Contact Solutions for Nonlinear Systems of PDE and Applications to the ∞-Laplacian

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“Nonlinear Systems of Elliptic Partial Differential Equations”

submitted at the Department of Mathematics, University of Athens, Greece.

What is this talk about?

A new systematic theory of non-differentiable solutions which applies to fully nonlinear PDE systems.

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The ∞-Laplacian: if $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$,

(tensorial form:)

$$\Delta_\infty u \ := \ Du \otimes Du : D^2 u \ = \ 0$$

(index form:)

$$D_i u_\alpha \, D_j u_\beta \, D^2_{ij} u_\beta \ = \ 0$$

$1 \leq i, j \leq n, \ \ 1 \leq \alpha, \beta \leq N.$

(condensed form:)

$$\Delta_\infty u \ = \ D\left(\frac{1}{2}|Du|^2\right) \ Du$$

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$$\Delta_\infty u \;=\; D\left(\frac{1}{2}|Du|^2\right) Du$$

Δ_∞: Quasilinear, 2nd order, Degenerate Elliptic operator in Non-Divergence form.

Δ_∞ arises

- As an "Euler-Lagrange" PDE in Calculus of Variations in L^∞:

$$E_\infty(u,\Omega) \;=\; \operatorname{ess\,sup}_\Omega |Du|$$

  for Absolute Minimizers, a version of local minimizers

- As the "limit" of p-Laplacian $\Delta_p u = \mathrm{Div}(|Du|^{p-2}Du)$ as $p \to \infty$:

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- Implicitly in Geometric Evolution Problems (level-set approach):

$$u_t \;=\; \Delta u \;-\; \frac{\Delta_\infty u}{|Du|^2}$$

(term of the 1-Laplacian $\mathrm{Div}(Du/|Du|)$)

- In Game Theory,
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The general Aronsson PDE system

If $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$,

$$\mathcal{A}[u] \;:=\; H_P(\cdot, u, Du)D\big(H(\cdot, u, Du)\big) \;=\; 0.$$

Solutions arise as Absolute Minimizers of the supremal functional

$$E_\infty(u, \Omega) \;=\; \operatorname*{ess\,sup}_{x\in\Omega} H\big(x, u(x), Du(x)\big)$$

placed in $L^\infty(\mathbb{R}^n)^N$.

If $H = H(Du)$, then

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A startling problem: singular "solutions" of Δ_∞

The scalar Δ_∞ for $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ well studied in the last 20 years.

We want to solve the problem of existence of $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$:

$$\begin{cases} \Delta_\infty u \;=\; 0, & \text{in } \Omega \subset\subset \mathbb{R}^n, \\ u \;=\; g & \text{on } \partial\Omega, \quad g \text{ Lipschitz.} \end{cases}$$

The vector Δ_∞ for $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ can not be studied rigorously!

Example. Let $u : \mathbb{R} \longrightarrow \mathbb{R}^2$ be the curve:

$$u(x) \;:=\; \int_0^x \Big(\cos\big(K(t)\big), \sin\big(K(t)\big)\Big)^\top dt$$

where $K \in C^0(\mathbb{R})$. Then, u is Eikonal:

$$|Du|^2 \;=\; 1.$$

Should be "∞-Harmonic":

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However, for appropriate K:

- $\nexists$ D^2u anywhere on $\mathbb{R}$!
- $\nexists$ D^2u as a Radon measure !
- $\exists$ D^2u only in $\mathcal{D}'$!

Generally, the vectorial Δ_∞:

- Δ_∞ has no classical solutions,
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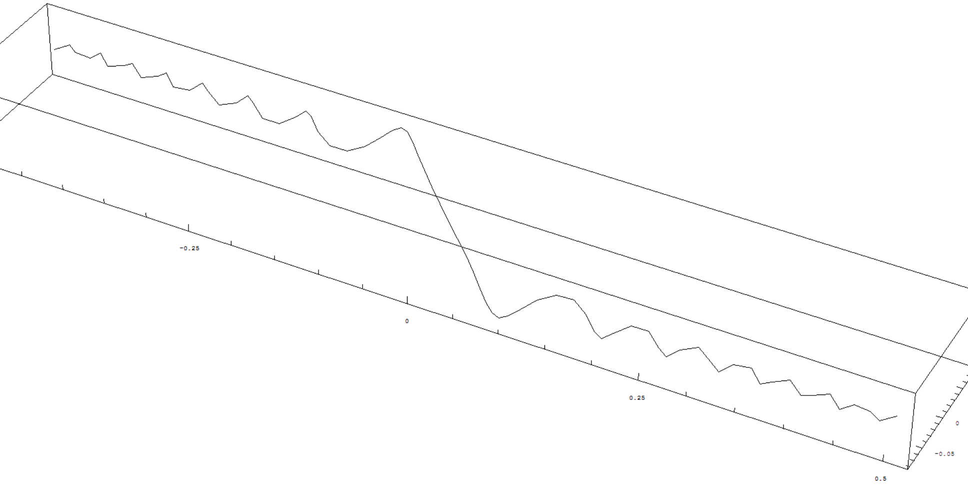

Simulation of the ∞-Harmonic curve $u(x) = \int_0^x \Big(\cos\big(K(t)\big), \sin\big(K(t)\big)\Big)^\top dt,\ u : \mathbb{R} \longrightarrow \mathbb{R}^2.$

A scalar digression: Δ_∞ for $N = 1$ & Viscosity Solutions

Scalar PDE ($N = 1$): if $u : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\Delta_\infty u \; = \; D_i u \, D_j u \, D^2_{ij} u = 0$.

Well studied in the context of Viscosity Solutions ('90 - , existence and uniqueness for the Dirichlet problem, C^1-regularity, ...).

Example (Aronsson '1984). Let $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be:

$$u(x, y) \; := \; |x|^{4/3} \; - \; |y|^{4/3}.$$

Then, u is $C^{1,\frac{1}{3}}$ and solves $\Delta_\infty u = 0$ only in the viscosity sense ($\nexists\ D^2 u$ on the axes).

Example (K. '2010). Let $H \in C^1(\mathbb{R}^n)$ be constant along $[a, b]$. Then:

$$u(x) \; := \; \frac{b + a}{2}^{\top} x \; + \; f\left(\frac{b - a}{2}^{\top} x\right).$$

is for all $f \in W^{1,\infty}_{loc}(\mathbb{R})$, $\|f\|_{L^\infty(\mathbb{R})} < 1$ a non-C^1 solution of $\mathcal{A}[u] = 0$ in the viscosity sense ($\nexists\ D^2 u$ anywhere).

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$$u(x) \;:=\; \frac{b+a}{2}^{\top} x \;+\; f\left(\frac{b-a}{2}^{\top} x\right).$$

is for all $f \in W^{1,\infty}_{loc}(\mathbb{R})$, $\|f\|_{L^\infty(\mathbb{R})} < 1$ a non-C^1 solution of $\mathcal{A}[u] = 0$ in the viscosity sense ($\nexists\ D^2 u$ anywhere).

Idea behind Viscosity Solutions $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ of

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$$\begin{aligned} Du(x) \ &= \ D\psi(x), \\ D^2u(x) \ &\leq \ D^2\psi(x). \end{aligned}$$

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Main Subject: Contact Solutions for systems of PDEs

Exists a theory of non-differentiable solutions which applies to fully nonlinear systems of PDEs

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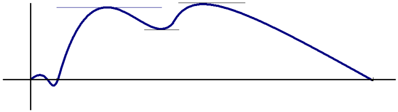
Functional notion, not pointwise!

Fact: If $u \in C^1_0(\Omega)$ and $u \not\equiv 0$, then u has interior extremum and $Du = 0$ there.



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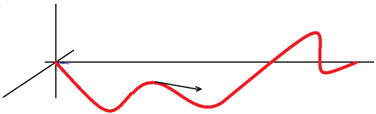

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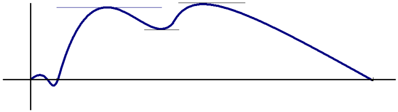

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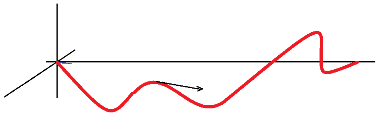

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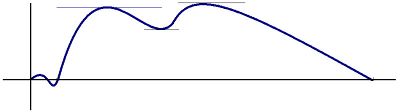

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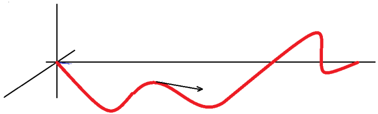

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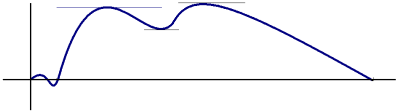

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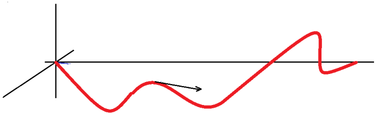

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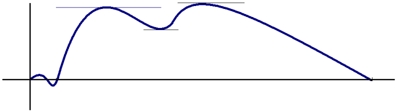

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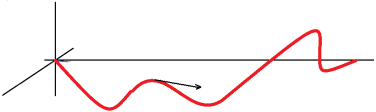

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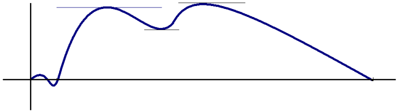

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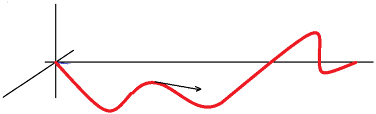

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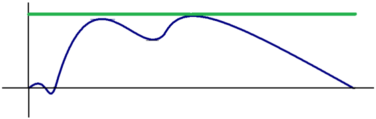

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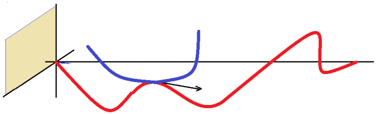

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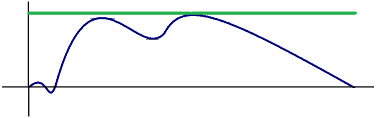

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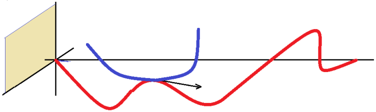

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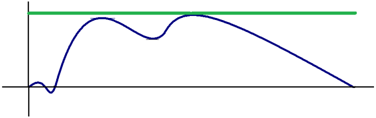

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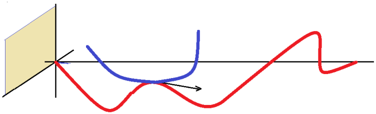

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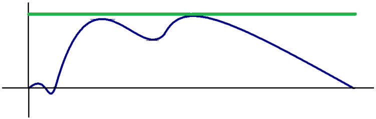

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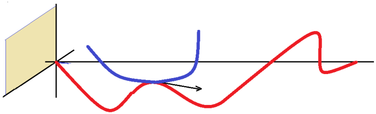

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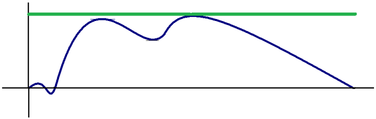

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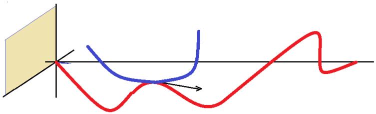

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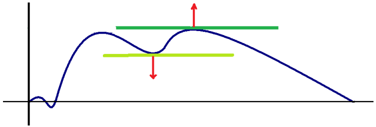

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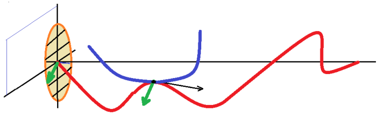

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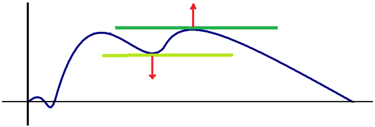

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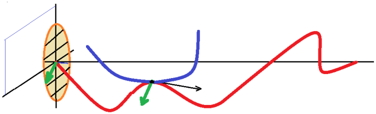

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Main Subject: Contact Solutions for systems of PDEs

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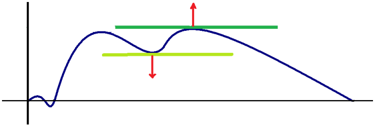

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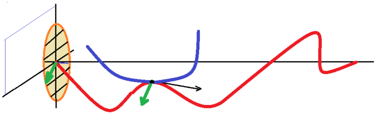

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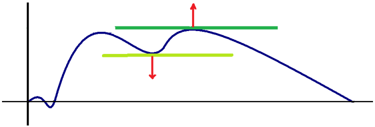

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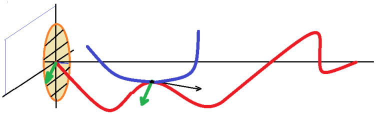

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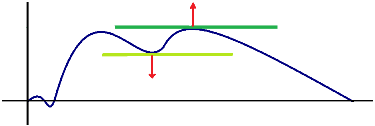

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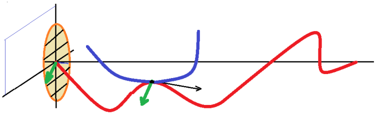

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Main Subject: Contact Solutions for systems of PDEs

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The Order has a consequence:

The Extremality imposes partial regularity of lower-dimensional projections of the function u:

ψ has 1st order Contact with u at $x \Longrightarrow$

u has a $C^{1/2}$ regular codimension-1 projection near x

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Sets of Pointwise Generalized Derivatives.

Definition(2nd Order Contact Jets). Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$, $x \in \mathbb{R}^n$ and $\xi \in \mathbb{S}^{N-1}$. Then,

$$(\mathrm{P}, \mathbf{X}) \in J^{2,\xi}u(x)$$

iff

$$\xi \vee \left[u(z) - u(x) - \mathrm{P}(z-x) - \frac{1}{2}\mathbf{X} : (z-x) \otimes (z-x) \right] \leq o(|z-x|^2)$$

as $z \to x$.

Here "$\leq$" is taken in Symmetric tensors of $\mathbb{R}^N \otimes \mathbb{R}^N$ and

$$a \vee b \ := \ \frac{1}{2}\big(a \otimes b + b \otimes a\big).$$

Equivalence:

$$J^{2,\xi}u(x) \ = \ \Big\{ \big(D\psi(x), D^2\psi(x)\big) \ : \ \psi \text{ is a Contact } \xi\text{-function} \Big\}.$$

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Main Subject: Contact Solutions for systems of PDEs

The PDE notions for systems:

Definition(Contact Solutions). Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$. Then u solves

$$F(\cdot, u, Du, D^2 u) \ = \ 0$$

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for all $x \in \mathbb{R}^n$, $\xi \in \mathbb{S}^{N-1}$.

Reduction to the scalar case $N = 1$: $\mathbb{S}^0 = \{-1, +1\}$ and

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Main Subject: Contact Solutions for systems of PDEs

Contact Solutions completely compatible with Classical Solutions for Degenerate Elliptic systems:

Definition. The system $F(D^2u) = 0$ is Degenerate Elliptic when $[F(\mathbf{X}) - F(\mathbf{Y})]_\alpha [\mathbf{X} - \mathbf{Y}]_{\alpha ij} w_i w_j \ \geq \ 0, \quad w \in \mathbb{R}^n$, i.e.

$$[F(\mathbf{X}) - F(\mathbf{Y})]^\top [\mathbf{X} - \mathbf{Y}] \ \geq \ 0$$

Example. (Quasi)linear case: $\mathbf{A} : D^2u = 0$, i.e.

$$\mathbf{A}_{\alpha i \beta j} : D^2_{ij} u_\beta \ = \ 0.$$

D.E. $\iff \mathbf{A} : (\eta \otimes w) \otimes (\eta \otimes w) \geq 0 \iff \begin{cases} \mathbf{A} \geq_\otimes 0 \\ \text{rank-1 positivity.} \end{cases}$

Example. A Fully nonlinear elliptic system:

$$F_\alpha(\cdot, u, Du, \sigma(D^2u_\alpha)) = 0.$$

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Example. (Quasi)linear case: $\mathbf{A} : D^2u = 0$, i.e.

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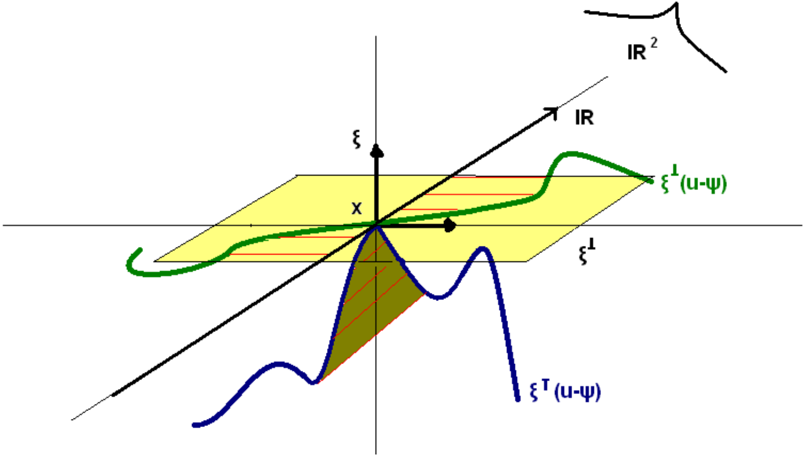
Contact Jets *implicitly* Equivalent to Contact Functions.



$\xi^\top(u-\psi)$ has vanishing max at x

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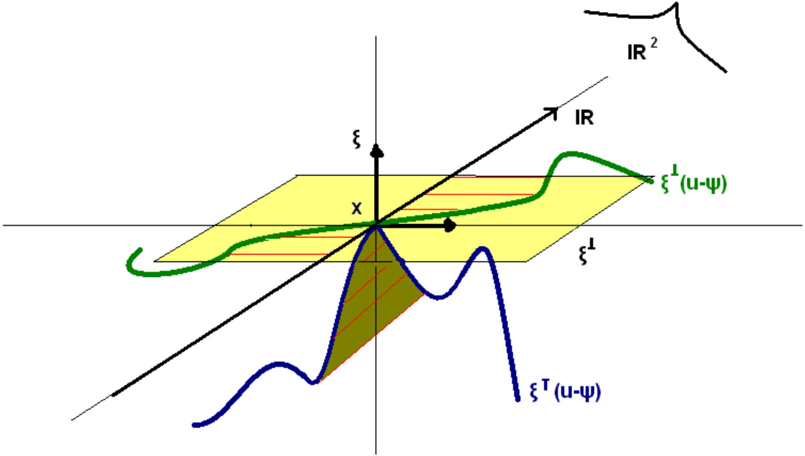

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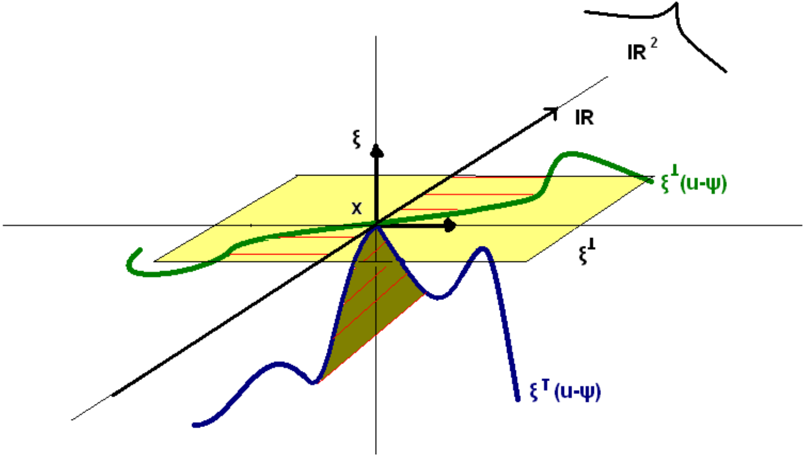

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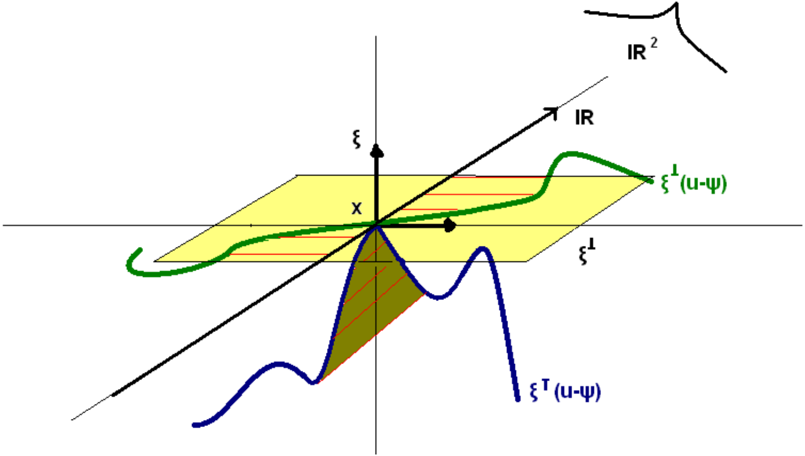

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- Equivalence between J^2 and Extremality notions is deeper (Extremality contains partial regularity info, J^2 does not).
- Extremality is characterized by the "Contact" Principle calculus: if $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$, $\exists Du(x), D^2u(x)$ & $\psi \in C^2(\mathbb{R}^n)^N$

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- Similarities with scalar case formal. Finer structure exists:

$F(\cdot, u, Du) = 0$ requires $C^{1/2}$ codimension-1 partial regularity,
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- Fundamental Solutions of scalar Δ_∞ for $N = 1$: Cones.
- Fundamental Solutions of vector Δ_∞ for $N > 1$: Generalized "Twisted Cones":

Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be radial. Then, $\Delta_\infty u = 0$ iff

$$u(z) \ := \ u_0 + L \int_0^{|z-x|} \nu(t)\, dt,$$

$u_0 \in \mathbb{R}^N$, $L \geq 0$ $\nu : (0, \infty) \longrightarrow \mathbb{S}^{N-1}$ curve in the sphere.

Fundamental Solutions of Δ_∞ are non-differentiable (at the "vertex" x) Contact Solutions of the Eikonal PDE:

$$|Du|^2 - L^2 \ = \ 0$$

If $N = 1$ then $\nu \equiv \pm 1$ and $u(z) = u_0 \pm L|z - x|$.
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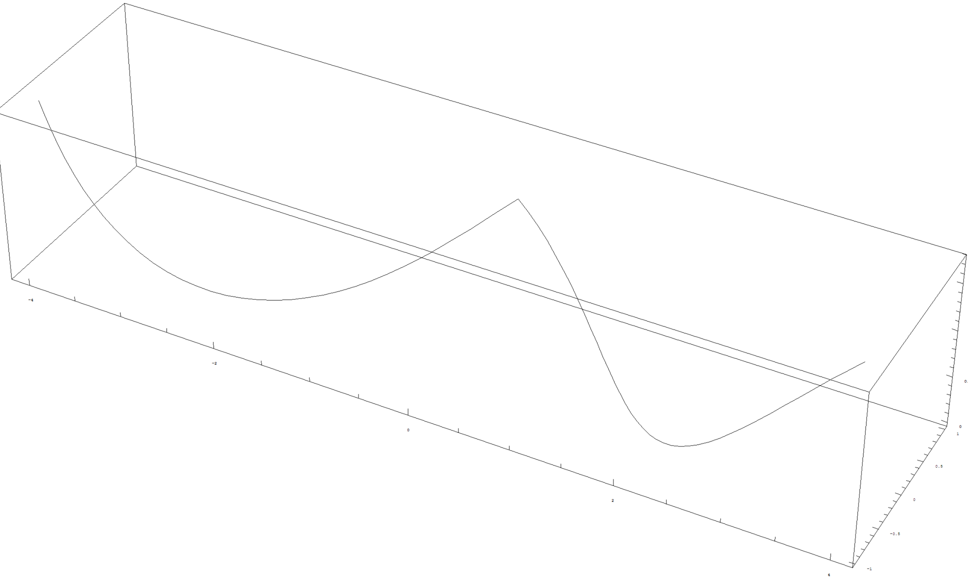

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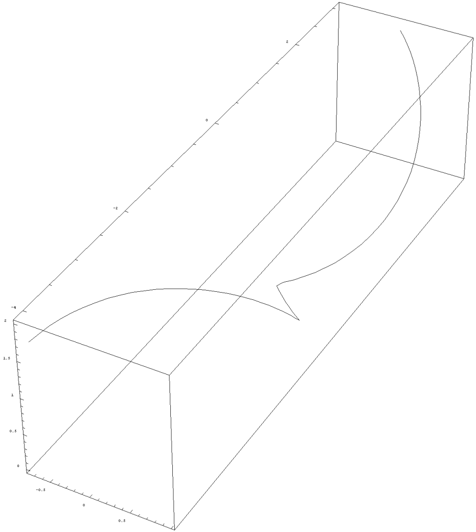

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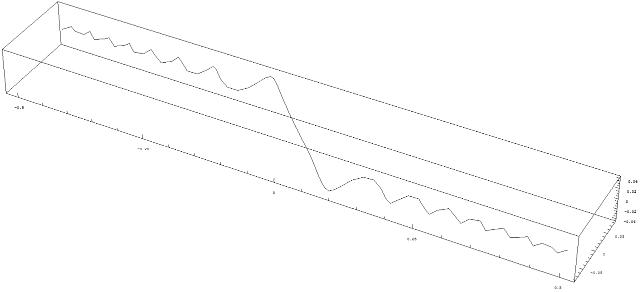
Application 2: A class of $C^{1,\frac{1}{2}+}$ ∞-Harmonic Functions

If $w \in \mathbb{R}^n$, $\nu : (0,\infty) \longrightarrow \mathbb{S}^{N-1}$ Lipschitz curve in the sphere, $K \in C^{\frac{1}{2}+}(\mathbb{R})$, then

$$u(z) \ := \ \int_0^{w^\top z} \nu\big(K(t)\big)\, dt$$

defines a Contact solution of $\Delta_\infty u = 0$ of regularity $C^{1,\frac{1}{2}+}(\mathbb{R})^N$.



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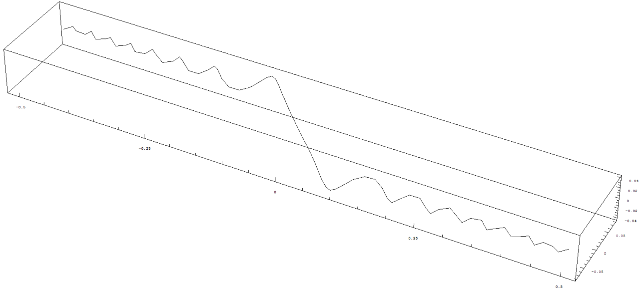

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Application 3: Pseudo-Gradient Flows for Δ_∞

Smooth Case. Let $u \in C^2(\mathbb{R}^n)^N$. Then, $\Delta_\infty u = 0$ iff $\forall x \in \mathbb{R}^n$, $\xi \in \mathbb{S}^{N-1}$ $\exists$ maximal curve

$$\begin{cases} \dot\gamma(t) = \xi^\top Du(\gamma(t)), \\ \gamma(0) = x, \end{cases}$$

such that

$$\begin{cases} \big|Du(\gamma(t))\big| = \big|Du(x)\big|, \quad t \in \mathbb{R}, \\ \quad t \mapsto \xi^\top u(\gamma(t)) \text{ increasing.} \end{cases}$$

Follows form

$$\frac{1}{2}\frac{d}{dt}\Big(\big|Du(\gamma(t))\big|^2\Big) \; = \; \xi^\top \Delta_\infty u(\gamma(t)),$$

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$$\frac{d}{dt}\Big(\eta^\top u(\gamma(t))\Big) \;=\; \Big(\xi \otimes \eta : Du(Du)^\top\Big)(\gamma(t)),$$

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General Case. Let $u \in C^0(\mathbb{R}^n)^N$. Then,

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Existence of ∞-Harmonic Vector Functions with prescribed boundary values:

- Problem: Exists a Lipschitz Contact Solution $u : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ to the Dirichlet Problem for the ∞-Laplacian

$$\begin{cases} \Delta_\infty u = 0, & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

for $\Omega \subset\subset \mathbb{R}^n$ and g Lipschitz.

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Application 4: In Progress!

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The End

THANK YOU !!!