2\sqrt{i\omega\nu} + p}\exp\left(-\sqrt{\frac{i\omega}{\nu}}L\right)\right|. \end{aligned}$$

Maximum Principle, 1860

H. Schwarz

In 1860, H. Schwarz proved the existence of a solution for the Poisson equation in a domain being a combination of a circle and a rectangle by a maximum principle.

$$
\begin{aligned}
-\triangle u &= f \text{ in } \Omega, \\
u &= g \text{ on } \partial\Omega.
\end{aligned}
$$

Maximum Principle

M. Gander and H. Zhao, 2000

In 2000, Martin Gander and Hongkai Zhao considered the linear heat equation

$$\frac{\partial u}{\partial t} - \triangle u = f.$$

Orthogonal Projection Technique

P. L. Lions, 1987

In 1987, P. L. Lions introduced the Orthogonal Projection Technique for linear Poisson equation.

Orthogonal Projection Technique

In the same paper, P. L. Lions considered the Stoke problem

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega, \\ div\, u = 0 & \text{on } \partial\Omega. \end{cases}$$

Monotone Iterations

S.-H. Liu

Lui Shiu-Hong, 2003, 2007

$$\boxed{\frac{\partial u}{\partial t} - \triangle u + c(x,t)u = f(x,t,u),}.$$

Energy Estimates Method

J.-D Benamou, B. Deprès

In 1997, Jean-David Benamou, and Bruno Desprès introduce the Energy Estimate Method.

$$-\Delta u - \omega^2 u = 0.$$

Kimn's Technique

J.-H. Kimn, 2010

J.-H. Kimn considers Poisson equation

$$-\Delta u = f \text{ in } \Omega.$$

Kimn's Technique

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J.-H. Kimn considers Poisson equation

$$-\Delta u = f \text{ in } \Omega.$$

S. Loisel and D. B. Szyld, 2010

$$\begin{cases} -\nabla(a\nabla u) + cu = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where a is a C^1-function and c is positive and belongs to $L^\infty(\Omega)$.

Parabolic equations

The Settings

We consider the following parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \sum_{i,j=1}^{n} a_{i,j}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i}(x,t) \\ \quad +c(x,t)u(x,t) = F(x,t,u(x,t)), \text{ in } \Omega \times (0,\infty), \\ u(x,t) = g(x,t), \text{ on } \partial\Omega \times (0,\infty), \\ u(x,0) = g(x,0), \text{ on } \Omega, \end{cases}$$

where Ω is a bounded and smooth enough domain in $\mathbb{R}^n$.

Parabolic equations

The Settings

(A1) For all i, j in $\{1, \ldots, I\}$, $a_{i,j}(x,t) = a_{j,i}(x,t)$. There exist strictly positive numbers λ, Λ such that $A = (a_{i,j}(x,t)) \geq \lambda I$ in the sense of symmetric positive definite matrices. Moreover, $a_{i,j}(x,t) < \Lambda$ a.e. in $\Omega \times (0, \infty)$.

(A2) The functions $a_{i,j}$, b_i, c are functions in $L^\infty(\mathbb{R}^{n+1})$.

(A3) There exists $C > 0$, such that $\forall\, t \in \mathbb{R}, \forall\, x \in \mathbb{R}^n, |F(x,t,z) - F(x,t,z')| \leq C|z - z'|, \forall\, z, z' \in \mathbb{R}$.

Parabolic equations

The Settings

We divide the domain Ω into I smooth overlapping subdomains $\{\Omega_l\}_{l\in\{1,I\}}$, such that

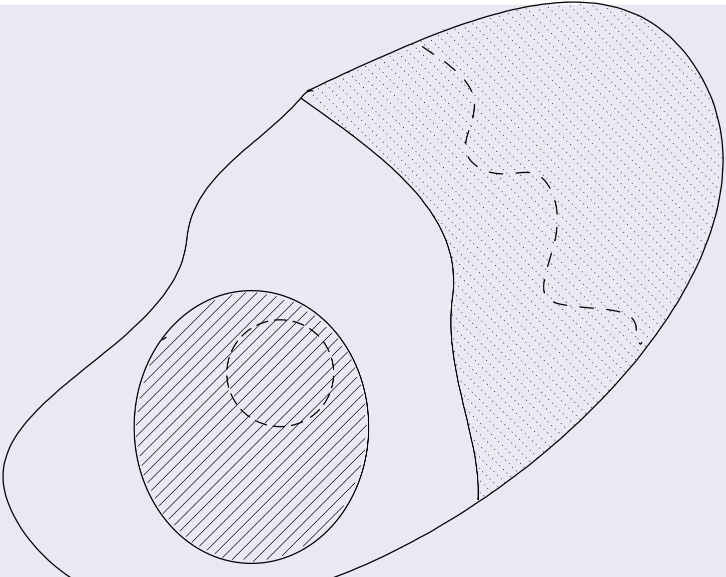
$$(\partial\Omega_l \backslash \partial\Omega) \cap (\partial\Omega_{l'} \backslash \partial\Omega) = \varnothing, \ \forall\, l, l' \in I, \ l \neq l',$$

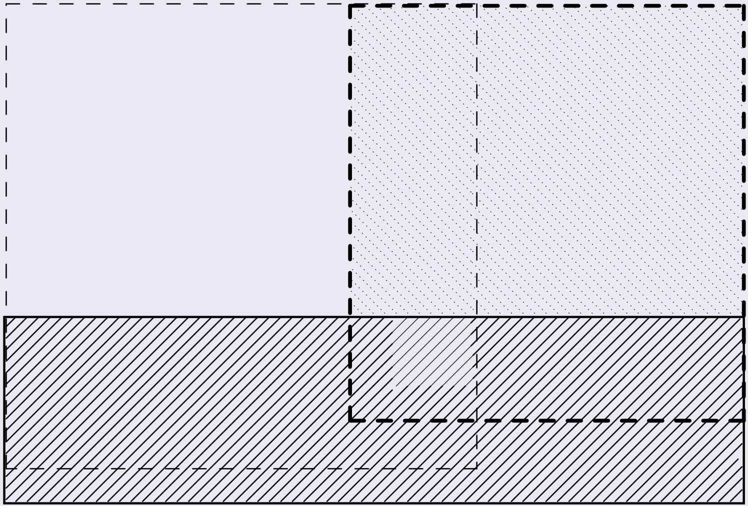
$$\forall l \in I, \forall l', l'' \in I_l, \ \Omega_{l'} \cap \Omega_{l''} = \varnothing,$$

and

$$\cup_{l=1}^{n} \Omega_l = \Omega.$$

Parabolic equations





Elliptic equations

General Setting: The same with parabolic equation $\rightarrow$ CONVERGENCE!

Main Idea of the Proof for Parabolic Equations

Define $e_l^k = u_l^k - u$ and

$$\Phi_l^k(x,t) := (e_l^k)^2 g(x) f(t),$$

$$\frac{\partial \Phi}{\partial t} - \sum_{i,j=1}^{n} a_{i,j} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} c_i(x,t) \frac{\partial \Phi}{\partial x_i} \leq (e_l^k)^2 \mathfrak{M},$$

where

$$\begin{aligned} \mathfrak{M} &= \left(- \sum_{i,j=1}^{n} a_{i,j} \frac{\frac{\partial^2 g}{\partial x_i \partial x_j}}{g} + \frac{f'}{f} + 2(C + ||c||_\infty) \right. \\ &\quad \left. + \sum_{i=1}^{n} b_i \frac{\frac{\partial g}{\partial x_i}}{g} + \sum_{i,j=1}^{n} 2a_{i,j} \frac{\frac{\partial g}{\partial x_j}}{g} \frac{\frac{\partial g}{\partial x_i}}{g} \right) fg. \end{aligned}$$

Main Idea of the Proof for Elliptic Equations

Define

$$\Phi_l^k(x,t) := (e_l^k)^2 g(x),$$

$$-\sum_{i,j=1}^{n} a_{i,j}\frac{\partial^2\Phi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} c_i(x,t)\frac{\partial\Phi}{\partial x_i} \leq (e_l^k)^2\mathfrak{M},$$

where

$$\mathfrak{M} = \left(\sum_{i,j=1}^{n} a_{i,j}\frac{\partial^2(g^{-1})}{\partial x_i \partial x_j} + 2(C + ||c||_\infty)(g^{-1}) - \sum_{i=1}^{n} b_i\frac{\partial(g^{-1})}{\partial x_i}\right) g^2.$$

Counter Example

We consider the following elliptic problem on the domain $\Omega = (0, L)$

$$\begin{cases} u'' - 3u' - 4u = f, & \text{in } (0, L), \\ u(0) = u(L) = 0, \end{cases}$$

where f belongs to $C^\infty([0, L])$.

Counter Example

$$\begin{cases} (u_1^k)'' - 3(u_1^k)' - 4u_1^k = f, \text{ in } (0, L_2), \\ u_1^k(0) = 0 \text{ and } (u_1^k)'(L_2) + pu_1^k(L_2) = (u_2^{k-1})'(L_2) + pu_2^{k-1}(L_2), \end{cases}$$

$$\begin{cases} (u_2^k)'' - 3(u_2^k)' - 4u_2^k = f, \text{ in } (L_1, L), \\ u_2^k(L) = 0 \text{ and } (u_2^k)'(L_1) - qu_2^k(L_1) = (u_1^{k-1})'(L_1) - qu_1^{k-1}(L_1), \end{cases}$$

where p, q are positive constants.

Counter Example

Denote the errors by e_1^k and e_2^k, we obtain

$$\begin{cases} (e_1^{k+1})'' - 3(e_1^{k+1})' - 4e_1^{k+1} = 0, \text{ in } (0, L_2), \\ e_1^{k+1}(0) = 0, (e_1^{k+1})'(L_2) + pe_1^{k+1}(L_2) = (e_2^k)'(L_2) + pe_2^k(L_2), \end{cases}$$

$$\begin{cases} (e_2^{k+1})'' - 3(e_2^{k+1})' - 4e_2^{k+1} = 0, \text{ in } (L_1, L), \\ e_2^{k+1}(L) = 0, (e_2^{k+1})'(L_1) - qe_2^{k+1}(L_1) = (e_1^k)'(L_1) - qe_1^k(L_1). \end{cases}$$

Counter Example

From the above equations, we get

$$e_1^{k+1} = A_{k+1}(\exp(4x) - \exp(-x)),$$

$$e_2^{k+1} = B_{k+1}(\exp(4(x - L)) - \exp(-(x - L))).$$

Counter Example

Setting

$$\tau = \left| \frac{A_{k+1} B_{k+1}}{B_k A_k} \right|,$$

we obtain

$$\begin{aligned} \tau \quad = \quad & \left| \frac{4\exp(5L_2) - \exp(5L) + p(\exp(5L_2) - \exp(5L))}{4\exp(5L_2) - 1 + p(\exp(5L_2) - 1)} \right| \\ & \times \left| \frac{4\exp(5L_1) - 1 - q(\exp(5L_1) - 1)}{4\exp(5L_1) - \exp(5L) - q(\exp(5L_1) - \exp(5L))} \right|. \end{aligned}$$

How to Fix the Situation?

Put a new parameter ρ into the Robin tranmissions conditions

$$\mathbb{B}^{\rho}_{l,l'}v = \sum_{i,j=1}^{n} a_{i,j}\frac{\partial v}{\partial x_i}n_{l,l',j} + \rho p_{l,l'}v.$$

How to Fix the Situation?

Put a new parameter ρ into the Robin tranmissions conditions

$$\mathbb{B}^{\rho}_{l,l'}v = \sum_{i,j=1}^{n} a_{i,j}\frac{\partial v}{\partial x_i} n_{l,l',j} + \rho p_{l,l'} v .$$

When ρ is large enough, the algorithm converges.

Main Idea of the Proof of Schwarz Methods with Robin Transmission Conditions for Parabolic Equations

Define $e_l^k = u_l^k - u$ and

$$\Phi_l^k(x) := \left(\int_0^\infty e_l^k \exp(-\alpha t) dt\right) g_l(x),$$

where g_l is a bounded function in $C^\infty(\mathbb{R}^n, \mathbb{R})$, g_l is greater than 1 and to be chosen later, and α is a postive constant large enough.

Main Idea of the Proof of Schwarz Methods with Robin Transmission Conditions for Parabolic Equations

Define $e_l^k = u_l^k - u$ and

$$\Phi_l^k(x) := \left(\int_0^\infty e_l^k \exp(-\alpha t) dt \right) g_l(x),$$

where g_l is a bounded function in $C^\infty(\mathbb{R}^n, \mathbb{R})$, g_l is greater than 1 and to be chosen later, and α is a postive constant large enough.

We get a new equation for Φ_l^k.

$$
\begin{aligned}
0 \;=\; & -\sum_{i,j=1}^{n} a_{i,j}\frac{\partial^2 \Phi_l^k}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i^l \frac{\partial \Phi_l^k}{\partial x_i} + C^l \Phi_l^k \\
& + \left\{ \int_0^\infty \left[\left(\frac{\alpha}{2} + c \right) e_l^k - F(u_l^k) + F(u) \right] \exp(-\alpha t) dt \right\} g_l .
\end{aligned}
$$

$$
\beta_l \mathfrak{B}_{l,l'}(\Phi_l^k) = \mathfrak{B}_{l,l'}(\Phi_{l'}^{k-1})
$$

Main Idea of the Proof of Schwarz Methods with Robin Transmission Conditions for Elliptic Equations

Define

$$\Phi_l^k(x,t) := e_l^k(x) g_l(x),$$

and derive a new equation for $\Phi_l^k(x,t)$.

$$
\begin{aligned}
0 \; = \; & -\sum_{i,j=1}^{n} a_{i,j} \frac{\partial^2 \Phi_l^k}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i^l \frac{\partial \Phi_l^k}{\partial x_i} \\
& + \left(\sum_{i,j=1}^{n} a_{i,j} \frac{\frac{\partial^2 g_l^k}{\partial x_i \partial x_j}}{g_l^k} - \sum_{i,j=1}^{n} a_{i,j} \frac{2 \frac{\partial g_l^k}{\partial x_i} \frac{\partial g_l^k}{\partial x_j}}{(g_l^k)^2} \right. \\
& \left. - \sum_{i=1}^{n} b_i \frac{\frac{\partial g_l^k}{\partial x_i}}{g_l^k} + (c - \overline{F})(g_l^k)^{-1} \right) \Phi_l^k .
\end{aligned}
$$

$$
\color{red}{\beta_l} \color{black} \mathfrak{B}_{l,l'}(\Phi_l^k) = \mathfrak{B}_{l,l'}(\Phi_{l'}^{k-1})
$$

THANK YOU FOR YOUR ATTENTION!