Zero pressure and viscosity limit of the 1D-Navier-Stokes-Poisson system

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Benasque Aug 2011

Outline

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2. The NS-Poisson system in 1D

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4. A rearrangement scheme
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MOTIVATION:THE EARLY UNIVERSE RECONSTRUCTION PB

Following Peebles 1989, Frisch and coauthors (Nature 417) 2002, one wants to reconstruct the history of the Universe from the knowledge of the present mass density field. Onn conside an expanding universe with self-gravitating matter.

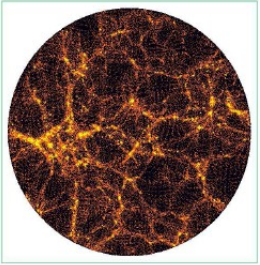

Figure 7. N-body simulation output in the Eulerian space used for testing our reconstruction method (shown is a projection onto the x-y plane of a 10% slice of the simulation box of size $200h^{-1}$ M[illegible] reconstruction fails by more than $6.25\,h^{-1}$ Mpc, which happens mostly in

THE PRESSURE-LESS EULER-POISSON MODEL SCALED BY GENERAL RELATIVITY (GR)

Denoting by $(\rho, \rho \mathbf{v}, \varphi)$ the mass, momentum and gravity potential, we use the pressure-less Euler Poisson model

$$\partial_t(t^{3/2}\rho \mathbf{v}) + \nabla \cdot (t^{3/2}\rho \mathbf{v} \otimes \mathbf{v}) = -\frac{3t^{1/2}}{2}\rho \nabla \varphi$$

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \quad \rho = 1 + t\, \nabla^2 \varphi$$

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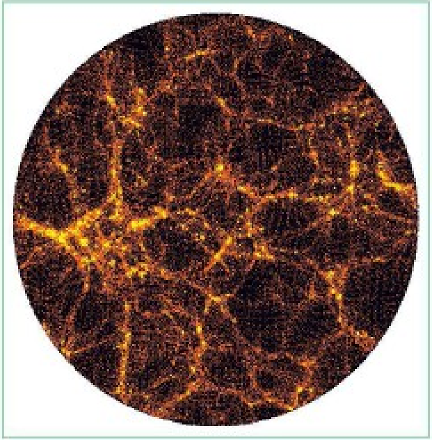
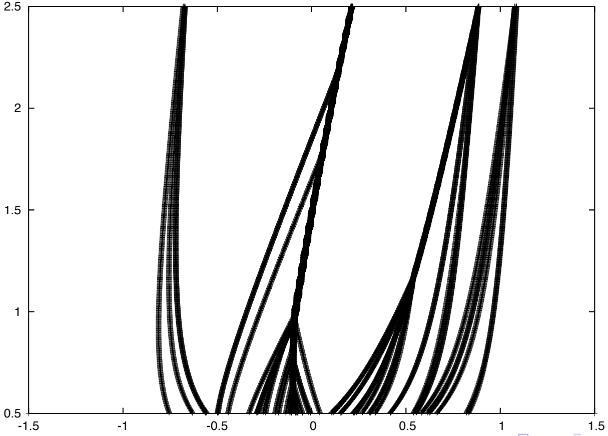
Typical solutions exhibit mass concentrations in finite time, i.e. $\rho = \rho(t, dx)$ becomes a measure with singular parts.
The inverse problem amounts to recovering the solution from the knowledge of the density field at the present time.



Trajectories with dynamical concentrations (1D pressure-less Euler Poisson system)

horizontal : space /vertical : time



THE 1D NAVIER-STOKES POISSON SYSTEM

Concentration mechanisms in the pressure-less Euler-Poisson model are unclear. One way to handle them is to go back to a Navier-Stokes model. To simplify the discussion, we ignore the general relativity terms. Denoting by $(\rho, \rho \mathbf{v}, \varphi)$ the mass, momentum and gravity potential, we consider the isentropic NS-Poisson model

$$\boxed{\partial_{\mathbf{t}}(\rho \mathbf{v}) + \partial_{\mathbf{x}}(\rho \mathbf{v}^2 + \mathbf{p}) = \partial_{\mathbf{x}}(\mu \partial_{\mathbf{x}} \mathbf{v}) - \rho \partial_{\mathbf{x}} \varphi}$$

$$\boxed{\partial_{\mathbf{t}} \rho + \partial_{\mathbf{x}}(\rho \mathbf{v}) = \mathbf{0}, \quad (\rho - \mathbf{1})\beta = \partial^{\mathbf{2}}_{\mathbf{xx}} \varphi}$$

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$$\partial_t(\rho v) + \partial_x(\rho v^2 + p) = \partial_x(\mu \partial_x v) - \rho \partial_x \varphi$$

$$\partial_t \rho + \partial_x(\rho v) = 0, \quad (\rho - 1)\beta = \partial^2_{xx}\varphi$$

with special (quite unphysical) pressure and viscosity law

$$p = \lambda \epsilon \rho, \quad \mu = \epsilon \rho$$

(λ, β) **are fixed constants and we are interested in the $\epsilon = 0$ limit**

THE TRAJECTORIAL VERSION OF THE 1D NS-POISSON SYSTEM

Following the fluid particle trajectories $(t, a) \to X(t, a)$ so that

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we get the trajectorial formulation of the (special) 1D NS-Poisson system

$$\partial_t X + \epsilon\, \partial_a\left(\frac{1}{\partial_a X}\right) = Z - \lambda X, \quad \partial_t Z + \lambda(Z - \lambda X) = (X - a)\beta$$

(this requires the special choice of the pressure and viscosity laws)

THE 1D NS-POISSON SYSTEM VIEWED AS A MODIFIED HEAT EQUATION

The trajectorial formulation of the (special) 1D NS-Poisson system

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$$\partial_t X + \epsilon\, \partial_a\left(\frac{1}{\partial_a X}\right) = 0$$

which is nothing but the trajectorial version of the heat equation $\partial_t \rho = \partial^2_{xx} \rho$
This suggests the use of a random particle scheme with rearrangement that goes back to Alexander Chorin 1979 (for reaction-diffusion equations) and Y.B. 1990 (for viscous scalar conservation laws).

A PSEUDO-RANDOM REARRANGEMENT SCHEME

Let us discretize the trajectories $(t, a) \to (X, Z)(t, a)$ by $(n, i) \to (X_i^n, Z_i^n)$

$$Z^{n+1} = Z^n + h((X^n - A)\beta - \lambda(Z^n - \lambda X^n))$$

$$X^{n+1} = (X^n + h(Z^n - \lambda X^n) + \sqrt{2\epsilon h}e)^*$$

where $*$ denotes the rearrangement in increasing order of a sequence, $A_i = i/N$, $e_i = (-1)^i$ respectively discretizes the variable a and “simulates” (in a deterministic way!) a random step

CONVERGENCE ANALYSIS

If the discretization parameters N, h are properly scaled -typically $Nh \to +\infty$ the scheme can be proven to be convergent to the trajectorial formulation

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In addition, the $\epsilon \to 0$ limit can be shown to correspond to the subdifferential system

$$-\partial_t X + Z - \lambda X \in \partial\, 1_{\{\partial_a X \geq 0\}}, \quad \partial_t Z + \lambda(Z - \lambda X) = (X - a)\beta$$

which is well posed in L^2

BALANCE BETWEEN PRESSURE AND VISCOSITY

The zero-pressure and viscosity limit of the (special) 1d NS-Poisson system is now clearly identified by the subdifferential system

$$-\partial_t \mathbf{X} + \mathbf{Z} - \lambda \mathbf{X} \in \partial\, \mathbf{1}_{\{\partial_a \mathbf{X} \geq 0\}}, \quad \partial_t \mathbf{Z} + \lambda(\mathbf{Z} - \lambda \mathbf{X}) = (\mathbf{X} - \mathbf{a})\beta$$

that *a priori* does depend on λ which is somewhat surprising, since it involves a fine balance between pressure and viscosity

$$\mathbf{p} = \lambda \epsilon \rho, \quad \mu = \epsilon \rho$$

DISCUSSION

For a special choice of vanishing pressure and viscosity laws, we get a limit that a priori depends on the balance between viscosity and pressure. Thus, one has to be very careful when dealing with pressureless Euler-Poisson models. Numerics suggests that there is indeed such a dependence in the repulsive case but not in the attractive case, which is good news for the early universe reconstruction problem.

DISCUSSION

For a special choice of vanishing pressure and viscosity laws, we get a limit that a priori depends on the balance between viscosity and pressure. Thus, one has to be very careful when dealing with pressureless Euler-Poisson models. Numerics suggests that there is indeed such a dependence in the repulsive case but not in the attractive case, which is good news for the early universe reconstruction problem.
Another issue is the possibility of using pseudo-random rearrangement schemes in the multi-d case. This seems difficult and, anyway, more related to Brenner's modification of the NS equations (as studied recently by Feireisl and Vasseur) than to the original NS equations.

THANK YOU FOR YOUR ATTENTION!

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Some references

The EUR problem and the pressure-less Euler-Poisson model

Y. Brenier, U. Frisch, M. Hénon, G. Loeper, S. Matarrese, Mohayaee, Sobolevskii, Mon. Not. R. Astron. Soc. (2003) and references included

G. Loeper, Arch. Ration. Mech. Anal.(2006)

Zeldovich approximation and burgurlence Y. Zeldovich, Astron. Astrophys. (1970)

E. Aurell, U. Frisch, J. Lutsko, M. Vergassola, J. Fluid Mech. (1992)

W. E, Y.Rykov,Y. Sinai, Comm. Math. Phys. (1996)