Control of the motion of a boat

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Partial Differential Equations, Optimal Design, and Numerics, Benasque 2011

Joint work with

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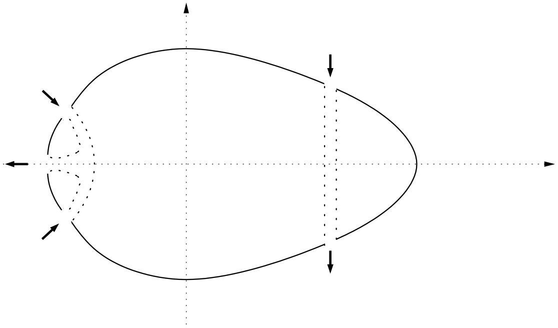
- We consider a rigid body $S \subset \mathbb{R}^2$ with one axis of symmetry, surrounded by a fluid, and which is controlled by two fluid flows, a longitudinal one and a transversal one.



Aims

- We aim to control the position and velocity of the rigid body by the control inputs. System of dimension 3+3 with a PDE in the dynamics. Control living in $\mathbb{R}^2$. No control objective for the fluid flow (exterior domain!!).
- Model for the motion of a boat with a longitudinal propeller, and a transversal one (thruster) in the framework of the theory of fluid-structure interaction problems. Rockets and planes also concerned.

Bowthruster

Longitudinal thruster

What is a fluid-structure interaction problem?

- Consider a rigid (or flexible) structure in touch with a fluid.
- The velocity of the fluid obeys **Navier-Stokes (or Euler)** equations in a **variable domain**
- The dynamics of the rigid structure is governed by **Newton** laws. Great role played by the pressure.
- Questions of interest: **existence** of (weak, strong, global) solutions of the system fluid+solid, **uniqueness**, **long-time behavior**, **control**, **inverse problems**, **optimal design**, ...

Some references

- Models for potential flows
  Kirchhoff, Kelvin, Lamb, Marsden (et al.) ,...
- Control problems for some models with potential flows
  N. Leonard [1997] N. Leonard, J. Marsden [1997],...
- Cauchy problem
  J. Ortega, LR, T. Takahashi [2005,2007], C. Rosier, LR [2009], O. Glass, F. Sueur, T. Takahashi [2012?],...
- Inverse Problems
  C. Conca, P. Cumsille, J. Ortega, LR [2008]
  C. Conca, M. Malik, A. Munnier [2010]

Main difficulties

1. The systems describing the motions of the fluid and the solid are nonlinear and **strongly coupled**; e.g., the **pressure of the fluid** gives rise to a force and a torque applied to the solid, and the fluid domain changes when the solid is moving.
2. The fluid domain $\mathbb{R}^N \setminus S(t)$ is an **unknown** function of time

Why to consider perfect fluids?

1. Euler equations provide a good model for the motion of boats or submarines in a reasonable time-scale.
2. **Explicit** computations may be performed with the aid of **Complex Analysis** when the flow is potential and 2D.
3. There is a **natural** choice for the boundary conditions $u_{rel} \cdot n = 0$ for Euler equations. For Navier-Stokes flows, one often takes $u_{rel} = 0$
4. The control theory of Euler flows is well understood (Coron, Glass).

System under investigation

$\Omega(t) = \mathbb{R}^2 \setminus S(t)$

Euler $\quad u_t + (u \cdot \nabla)u + \nabla p = 0, \ x \in \Omega(t)$

$$\mathrm{div}\, u = 0, \ x \in \Omega(t)$$

$$u \cdot \vec{n} = (h' + r(x-h)^{\perp}) \cdot \vec{n} + w(x,t), \ x \in \partial\Omega(t)$$

$$\lim_{|x| \to \infty} u(x,t) = 0$$

Newton $\quad m\, h''(t) = \int_{\partial\Omega(t)} p\, \vec{n}\, d\sigma$

$$J\, r' = \int_{\partial\Omega(t)} (x-h)^{\perp} \cdot p\vec{n}\, d\sigma$$

System supplemented with **Initial Conditions**, and with the value of the vorticity at the **incoming flow** (in $\Omega(t)$) for the uniqueness

System in a frame linked to the solid

After a change of variables and unknown functions, we obtain in $\Omega := \mathbb{R}^2 \setminus S(0)$

$$v_t + (v - l - ry^\perp) \cdot \nabla v + rv^\perp + \nabla q = 0, \; y \in \Omega$$
$$\mathrm{div}\; v = 0, \; y \in \Omega$$
$$v \cdot \vec{n} = (l' + ry^\perp) \cdot \vec{n} + \sum_{1 \leq j \leq 2} w_j(t)\chi_j(y), \; y \in \partial\Omega$$
$$\lim_{|y| \to \infty} v(y,t) = 0$$

$$m\, l'(t) = \int_{\partial\Omega} q\, \vec{n}\, d\sigma - mrl^\perp$$
$$J\, r' = \int_{\partial\Omega} qn \cdot y^\perp\, d\sigma$$

where $l(t) := Q(\theta(t))^{-1} h'(t)$, $r(t) = \theta'(t)$.

Potential flows

Assuming that the initial vorticity and circulation are null

$$\omega_0 := \mathrm{curl}\ u_0 \equiv 0, \qquad \Gamma_0 := \int_{\partial\Omega} u_0 \cdot n^\perp d\sigma = 0$$

and that the vorticity at the inflow part of $\partial\Omega$ is null

$$\omega(y,t) = 0 \quad \text{if} \;\; w_i(t)\chi_i(y) < 0 \qquad \text{for some } i = 1,2$$

then the flow remains potential, i.e. $v = \nabla\phi$ where ϕ solves

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega \times [0,T] \\ \dfrac{\partial\phi}{\partial n} = (l + ry^\perp) \cdot n + \displaystyle\sum_{i=1,2} w_i(t)\chi_i(y) & \text{on } \partial\Omega \times [0,T] \\ \lim_{|y|\to\infty} \nabla\phi(y) = 0 & \text{on } [0,T] \end{cases}$$

Potential flows (continued)

$v = \nabla\phi$ decomposed as

$$\nabla\phi = \sum_{i=1,2} l_i(t)\nabla\psi_i(y) + r(t)\nabla\varphi(y) + \sum_{i=1,2} w_i(t)\nabla\theta_i(y)$$

where the functions $\varphi,\ \psi_i$ and θ_i are harmonic on Ω and fulfill the following boundary conditions on $\partial\Omega$

$$\frac{\partial\varphi}{\partial n} = y^\perp \cdot n, \qquad \frac{\partial\psi_i}{\partial n} = n_i(y), \qquad \frac{\partial\theta_i}{\partial n} = \chi_i(y)$$

This gives the following expression for the pressure

$$q = -\{\sum_{i=1,2} l_i'\psi_i + r'\varphi + \sum_{i=1,2} w_i'\theta_i + \frac{|v|^2}{2} - l \cdot v - ry^\perp \cdot v\}$$

Plugging this expression in Newton's law yields a

Control system in finite dimension

$$\begin{aligned} h' &= \mathcal{Q} l \\ \mathcal{J} l' &= \mathcal{C} w' + B(l, w) \end{aligned}$$

where $\mathcal{Q} = diag(Q, 1)$, $h = [h_1, h_2, \theta]^T$, $l = [l_1, l_2, r]^T$, $w = [w_1, w_2]^T$ is the control input, and

$$\mathcal{J} = \begin{bmatrix} m + \int \psi_1 n_1 & 0 & 0 \\ 0 & m + \int \psi_2 n_2 & \int \psi_2 y^\perp \cdot n \\ 0 & \int \psi_2 y^\perp \cdot n & J + \int \varphi y^\perp \cdot n \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} -\int \theta_1 n_1 & 0 \\ 0 & -\int \theta_2 n_2 \\ 0 & -\int \theta_2 y^\perp \cdot n \end{bmatrix} = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \\ 0 & \tilde{c}_2 \end{bmatrix}$$

where $\int = \int_{\partial\Omega}$ and $B(l, w)$ is bilinear in (l, w)

Toy problem $w_2 = 0, \; h_2 = l_2 = 0$

$$(*) \begin{cases} h_1' = & l_1 \\ l_1' = & \alpha w_1' + \beta w_1 l_1 + \gamma w_1^2 \end{cases}$$

where

$$(\alpha, \beta, \gamma) := (m + \int_{\partial\Omega} \psi_1 n_1)^{-1} (\int_{\partial\Omega} \theta_1 n_1, \int_{\partial\Omega} \chi_1 \partial_1 \psi_1, \int_{\partial\Omega} \chi_1 \partial_1 \theta_1)$$

Claims

- If we add the equation ${w_1}' = v_1$ to $(*)$, the system with state (h_1, l_1, w_1) and input v_1 is NOT controllable!
- **In general** we cannot impose the condition $w_1(0) = w_1(T) = 0$ when $l_1(0) = l_1(T) = 0$ (i.e. fluid at rest at $t = 0, T$). Actually we can do that if and only if $\gamma + \alpha\beta = 0$.

Proof of the claims

Introduce $z_1 := l_1 - \alpha w_1$ From

$$l_1' = \alpha w_1' + \beta w_1 l_1 + \gamma w_1^2$$

we derive

$$z_1' = \beta w_1 z_1 + (\gamma + \alpha\beta) w_1^2$$

hence

$$z_1(t) = [z_1(0) + (\gamma + \alpha\beta) \int_0^t w_1^2(\tau) e^{-\int_0^\tau \beta w_1(s) ds} d\tau] e^{\int_0^t \beta w_1(s) ds}$$

Generic assumption

We shall assume that $c_1 \neq 0$ and that

$$\det \begin{bmatrix} c_2 & b_3 \\ \tilde{c}_2 & b_5 \end{bmatrix} \neq 0$$

where

$$\begin{aligned}
c_1 &= -\int \theta_1 n_1 \\
c_2 &= -\int \theta_2 n_2 \\
\tilde{c}_2 &= -\int \theta_2 y^\perp \cdot n \\
b_3 &= -\int \chi_1 \partial_2 \theta_2 - \int \chi_2 \partial_2 \theta_1 \\
b_5 &= -\int \chi_1 \nabla \theta_2 \cdot y^\perp - \int \chi_2 \nabla \theta_1 \cdot y^\perp
\end{aligned}$$

Control result for potential flows

Thm (O Glass, LR)

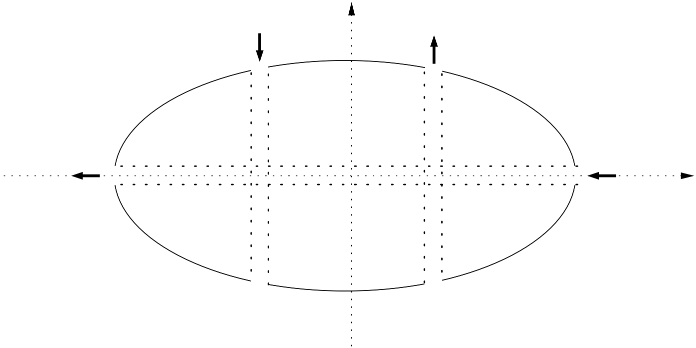
- If the “generic” assumption holds with $m >> 1$, $J >> 1$, then the system

$$
\begin{aligned}
h' &= \mathcal{Q} l \\
\mathcal{J} l' &= \mathcal{C} w' + B(l, w)
\end{aligned}
$$

  with state $(h, l) \in \mathbb{R}^6$ and control $w \in \mathbb{R}^2$ is **locally controllable** around 0.

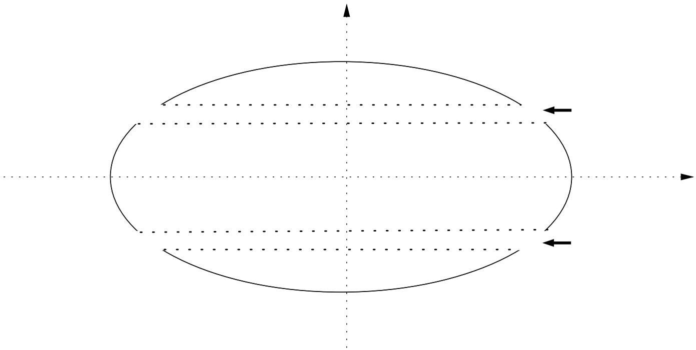
- If, in addition, $\gamma + \alpha\beta = 0$, then we have a **global** controllability for steady states

Example 1: Elliptic boat with 3 controls



Actually, the linearized system around the null trajectory is controllable!

Example 2: Elliptic boat with 2 longitudinal controls



Generic condition fulfilled iff

$$b_3 = -\frac{1}{2} \int |\nabla \Psi|^2 n_2 \neq 0$$

where $-\Delta \Psi = 0$, $\partial \Psi / \partial n = \chi 1_{y_2 > 0}$

Step 1. Loop-shaped trajectory

We consider a special trajectory of the toy problem ($w_2 \equiv 0$) constructed as in the **flatness approach** due to M. Fliess, J. Levine, P. Martin, P. Rouchon

- We **first define** the trajectory

$$\begin{aligned} \overline{h_1}(t) &= \lambda(1 - \cos(2\pi t/T)) \\ \overline{l_1}(t) &= \lambda(2\pi/T)) \sin(2\pi t/T) \end{aligned}$$

- We **next solve** the Cauchy problem

$$\begin{cases} \overline{w_1}' & = \alpha^{-1}\{\overline{l_1}' - \gamma \overline{w_1}^2 - \beta \overline{w_1}\,\overline{l_1}\} \\ \overline{w_1}(0) & = 0 \end{cases}$$

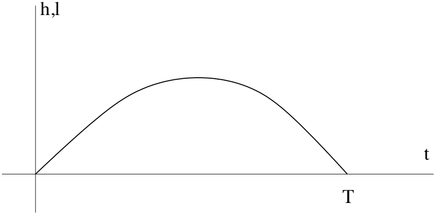
  to design the control input.

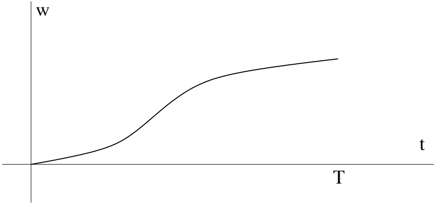
- Then $\overline{w_1}$ exists on $[0, T]$ for $0 < \lambda << 1$. $(\overline{h_1}, \overline{l_1}) = 0$ at $t = 0, T$. **Nothing** can be said about $\overline{w_1}(T)$.

Step 2. Return Method

We linearize along the above (non trivial) reference trajectory to use the nonlinear terms. We obtain a system of the form

$$x' = A(t)x + B(t)u + Cu'$$





Linearization along the reference trajectory

Fact. The reachable set from the origin for the system

$$x' = A(t)x + B(t)u + Cu',\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

is

$$\mathcal{R} = \mathcal{R}_T(A, B + AC) + C\mathbb{R}^m + \Phi(T, 0)C\mathbb{R}^m$$

where $\Phi(t, t_0)$ is the resolvent matrix associated with the system $x' = A(t)x$, and $\mathcal{R}_T(A, B)$ denotes the reachable set in time T from 0 for $x' = A(t)x + B(t)u$, i.e.

$$\mathcal{R}_T(A, B) = \{x(T);\ x' = A(t)x + B(t)u,\ x(0) = 0,\ u \in L^2(0, T, \mathbb{R}^m)\}$$

Silverman-Meadows test of controllability

Consider a C^ω time-varying control system

$$\dot{x} = A(t)x + B(t)u, \qquad x \in \mathbb{R}^n,\ t \in [0, T],\ u \in \mathbb{R}^m.$$

Define a sequence $(M_i(\cdot))_{i\geq 0}$ by

$$M_0(t) = B(t), \qquad M_i(t) = \frac{dM_{i-1}}{dt} - A(t)M_{i-1}(t) \qquad i \geq 1,\ t \in [0, T]$$

Then for any $t_0 \in [0, T]$

$$\sum_{i\geq 0} \Phi(T, t_0)M_i(t_0)\mathbb{R}^m = \mathcal{R}_T(A, B)$$

Proof of the main result (continued)

To complete the proof of the theorem we use

- the generic assumption to prove that the linearized system is controllable. Compute M_0, M_1, M_2 in Silverman-Meadows test (and also M_3 for the global controllability)
- the Inverse Mapping Theorem to conclude.

Control result for general flow

Thm (O. Glass, LR)
Under the same rank condition as above, for any $T_0 > 0$, any initial vorticity $\omega_0 \in W^{1,\infty}(\Omega) \cap L^1_{(1+|y|)^\theta dy}(\Omega)$ with $\theta > 2$, there is some $\delta > 0$ such that for $(h_0, l_0), (h_1, l_1) \in \mathbb{R}^6$ with

$$|(h_0, l_0)| < \delta, \quad |(h_1, l_1)| < \delta$$

there is some control $w \in H^2(0, T, \mathbb{R}^2)$ driving the solid from (h_0, l_0) at $t = 0$ to (h_1, l_1) at $t = T \leq T_0$ for the complete fluid-structure system.

Proof of the main result (continued)

In the general case (vorticity $+$ circulation), we prove/use

- a Global Well-Posedness result using an **extension** argument (which enables us to define the vorticity at the incoming part of the flow), and **Schauder** fixed-point Theorem in **Kikuchi**'s spaces;
- Lipschitz **estimates** for the difference of the velocities corresponding to potential (resp. general) flows in terms of the vorticity and circulation at time 0;
- a **topological** argument to conclude when the vorticity and the circulation are small;
- a **scaling** argument due to J.-M. Coron

A Topological Lemma

Let $B = \{x \in \mathbb{R}^n;\ |x| < 1\}$, and let $f : \overline{B} \to \mathbb{R}^n$ be a continuous map such that for some constant $\varepsilon \in (0, 1)$

$$|f(x) - x| \le \varepsilon \qquad \forall x \in \partial B.$$

Then

$$(1 - \varepsilon)B \subset f(\overline{B}).$$

Conclusion

- Local exact controllability result for a boat with a general shape
- Two linearization arguments: in $\mathbb{R}^6$ (for potential flows) and next to deal with general flows
- Prospects:
  - Motion planning
  - 3D (submarine) (work in progress with Rodrigo Lecaros, CMM, Santiago of Chili)
  - Numerics??