Explicit approximate controllability of the Schrödinger equation with a polarizability term.

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Sept. 2011

Control of dispersive equations, Benasque.

Outline

1. Model studied and strategy
   - Bilinear controlled Schrödinger equation and polarizability
   - LaSalle invariance principle
2. Previous results
   - Semiglobal weak stabilization in the dipolar approximation
   - Finite dimension approximation of the polarizability system
3. Explicit approximate controllability with polarizability
   - Study of the averaged system
   - The averaging strategy in infinite dimension

Model studied

- Model of a quantum particle in a potential V

$$\begin{cases} i\partial_t \psi = \big(-\Delta + V(x) \big)\psi + u(t) Q_1(x) \psi & , \quad x \in D, \\ \psi_{|\partial D} = 0, \\ \psi(0,\cdot) = \psi^0, \end{cases} \tag{1.1}$$

where

- ψ is the wave function,
- $D \subset \mathbb{R}^m$ is a bounded regular domain,
- $V \in C^\infty(\overline{D}, \mathbb{R})$ is the potential,
- the control u is the real amplitude of the electric field,
- $Q_1 \in C^\infty(\overline{D}, \mathbb{R})$ is the dipolar moment,

Model studied

- Model of a quantum particle in a potential V with a polarizability term.

$$\begin{cases} i\partial_t\psi = \big(-\Delta + V(x)\big)\psi + u(t)Q_1(x)\psi + u(t)^2 Q_2(x)\psi, \quad x \in D, \\ \psi_{|\partial D} = 0, \\ \psi(0,\cdot) = \psi^0, \end{cases} \tag{1.1}$$

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- ψ is the wave function,
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- the control u is the real amplitude of the electric field,
- $Q_1 \in C^\infty(\overline{D}, \mathbb{R})$ is the dipolar moment,
- $Q_2 \in C^\infty(\overline{D}, \mathbb{R})$ is the polarizability moment.

Notations

- $\mathcal{S} := \left\{\psi \in L^2(D, \mathbb{C}); ||\psi||_{L^2} = 1\right\}$.
- $< f, g >:= \displaystyle\int_D f(x)\overline{g(x)}\mathrm{d}x$, for $f, g \in L^2$.
- Let $(\lambda_k)_{k\in\mathbb{N}^*}$ be the non decreasing sequence of eigenvalues of the operator $\big(-\Delta + V\big)$ with domain $H^2 \cap H^1_0$.
- Let $(\varphi_k)_{k\in\mathbb{N}^*}$ be the associated sequence of eigenvectors in $\mathcal{S}$.
- $\mathcal{C} := \{c\varphi_1; c \in C, |c| = 1\}$.

Goal : Find a control u such that $\psi \to \varphi_1$.

LaSalle invariance principle in infinite dimensions. I

1. Lyapunov function. $\mathcal{L} : H \to \mathbb{R}$ non negative, $\mathcal{L}(x) = 0 \Longleftrightarrow x = \tilde{x}$ and $\mathcal{L}(x) \underset{x\to\infty}{\longrightarrow} +\infty$.
2. Non increasing along trajectories

$$t \mapsto \mathcal{L}(x(t)) \text{ non increasing },$$

so

$$\mathcal{L}(x(t)) \underset{t\to+\infty}{\to} \alpha.$$

3. Invariant set. We assume x solution of the PDE and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(x(t)) \equiv 0, \forall t \geq 0 \Longrightarrow x(t) \equiv \tilde{x}, \forall t \geq 0.$$

4. Let $(t_n)_{n\in\mathbb{N}} \nearrow +\infty$. $\mathcal{L}(x(t_n)) \leq \mathcal{L}(x(t_0))$ so $(x(t_n))_{n\in\mathbb{N}}$ is bounded.

$$x(t_n) \underset{n\to+\infty}{\rightharpoonup} x_\infty.$$

5. Continuity with respect to the initial condition. Let $x_\infty(\cdot)$ initiated from x_∞.

$$x_n(t) := x(t + t_n) \underset{n\to\infty}{\rightharpoonup} x_\infty(t), \forall t \geq 0.$$

6. Conclusion. Continuity of the Lyapunov function for the weak topology.

$$\mathcal{L}(x_n(t)) \underset{n\to\infty}{\to} \mathcal{L}(x_\infty(t)), \quad \forall t \geq 0,$$
$$\mathcal{L}(x(t_n + t)) \underset{n\to\infty}{\to} \alpha, \quad \forall t \geq 0.$$

So (invariant set)

$$x_\infty(t) = \tilde{x}, \quad \forall t \geq 0,$$

hence

$$x(t) \underset{t\to\infty}{\rightharpoonup} \tilde{x}.$$

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System under the dipole approximation

- System studied in [Beauchard and Nersesyan, 2010].

$$\begin{cases} i\partial_t\psi = \big(-\Delta + V(x)\big)\psi + u(t)Q(x)\psi, \\ \psi_{|\partial D} = 0, \\ \psi(0,\cdot) = \psi^0. \end{cases} \tag{2.1}$$

- Lyapunov function

$$\mathcal{L}(\psi) := \gamma||(-\Delta + V)P\psi||^2_{L^2} + \big(1 - |\langle\psi,\varphi_1\rangle|^2\big),$$

with P the orthogonal projection on Span $\{\varphi_k, k \geq 2\}$ and $\gamma > 0$.

- Hypotheses
  - $\langle Q\varphi_1, \varphi_k\rangle \neq 0$, for all $k \geq 2$.
  - $\lambda_1 - \lambda_j \neq \lambda_p - \lambda_q$, for all $\{1, j\} \neq \{p, q\}$ and $j \neq 1$.

Semiglobal weak stabilization using feedback law

- We consider the feedback law

$$u(\psi) := -\mathrm{Im}\left[\langle\gamma(-\Delta+V)P(Q\psi),(-\Delta+V)P\psi\rangle - \langle Q\psi,\varphi_1\rangle\langle\varphi_1,\psi\rangle\right]. \tag{2.2}$$

Theorem

Under the previous hypotheses, there exists $J \subset \mathbb{R}_+^*$ finite or countable such that for any $\psi^0 \in \mathcal{S} \cap H_0^1 \cap H^2$ not belonging to $\mathcal{C}$, there exists $\gamma^* := \gamma^*(||\psi^0||_{L^2}) > 0$ such that the solution of the system (2.1) with control u defined in (2.2) with $\gamma \in (0,\gamma^*)\backslash J$ and initial condition ψ^0 satisfies (up to a global phase)

$$\psi(t) \underset{t\to\infty}{\rightharpoonup} \varphi_1, \quad \text{in } H_w^2.$$

Beyond the dipolar approximation I

$$\begin{cases} i\dfrac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \big(H_0 + u(t)H_1 + u(t)^2 H_2\big)\psi(t), \\ \psi(0,\cdot) = \psi^0. \end{cases} \tag{2.3}$$

with $\psi(\cdot) \in \mathbb{C}^n$, H_0, H_1 and H_2 are $n \times n$ Hermitian matrices. $\lambda_1, \cdots, \lambda_n$ eigenvalues of H_0 and $\varphi_1, \cdots, \varphi_n$ the associated eigenvectors.

- Studied in [Grigoriu et al., 2009].
- Improved in [Coron et al., 2009].

Strategy : Use of a time periodic feedback

$$u(t,\psi) := \alpha(\psi) + \beta(\psi)\sin\left(\frac{t}{\varepsilon}\right).$$

Beyond the dipolar approximation II

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \Big(H_0 + \alpha(\psi)H_1 + \beta(\psi)\sin\left(\frac{t}{\varepsilon}\right)H_1 + \alpha^2(\psi)H_2 + 2\alpha(\psi)\beta(\psi)\sin\left(\frac{t}{\varepsilon}\right)H_2 + \beta^2(\psi)\sin^2\left(\frac{t}{\varepsilon}\right)H_2\Big)\psi(t). \tag{2.4}$$

- Use of the averaged system. Let f be T periodic and $f_{av}(x) = \frac{1}{T}\int_0^T f(t,x)\mathrm{d}t$.

$$\dot{x}(t) = f(t,x(t)) \Longrightarrow \dot{x}_{av}(t) = f_{av}(x_{av}(t)).$$

  This leads to

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi_{av}(t) = \Big(H_0 + \alpha(\psi_{av})H_1 + \left(\alpha^2(\psi_{av}) + \frac{1}{2}\beta^2(\psi_{av})\right)H_2\Big)\psi_{av}(t). \tag{2.5}$$

- Stabilization of the averaged system.
- Approximation by averaging.

Application of the LaSalle invariance principle I

- Lyapunov function.

$$\mathcal{L}(\psi_{av}(t)) := ||\psi_{av}(t) - \varphi_1||^2.$$

- Choice of the feedbacks.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\psi_{av}(t)) = 2\alpha I_1(\psi_{av}(t)) + (2\alpha^2 + \beta^2) I_2(\psi_{av}(t)),$$

where $I_j(\psi_{av}(t)) = \mathrm{Im}(\langle H_j \psi_{av}(t), \varphi_1 \rangle$.
Let $k \in \left(0, \frac{1}{||H_2||}\right)$. The choice of feedbacks

$$\alpha(\psi_{av}(t)) := -k I_1(\psi_{av}(t)),$$
$$\beta(\psi_{av}(t)) := (I_2(\psi_{av}(t)))^-,$$

leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\psi_{av}(t)) = -2\Big(k I_1(\psi_{av}(t))^2(1 - k I_2(\psi_{av}(t))) + \frac{1}{2}(I_2(\psi_{av}(t))^-)^3\Big) \leq 0.$$

Application of the LaSalle invariance principle II

- Invariant set. Assume $\lambda_j \neq \lambda_l$ for $j \neq l$ and for any $j \in \{2, \cdots, n\}$, $\langle H_1\varphi_j, \varphi_1\rangle \neq 0$ or $\langle H_2\varphi_j, \varphi_1\rangle \neq 0$. Then

$$\psi_{av}(\cdot) \text{ solution of (2.5) with } \mathcal{L}(\psi_{av}(\cdot)) \text{ constant } \text{ implies } \psi_{av}(\cdot) \equiv \pm\varphi_1.$$

Under the previous hypothesis, the averaged system is globally asymptotically stable on $\mathbb{S}^{2n-1}\backslash\{-\varphi_1\}$.

Approximation by averaging

Lemma of approximation

Let $T > 0$. There exists C and $\varepsilon_0 > 0$ such that, for every $\tau \in \mathbb{R}$ and for every $\varepsilon \in (0, \varepsilon_0)$, if $\psi : [\tau, \tau + T] \to \mathbb{S}^{2n-1}$ is a solution of (2.4), and ψ_{av} is the solution of (2.5) such that $\psi_{av}(\tau) = \psi(\tau)$, then

$$||\psi(t) - \psi_{av}(t)|| < C\varepsilon, \quad \forall t \in [\tau, \tau + T].$$

- Combining this with the convergence of ψ_{av} we obtain

Main result

Assume that the coupling assumption and the non degeneracy of the spectrum hold. Let $\mathcal{V}$ be a neighborhood of $-\varphi_1$ and $\delta > 0$. There exists a time $T > 0$ and $\varepsilon_0 > 0$ such that every solution of (2.4) with $\varepsilon \in (0, \varepsilon_0)$ that satisfies $\psi(\tau) \in \mathbb{S}^{2n-1} \backslash \mathcal{V}$ for some $\tau > 0$ also satisfies

$$||\psi(t) - \varphi_1|| < \delta, \quad \forall t \geq \tau + T.$$

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Averaged system

$$\begin{cases} i\partial_t \psi = \big(-\Delta + V(x)\big)\psi + u(t)Q_1(x)\psi + u(t)^2 Q_2(x)\psi, \\ \psi_{|\partial D} = 0, \end{cases}$$

with feedback control $u(t,\psi(t)) := \alpha(\psi(t)) + \beta(\psi(t))\sin(t/\varepsilon)$ leads to the averaged system

$$\begin{cases} i\partial_t \psi_{av} = \big(-\Delta + V(x)\big)\psi_{av} + \alpha(\psi_{av})Q_1\psi_{av} \\ \qquad\qquad + \left(\alpha(\psi_{av})^2 + \frac{1}{2}\beta(\psi_{av})^2\right) Q_2\psi_{av}, \\ \psi_{av_{|\partial D}} = 0. \end{cases} \tag{3.1}$$

- Lyapunov function.

$$\mathcal{L}(z) := \gamma ||(-\Delta + V)Pz||^2_{L^2} + \left(1 - |\langle z, \varphi_1 \rangle|^2\right).$$

- Feedback laws

$$\alpha(z) := -kI_1(z), \quad \beta(z) := g(I_2(z)),$$

with

$$I_j(z) := \text{Im}\left(\gamma \langle (-\Delta + V)P(Q_j z), (-\Delta + V)Pz \rangle - \langle Q_j z, \varphi_1 \rangle \langle \varphi_1, z \rangle \right),$$

$k > 0$ small enough and $g \in C^2(\mathbb{R}, \mathbb{R}^+)$ satisfying $g(x) = 0$ if and only if $x \geq 0$, g' bounded.

- Then, $\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}(\psi_{av}(t)) \leq 0$.
- Under the following assumptions
  - **(H1)** $\langle Q_1 \varphi_1, \varphi_k \rangle = 0 \Longrightarrow \langle Q_2 \varphi_1, \varphi_k \rangle \neq 0$,
  - **(H2)** Card $\{k \geq 2; \langle Q_1 \varphi_1, \varphi_k \rangle = 0\} < \infty$,

Study of the averaged system and choice of the feedbacks II

- **(H3)** $\lambda_1 - \lambda_k \neq \lambda_p - \lambda_q$ for $\{1, k\} \neq \{p, q\}$ and $k \neq 1$,
- **(H4)** $\lambda_p \neq \lambda_q$ for $p \neq q$,

the invariant set is included in $\mathcal{C}$.

- Continuity with respect to the initial condition and continuity of the feedback law for the weak H^2 topology.

Assume that hypotheses **(H1)-(H4)** hold. If $\psi^0 \in X_0 := \left\{z \in \mathcal{S} \cap H_0^1 \cap H^2; \Delta z \in H_0^1 \cap H^2\right\}$ with $0 < \mathcal{L}(\psi^0) < 1$, the solution of (3.1) satisfies (up to a global phase)

$$\psi_{av}(t) \underset{t \to +\infty}{\rightharpoonup} \varphi_1, \quad \text{in } H^2.$$

Approximation by averaging I

For an initial condition $\psi^0 \in X_0$, we consider the control

$$u^\varepsilon(t) := \alpha(\psi_{av}(t)) + \beta(\psi_{av}(t)) \sin(t/\varepsilon),$$

with ψ_{av} the solution of (3.1) satisfying $\psi_{av}(0,\cdot) = \psi^0$.

Let $L > 0$, $\psi^0 \in X_0$ with $0 < \mathcal{L}(\psi^0) < 1$. Let ψ_{av} be the solution of the closed loop system (3.1) with initial condition ψ^0. For any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that if ψ_ε is the solution of (1.1) with initial condition ψ^0 and control u^ε with $\varepsilon \in (0, \varepsilon_0)$, then

$$||\psi_\varepsilon(t) - \psi_{av}(t)||_{H^2} \leq \delta, \quad \forall t \in [0, L].$$

Explicit approximate controllability

Main result

Assume that hypotheses **(H1)-(H4)** hold. For any $s < 2$, for any $\psi^0 \in X_0$ with $0 < \mathcal{L}(\psi^0) < 1$, there exist a strictly increasing time sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+^*$ tending to $+\infty$ and a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in R_+^* such that if ψ_ε is the solution of (1.1) associated to the control u^ε with $\varepsilon \in (0, \varepsilon_n)$ and initial condition ψ^0,

$$\mathrm{dist}_{H^s}(\psi_\varepsilon(t), \mathcal{C}) \leq \frac{1}{2^n}, \quad \forall t \in [T_n, T_{n+1}].$$

Open Problems

- Convergence in the H^2 norm.
- Card $\{j \geq 2; \langle Q_1\varphi_1, \varphi_j\rangle = 0\} = \infty$.
- Approximation property on infinite time interval $[s, +\infty)$.
- Semi global exact controllability using [Beauchard and Laurent, 2010] in the 1D case with $V = 0$.

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Thank you for you attention.

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