Some Inverse Problems for Stochastic Partial Differential Equations

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Partial Differential Equations, Shape Optimization and Numerics

Aug 2011, Benasque

Stochastic partial differential equations (PDEs for short) are used to describe a lot of random phenomena appeared in physics, chemistry, biology, control theory and so on. In many situations, stochastic PDEs are more realistic mathematical models than the deterministic ones. Nevertheless, compared to the deterministic setting, there exist a very limited works addressing inverse problems for stochastic PDEs. Some interesting results are presented in (G. Bao, S.-N. Chow, P. Li and H. Zhou, 2010), (L. Cavalier and A. Tsybakov, 2002) and (I.A. Ibragimov and R.Z. Khas'minskii, 1999). However, as far as I know, there is no paper considering the inverse problem for stochastic wave equations and stochastic heat equations.

In this talk, I will give some results for some inverse problems for stochastic wave equations and stochastic heat equations.

Let $T > 0$, $G \in \mathbb{R}^n$ $(n \in \mathbb{N})$ be a given bounded domain with an C^2 boundary Γ. Let Γ_0 be a suitable chosen nonempty subset of Γ, whose definition will be given later.

Fix a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, on which a one dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$ is defined.

Consider the following stochastic wave equation:

$$
\begin{cases}
dz_t - \Delta z dt = \Big[b_1 z_t + (b_2, \nabla z) + b_3 z\Big] dt + (b_4 z + g) dB(t) & \text{in } (0,T)\times G,\\
z = 0 & \text{on } (0,T)\times\Gamma,\\
z(0) = z_0, z_t(0) = z_1 & \text{in } G,
\end{cases}
\tag{1}
$$

Let $p \in [n, \infty]$. Let

$$
\begin{aligned}
&b_1 \in L^\infty_{\mathcal{F}}(0,T; L^\infty(G)), \qquad && b_2 \in L^\infty_{\mathcal{F}}(0,T; L^\infty(G;\mathbb{R}^n)),\\
&b_3 \in L^\infty_{\mathcal{F}}(0,T; L^p(G)), \qquad && b_4 \in L^\infty_{\mathcal{F}}(0,T; L^\infty(G)).
\end{aligned}
\tag{2}
$$

$(z_0, z_1) \in L^2(\Omega, \mathcal{F}_0, P; H^1_0(G)\times L^2(G))$ and $g \in L^2_{\mathcal{F}}(0,T; L^2(G))$ are unknowns.

Put

$$H_T \triangleq L^2_{\mathcal{F}}(\Omega; C([0,T]; H^1_0(G))) \bigcap L^2_{\mathcal{F}}(\Omega; C^1([0,T]; L^2(G))). \tag{3}$$

It is clear that H_T is a Banach space with the canonical norm. Under suitable assumptions (the assumptions in this paper are enough), for any given (z_0, z_1) and g, one can show that the equation (1) admits one and only one solution $z = z(z_0, z_1, g)(t, x, \omega) \in H_T$. We will also denote by $z(z_0, z_1, g)$ or $z(z_0, z_1, g)(t)$ the solution of (1).

The random force $\int_0^t g dB$ is assumed to cause the random vibration starting from some initial state (z_0, z_1). Roughly speaking, our aim is to determine the unknown random force intensity g and the unknown initial displacement z_0 and initial velocity z_1 from the (partial) boundary observation $\frac{\partial z}{\partial \nu}\Big|_{(0,T)\times\Gamma_0}$ and the measurement on the terminal displacement $z(T)$, where $\nu = \nu(x)$ denotes the unit outer normal vector of G at $x \in \Gamma$, and Γ_0 is a suitable open subset (to be specified later) of Γ. More precisely, we are concerned with the following global uniqueness problem:

Do $\frac{\partial z}{\partial \nu}(z_0, z_1, g)\Big|_{(0,T)\times\Gamma_0} = 0$ *and* $z(z_0, z_1, g)(T) = 0$ *in* G*,* P*-a.s. imply that* $g = 0$ *in* $(0, T) \times G$ *and* $z_0 = z_1 = 0$ *in* G*,* P*-a.s.?*

In the deterministic setting, there exist numerous literatures addressing the inverse problem of PDEs. A typical (deterministic) inverse problem close to the above one is as follows: Fix suitable (known) functions $a(\cdot,\cdot)$ and $f_1(\cdot,\cdot)$ satisfying $\min\limits_{(t,x)\in(0,T)\times G}|f_1(t,x)|>0$, and consider the following hyperbolic equation:

$$\begin{cases} z_{tt}-\Delta z=a(t,x)z+f_1(t,x)f_2(x) & \text{in } (0,T)\times G, \\ z=0 & \text{on } (0,T)\times \Gamma, \\ z(0)=0,\ \ z_t(0)=z_1 & \text{in } G. \end{cases} \tag{4}$$

In (4), both z_1 and f_2 are unknown and one expects to determine them through the boundary observation $\frac{\partial z}{\partial \nu}\Big|_{(0,T)\times\Gamma_0}$.

As shown by Yamamoto and Zhang, by assuming suitable regularity on functions $a(\cdot,\cdot)$, $f_i(\cdot,\cdot)$ $(i=1,2)$ and $z_1(\cdot)$, and using the transformation

$$y = y(t,x) = \frac{d}{dt}\left(\frac{z(t,x)}{f_1(t,x)}\right), \tag{5}$$

this inverse problem can be reduced to deriving the so-called observability for the following wave equation with memory

$$\begin{cases} y_{tt} - \Delta y = a_1 y_t + a_2 \cdot \nabla y + a_3 y \\ \qquad\qquad + \displaystyle\int_0^t \left[c_1(t,s,x)y(s,x) + c_2(t,s,x)\cdot\nabla y(s,x)\right] ds & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0,x) = \dfrac{z_1(x)}{f_1(0,x)}, \quad y_t(0,x) = f_2(x) - \dfrac{2\partial_t f_1(0,x)}{|f_1(0,x)|^2} z_1(x) & \text{in } G, \end{cases}$$

where $a_i(\cdot,\cdot)$ $(i=1,2,3)$ and $c_i(\cdot,\cdot,\cdot)$ $(i=1,2)$ are suitable functions.

Obviously, we have $\left.\dfrac{\partial y}{\partial \nu}\right|_{\Gamma_0} = 0.$

In (M. Yamamoto and X. Zhang, 2001), the authors proved

$$|y(0), y_t(0)|_{H^1(G)\times L^2(G)} \leq C\left|\frac{\partial y}{\partial \nu}\right|_{L^2(0,T;L^2(\Gamma_0))}.$$

Then they get $z_1 = f_2 = 0$ in G.

One will meet substantially new difficulties in the study of inverse problems for stochastic PDEs. For instance, unlike the deterministic PDEs, the solution of a stochastic PDE is usually non-differentiable with respect to the variable with noise (say, the time variable considered in this paper). Also, the usual compactness embedding result does not remain true for the solution spaces related to stochastic PDEs. These new phenomenons lead that some useful methods for solving inverse problems for deterministic PDEs cannot be used to solve the corresponding inverse problems in the stochastic setting. Especially, one can see that none of the methods for solving the above inverse problem for the equation (4) can be easily adopted to solve our inverse problem for the stochastic hyperbolic equation (1), even if g is assumed to be of the form

$$g(t,x,\omega) = g_1(t,\omega)g_2(x), \qquad \forall\, (t,x,\omega) \in (0,T) \times G \times \Omega, \tag{6}$$

with a known stochastic process $g_1(\cdot,\cdot) \in L^2_{\mathcal{F}}(0,T)$ and an unknown function $g_2(\cdot) \in L^2(G)$. For these reasons, it is necessary to develop new methodology and technique for treating inverse problems for stochastic PDEs.

We solve the above formulated inverse problem for the equation (1) by employing a suitable Carleman estimate. To the best of our knowledge, the only published reference addressing the Carleman estimate for stochastic hyperbolic equations is (X. Zhang, 2009). In that paper, under suitable assumptions, the following estimate was proved for the solution z of (1):

$$|(z(T), z_t(T))|_{L^2(\Omega,\mathcal{F}_T,P;H_0^1(G)\times L^2(G))} \le C\left[\left|\frac{\partial z}{\partial \nu}\right|_{L^2_{\mathcal{F}}(0,T;L^2(\Gamma_0))} + |g|_{L^2_{\mathcal{F}}(0,T;L^2(G))}\right]. \tag{7}$$

Noting that g appears in the right hand side of (7), and the left hand side of (7) is $(z(T), z_t(T))$, which cannot be used to estimate (z_0, z_1) owing to the time irreversible, therefore, inequality (7) does not apply to the inverse problem in this paper.

In order to solve our stochastic inverse problem, we need to establish a new Carleman estimate for (1) so that the source term g can be bounded above by the observed data. Hence, we need to avoid employing the usual $energy\ estimate$ because, when applying this estimate to (1), the source term g would appear as a bad term. Meanwhile, noting that we are also expected to identity the initial data, hence we need to bound above the initial data by the observed data, too. Because of this, we need to obtain the estimate on the initial data and source term in the Carleman inequality simultaneously. Therefore we cannot use the usual "Carleman estimate" $+$ "energy estimate" method (which works well for the deterministic wave equation) to derive the desired estimates. This is the main difficulty that we need to overcome for solving this problem.

Choose $x_0 \notin \overline{G}$ and let $\Gamma_0 \stackrel{\triangle}{=} \{x: (x-x_0)\cdot\nu \geq 0\}$.

Let $T>0$, $0<c_1<1$ and $a>1$ satisfy

$$\begin{cases} 1. \dfrac{4a}{8c_1^3+c_1^2}\min\limits_{x\in\overline{G}}|x-x_0|^2 > c_1^2T^2 > 4a\max\limits_{x\in\overline{G}}|x-x_0|^2; \\ 2. 4a-4c_1-1>0. \end{cases}$$

Let $l=\lambda\big[a|x-x_0|^2-c_1(t-T)^2\big]$ and $\theta=e^l$. The desired Carleman estimate for (1) is as follows.

Theorem 1. *There exists a constant $\tilde{\lambda}>0$ such that for any $\lambda\geq\tilde{\lambda}$ and any solution $z\in H_T$ of the equation (1) satisfying $z(T)=0$ in G, P-a.s., it holds that*

$$\begin{aligned} &\mathbb{E}\int_G \theta^2(\lambda|z_1|^2+\lambda|\nabla z_0|^2+\lambda^3|z_0|^2)dx+\lambda\mathbb{E}\int_0^T\int_G(T-t)\theta^2g^2dxdt \\ &\leq C\lambda\mathbb{E}\int_0^T\int_{\Gamma_0}\theta^2\Big|\frac{\partial z}{\partial\nu}\Big|^2d\Gamma dt. \end{aligned} \tag{8}$$

From the above Carleman estimate, we obtain the following result.

Theorem 2. *Assume that the solution z of (1) satisfies that $\frac{\partial z}{\partial \nu} = 0$ on $(0,T) \times \Gamma_0$ and $z(T) = 0$ in G, P-a.s. Then $g = 0$ in Q and $z_0 = z_1 = 0$ in G, P-a.s.*

Remark 1. *Similar to the inverse problem for (4), and stimulated by Theorem 2, it seems natural to expect a similar uniqueness result for the following equation*

$$\begin{cases} dz_t - \Delta z dt = (b_1 z_t + b_2 \cdot \nabla z + b_3 z + f)\, dt + b_4 z dB(t) & \text{in } (0,T) \times G, \\ z = 0 & \text{on } (0,T) \times \Gamma, \\ z(0) = z_0, \;\; z_t(0) = z_1 & \text{in } G, \end{cases} \tag{9}$$

in which z_0, z_1 and f are unknown and one expects to determine them through the boundary observation $\frac{\partial z}{\partial \nu}\big|_{(0,T)\times\Gamma_0}$ and the terminal measurement $z(T)$.

However the same conclusion as that in Theorem 2 does not hold true even for the deterministic wave equation. Indeed, we choose any $y \in C_0^\infty((0,T)\times G)$ *so that it does not vanish in some subdomain of* $(0,T)\times G$*. Putting* $f = y_{tt} - \Delta y$*, it is obvious that* y *solves the following wave equation*

$$\begin{cases} y_{tt} - \Delta y = f & \textit{in } (0,T)\times G, \\ y = 0, & \textit{on } (0,T)\times \Gamma, \\ y(0) = 0,\ y_t(0) = 0 & \textit{in } G. \end{cases}$$

One can show that $y(T) = 0$ *in* G *and* $\frac{\partial y}{\partial \nu} = 0$ *on* $(0,T)\times\Gamma$*. However, it is clear that* f *does not vanish in* $(0,T)\times G$*. This counterexample shows that the formulation of the stochastic inverse problem may differ considerably from its deterministic counterpart.*

Remark 2. *From the computational point of view, it is quite interesting to study the following stability problem (for the inverse stochastic wave equation (1)): Is the map*

$$\frac{\partial z}{\partial \nu}(z_0, z_1, g)\Big|_{(0,T)\times\Gamma_0} \times z(z_0, z_1, g)(T) \longrightarrow (z_0, z_1, g)$$

continuous in some suitable Hilbert spaces?

Instead, it is easy to show the following partial stability result, i.e., for any solution $z \in H_T$ of the equation (1) satisfying $z(T) = 0$ in G, P-a.s., it holds that

$$|(z_0, z_1)|_{L^2(\Omega,\mathcal{F}_0,P;H_0^1(G)\times L^2(G))} + |\sqrt{T-t}g|_{L^2_{\mathcal{F}}(0,T;L^2(G))} \le C\left|\frac{\partial z}{\partial \nu}\right|_{L^2_{\mathcal{F}}(0,T;L^2(\Gamma_0))}.$$

Especially, if $g(t,x,\omega) = g_1(t,\omega)g_2(x)$ (with $g_1(\cdot,\cdot) \in L^2_{\mathcal{F}}(0,T)$ and $g_2(\cdot) \in L^2(G)$), then the following estimate holds

$$|(z_0, z_1)|_{L^2(\Omega,\mathcal{F}_0,P;H_0^1(G)\times L^2(G))} + |g_2|_{L^2(G)} \le C\left|\frac{\partial z}{\partial \nu}\right|_{L^2_{\mathcal{F}}(0,T;L^2(\Gamma_0))}.$$

Let $x = (x_1, x') \in \mathbb{R}^n$ and $x' = (x_2, \cdots, x_n) \in \mathbb{R}^{n-1}$. Consider a special G as $G = (0, l) \times G'$, where $G' \subset \mathbb{R}^{n-1}$ be a bounded domain with a C^2 boundary. Denote also by Γ the boundary of G. We consider the following stochastic heat equation:

$$
\begin{cases}
dy - \Delta y = [(b_1, \nabla y) + b_2 y + h(t, x')R(t, x)]dt + b_3 y dB(t) & \text{in } (0, T) \times G, \\
y = 0 & \text{on } (0, T) \times \Gamma, \\
y(0) = 0 & \text{in } G.
\end{cases}
\tag{10}
$$

Here

$$
b_1 \in L^\infty_{\mathcal{F}}(0, T; W^{1,\infty}(G; \mathbb{R}^n)), \quad b_2 \in L^\infty_{\mathcal{F}}(0, T; W^{1,\infty}(G)), \quad b_3 \in L^\infty_{\mathcal{F}}(0, T; W^{2,\infty}(G)),
$$

and

$$
R \in C^2([0, T] \times \overline{G}), \quad h \in L^2_{\mathcal{F}}(0, T; H^1(G)).
$$

The inverse source problem studied here is as follows:

Let R be given and $t_0 > 0$. Determine the source function $h(t, x')$, $(t, x') \in (0, t_0) \times G'$, by means of the observation of $\left.\dfrac{\partial y}{\partial \nu}\right|_{[0,t_0]\times\Gamma}$.

We have the following uniqueness result about the above problem.

Theorem 3. *Assume that* $y \in L^2_{\mathcal{F}}(\Omega; C([0,T]; H^1_0(G)))$, $y_{x_1} \in L^2_{\mathcal{F}}(\Omega; C([0,T]; H^1_0(G)))$ *and*

$$|R(t,x)| \neq 0 \ \ \textit{for all } (t,x) \in [0,t_0] \times \overline{G}. \tag{11}$$

If

$$\frac{\partial y}{\partial \nu} = 0 \ \textit{on } [0,t_0] \times \Gamma, \ \ P\textit{-a.s.},$$

then

$$h(t,x') = 0 \ \ \textit{for all } (t,x') \in [0,t_0] \times G', P\textit{-a.s.}$$

Remark 3. *In the literature, determining a spacewise dependent source function for parabolic equations has been considered comprehensively. A classical result for the deterministic setting is as follows.*

Consider the following parabolic equation:

$$
\begin{cases} y_t - \Delta y = c_1 \nabla y + c_2 y + Rf & \text{in } (0,T) \times G, \\ y = 0 & \text{on } (0,T) \times \Gamma. \end{cases} \tag{12}
$$

Here c_1 and c_2 are suitable functions on $(0,T) \times G$. $R \in L^\infty((0,T) \times G)$, $R_t \in L^\infty((0,T) \times G)$ and $R(t_0, x) \neq 0$ in $\overline{G}$ for some $t_0 \in (0,T]$. $f \in L^2(G)$ is independent of t. O. Yu. Imanuvilov and M. Yamamoto proved the following result:

Assume that $y \in H^{1,2}((0,T) \times G)$ and $y_t \in H^{1,2}((0,T) \times G)$, then there exists a constant $C > 0$ such that

$$
|f|_{L^2(G)} \leq C \left(|y(t_0)|_{H^2(G)} + \left| \frac{\partial y_t}{\partial \nu} \right|_{L^2(0,T;L^2(\Gamma_0))} \right), \tag{13}
$$

where Γ_0 is any open subset of ∂G.

Compared with Theorem 3, inequality (13) *gives an explicit estimate for the source term by* $|y(t_0)|_{H^2(G)}$ *and* $\left|\frac{\partial y_t}{\partial \nu}\right|_{L^2(0,T;L^2(\Gamma_0))}$. *A key step in the proof of equality* (13) *is to differentiate the solution of* (12) *with respect to* t. *Unfortunately, the solution of* (10) *does not enjoy differentiability with respect to* t. *This leads to the difficulty to follow the proof for inequality* (13) *to solve our problem.*

The problem is solved by a global Carleman estimate for the following equation.

$$
\begin{cases} dy - \Delta y dt = [(a_1, \nabla y) + a_2 y + f]dt + (a_3 y + g)dB(t) & \text{in } (0,T)\times G, \\ y = 0 & \text{on } (0,T)\times \Gamma, \end{cases} \tag{14}
$$

where $f \in L^2_{\mathcal{F}}(0,T;L^2(G))$ *and* $g \in L^2_{\mathcal{F}}(0,T;H^1(G))$.

Let $\psi \in C^\infty(\mathbb{R})$ with $|\psi_t| \geq 1$, which is independent of the x-variable. Put

$$\varphi = e^{\lambda\psi} \text{ and } \theta = e^{s\varphi}. \tag{15}$$

We have the following result.

Theorem 4. *Let $\delta \in [0,T)$. For all $y \in L^2_{\mathcal{F}}(\Omega; C([0,T]; H^1_0(G)))$ solve equation* (12)*, there exists a $\lambda_1 > 0$ such that for all $\lambda \geq \lambda_1$, there exists an $s_0(\lambda_1) > 0$ so that for all $s \geq s_0(\lambda_1)$, it holds that*

$$\begin{aligned}
&\lambda\mathbb{E}\int_\delta^T\int_G \theta^2|\nabla y|^2 dxdt + s\lambda^2\mathbb{E}\int_\delta^T\int_G \varphi\theta^2 y^2 dxdt\\
&\leq C\,\mathbb{E}\Big[\theta^2(T)|\nabla y(T)|^2_{L^2(G)} + \theta^2(\delta)|\nabla y(\delta)|^2_{L^2(G)} + s\lambda\varphi(T)\theta^2(T)|y(T)|^2_{L^2(G)}\\
&\quad + s\lambda\varphi(\delta)\theta^2(\delta)|y(\delta)|^2_{L^2(G)} + \int_\delta^T\int_G \big(f^2 + g^2 + |\nabla g|^2\big)dxdt\Big].
\end{aligned} \tag{16}$$

From the above Carleman estimate, For arbitrary small $\varepsilon > 0$, we choose t_1 and t_2 such that

$$0 < t_0 - \varepsilon < t_1 < t_2 < t_0.$$

Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function such that $0 \leq \chi \leq 1$ and that

$$\chi = \begin{cases} 1, & t \leq t_1, \\ 0, & t \geq t_2. \end{cases} \tag{17}$$

Put $z=\frac{y}{R}$ and $w=\chi z_{x_1}$ (recall (11) for R) in $[0,t_2]\times G$, by virtue of y is a solution of equation (10), we know that w solves

$$\begin{cases} dw-\Delta wdt=\Big[((b_1)_{x_1},\chi\nabla z)+(b_1,\nabla w)+\Big(\Big(\dfrac{2\nabla R}{R}\Big)_{x_1},\chi\nabla z\Big)+\Big(\dfrac{2\nabla R}{R},\nabla w\Big)\\ \qquad\qquad +\Big(b_2+\dfrac{\Delta R}{R}-\dfrac{2(\nabla R,\nabla R)}{R^2}-\dfrac{R_t}{R}+\Big(\dfrac{\nabla R}{R},b_1\Big)\Big)_{x_1}\chi z\\ \qquad\qquad +\Big(b_2+\dfrac{\Delta R}{R}-\dfrac{2(\nabla R,\nabla R)}{R^2}-\dfrac{R_t}{R}+\Big(\dfrac{\nabla R}{R},b_1\Big)\Big)w\Big]dt\\ \qquad\qquad +(b_3)_{x_1}\chi z dB(t)+b_3 w dB(t)-\chi' z_{x_1}dt \qquad \text{in } [0,t_0]\times G,\\ w=0 \qquad \text{on } [0,t_0]\times\Gamma. \end{cases} \tag{18}$$

Applying Theorem 4 to equation (18) with $\psi(t) = -t$, we have

$$|w|^2_{L^2_{\mathcal{F}}(0,T;H^1(G))} \leq Ce^{2s(e^{-\lambda t_1} - e^{-\lambda(t_0-\varepsilon)})}|y_{x_1}|^2_{L^2_{\mathcal{F}}(0,T;L^2(G))}. \tag{19}$$

Recalling that $t_0 - \varepsilon < t_1$, we know $e^{-\lambda t_1} - e^{-\lambda(t_0-\varepsilon)} < 0$. Letting $s \to +\infty$, we obtain that

$$w = 0 \ \text{ in } \ (0, t_0 - \varepsilon) \times G, \ \ P\text{-a.s.}$$

This implies that

$$z = 0 \ \text{ in } \ (0, t_0 - \varepsilon) \times G, \ \ P\text{-a.s.},$$

which means

$$h = 0 \ \text{ in } \ (0, t_0 - \varepsilon) \times G', \ \ P\text{-a.s.}$$

Since $\varepsilon > 0$ is arbitrary, Theorem 3 is proved.

Thank you!

Gracias!

Merci!

Danke!