A system of Schrödinger equations modeling two trapped ions. Some controllability results and open problems.

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Outline

Trapped ions or Qubits

- The goal is to create quantum logic gates like the phase gate or the C-Not gate.
  See S.Haroche lectures at College de France on Quantum Information Theory (available on the web) and experiments by the group S.Haroche, J.M.Raimond and collaborators at ENS Paris.
- Experiments are based on trapped ions (qubits) with the case of one single trapped ion (one qubit problem) or two coupled trapped ions (two qubits problem).

Trapped ions or Qubits

- Each ion is a two level system, trapped in an electromagnetic cavity, all ions are stabilized by the same spatial oscillations, here a harmonic oscillator with vibration quantum ω (phonon).
- The system is submitted to a superposition of electromagnetic waves of complex amplitude u_1 and u_2. The phases depend on the spatial coordinate in order to be able to conserve the impulsion : when an ion absorbs a photon, its energy changes and its impulsion captures the photon impulsion and excites the (quantized) vibration modes (phonon) inside the trap.

Mathematical model

- Two ions.
- Each ion is a two level system.
- Coupled to the same quantized harmonic oscillator

$$A = \frac{1}{2}(-\partial_{xx}^2 + x^2)$$

with vibration quantum ω.
We have

$$A = \mathbf{a}^\dagger \mathbf{a} + \frac{1}{2} = \mathbf{a}\mathbf{a}^\dagger - \frac{1}{2}$$

where

$$\mathbf{a} = \frac{1}{\sqrt{2}}(x + \frac{\partial}{\partial x})$$

is the annihilation operator and

$$\mathbf{a}^\dagger = \frac{1}{\sqrt{2}}(x - \frac{\partial}{\partial x})$$

is the creation operator.

Mathematical model

- Controls : two electromagnetic waves of complex amplitude u_1 and u_2 and phases depending on spatial coordinate :

$$u_j(t)e^{i(\Omega_j^L t - k_j x)},\ j = 1, 2,$$

- State of the system : 4-d vector-wave function

$$|\psi >= \psi =^t (\psi_{gg}, \psi_{ge}, \psi_{eg}, \psi_{ee})$$

- Dynamics of the system described by the Hamiltonian H

$$i\bar{h}\frac{\partial}{\partial t}|\psi >= H|\psi >,$$

where

$$\frac{H}{\bar{h}} = \omega A + \frac{\Omega}{2}(\sigma_{1,z} + \sigma_{2,z}) + \left(u_1 e^{i(\Omega_1^L t - k_1 x)} + u_1^* e^{-i(\Omega_1^L t - k_1 x)}\right)\sigma_{1,x} + \left(u_2 e^{i(\Omega_2^L t - k_2 x)} + u_2^* e^{-i(\Omega_2^L t - k_2 x)}\right)\sigma_{2,x}.$$

Mathematical model

Pauli matrices :

$$\sigma_{1,z} = (|e >< e| - |g >< g|)_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_{2,z} = (|e >< e| - |g >< g|)_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_{1,x} = (|g >< e| + |e >< g|)_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\sigma_{2,x} = (|g >< e| + |e >< g|)_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Schrödinger system

$$
\begin{aligned}
i\frac{\partial\psi}{\partial t} = \quad & \omega A\psi + \frac{\Omega}{2}\sigma_{1,z}\psi + \frac{\Omega}{2}\sigma_{2,z}\psi \\
& +(u_1 e^{i(\Omega_1^L t - k_1 x)} + u_1^* e^{-i(\Omega_1^L t - k_1 x)})\sigma_{1,x}\psi \\
& +(u_2 e^{i(\Omega_2^L t - k_2 x)} + u_2^* e^{-i(\Omega_2^L t - k_2 x)})\sigma_{2,x}\psi, \\
\psi(0) = \psi^0 \quad & .
\end{aligned}
$$

Question : Given an initial configuration ψ^0 and a final configuration ψ^1, can we find control amplitudes u_1 and u_2 in order to drive the system at time T "close" to ψ^1?

Parameters :

$$\omega \text{ large and } \Omega \text{ very large,}$$

$$|\Omega_1^L - \Omega| << \Omega, \quad |\Omega_2^L - \Omega| << \Omega, \quad \omega << \Omega,$$

$$|u_1| << \Omega, \quad |u_2| << \Omega, \quad |\frac{du_1}{dt}| << \Omega, \quad |\frac{du_2}{dt}| << \Omega.$$

Laser frame

Set

$$\psi = e^{-i\frac{\Omega_1^L}{2}t\sigma_{1,z}}.e^{-i\frac{\Omega_2^L}{2}t\sigma_{2,z}}\varphi$$

or

$$\varphi = e^{i\frac{\Omega_2^L}{2}t\sigma_{2,z}}.e^{-i\frac{\Omega_1^L}{2}t\sigma_{1,z}}\psi.$$

And

$$\Delta_1 = \frac{\Omega - \Omega_1^L}{2},\ \Delta_2 = \frac{\Omega - \Omega_2^L}{2},$$

$$k_1 x = \eta_1(\mathbf{a} + \mathbf{a}^\dagger),\ k_2 x = \eta_2(\mathbf{a} + \mathbf{a}^\dagger),$$

where η_j, $j = 1, 2$ are the Lamb-Dicke parameters with

$$\eta_j << 1.$$

Interaction frame

$$A = \mathbf{a}^\dagger \mathbf{a} + \frac{1}{2},$$

$$S(t) = e^{-i\omega t A}.e^{-i\Delta_1 t\sigma_{1,z}}.e^{-i\Delta_2 t\sigma_{2,z}}$$

(A, $\sigma_{1,z}$ and $\sigma_{2,z}$ commute.)

$$\xi(t) = S(-t)\varphi(t) \quad , \quad \varphi(t) = S(t)\xi(t).$$

$$\begin{aligned}
i\frac{\partial \xi}{\partial t} &= S(-t)\Big(u_1 e^{2i\Omega_1^L t - i\eta_1(\mathbf{a}+\mathbf{a}^\dagger)} + u_1^* e^{i\eta_1(\mathbf{a}+\mathbf{a}^\dagger)}\Big)(|e><g|)_1 S(t)\xi \\
&+ S(-t)\Big(u_1 e^{-i\eta_1(\mathbf{a}+\mathbf{a}^\dagger)} + u_1^* e^{-2i\Omega_1^L t + i\eta_1(\mathbf{a}+\mathbf{a}^\dagger)}\Big)(|g><e|)_1 S(t)\xi \\
&+ S(-t)\Big(u_2 e^{2i\Omega_2^L t - i\eta_2(\mathbf{a}+\mathbf{a}^\dagger)} + u_2^* e^{i\eta_2(\mathbf{a}+\mathbf{a}^\dagger)}\Big)(|e><g|)_2 S(t)\xi \\
&+ S(-t)\Big(u_2 e^{-i\eta_2(\mathbf{a}+\mathbf{a}^\dagger)} + u_2^* e^{-2i\Omega_2^L t + i\eta_2(\mathbf{a}+\mathbf{a}^\dagger)}\Big)(|g><e|)_2 S(t)\xi
\end{aligned}$$

Lamb-Dicke approximation

$$|\eta_1|, |\eta_2| << 1.$$

$$e^{i\eta_j(\mathbf{a}+\mathbf{a}^\dagger)} \sim \Big(Id + i\eta_j(\mathbf{a}+\mathbf{a}^\dagger)\Big),\ e^{-i\eta_j(\mathbf{a}+\mathbf{a}^\dagger)} \sim \Big(Id - i\eta_j(\mathbf{a}+\mathbf{a}^\dagger)\Big).$$

We then have (for example)

$$e^{i\omega tA}(e^{i\eta_1(\mathbf{a}+\mathbf{a}^\dagger)})e^{-i\omega tA} \sim Id + i\eta_1(\mathbf{a}e^{-i\omega t} + \mathbf{a}^\dagger e^{i\omega t}).$$

We obtain

$$\begin{aligned} i\frac{\partial \xi}{\partial t} = \ & \Big(u_1 e^{2i\Omega_1^L t}\Big(Id - i\eta_1(\mathbf{a}e^{-i\omega t} + \mathbf{a}^\dagger e^{i\omega t})\Big) \\ & + u_1^*\Big(Id + i\eta_1(\mathbf{a}e^{-i\omega t} + \mathbf{a}^\dagger e^{i\omega t})\Big)\Big)e^{2i\Delta_1 t}(|e><g|)_1\xi \\ & + \Big(u_1\Big(Id - i\eta_1(\mathbf{a}e^{-i\omega t} + \mathbf{a}^\dagger e^{i\omega t})\Big) \\ & + u_1^* e^{-2i\Omega_1^L t}\Big(Id + i\eta_1(\mathbf{a}e^{-i\omega t} + \mathbf{a}^\dagger e^{i\omega t})\Big)\Big)e^{-2i\Delta_1 t}(|g><e|)_1\xi \\ & + \cdots \end{aligned}$$

Averaging approximation

First of all we take each control u_j to be a superposition of 3 monochromatic waves, two of them having a pulsation shifted by $\pm$ a vibration quantum ω. In fact we take

$$u_1(t)e^{-2i\Delta_1 t} = v_0(t) + \tilde{v}_r(t)e^{-i\omega t} + \tilde{v}_b(t)e^{i\omega t}$$

$$u_2(t)e^{-2i\Delta_2 t} = w_0(t) + \tilde{w}_r(t)e^{-i\omega t} + \tilde{w}_b(t)e^{i\omega t}.$$

Then, using the averaging approximation, we can show that we can neglect the rapidly oscillating terms as ω, Ω_1^L, Ω_2^L and Ω are very large.

Approximate model

Similar to Law-Eberly equations in the case of one qubit.

$$
\begin{aligned}
i\frac{\partial y}{\partial t} = \quad & (v_0 - i\eta_1 \tilde{v}_r \mathbf{a}^\dagger - i\eta_1 \tilde{v}_b \mathbf{a})(|g><e|)_1 y \\
& (v_0^* + i\eta_1 \tilde{v}_r^* \mathbf{a} + i\eta_1 \tilde{v}_b^* \mathbf{a}^\dagger)(|e><g|)_1 y \\
& (w_0 - i\eta_2 \tilde{w}_r \mathbf{a}^\dagger - i\eta_2 \tilde{w}_b \mathbf{a})(|g><e|)_2 y \\
& (w_0^* + i\eta_2 \tilde{w}_r^* \mathbf{a} + i\eta_2 \tilde{w}_b^* \mathbf{a}^\dagger)(|e><g|)_2 y.
\end{aligned}
$$

Writing

$$v_r = -i\eta_1 \tilde{v}_r, \;\; v_b = -i\eta_1 \tilde{v}_b,$$

$$w_r = -i\eta_1 \tilde{w}_r, \;\; w_b = -i\eta_1 \tilde{w}_b,$$

and

$$y =^t (y_{gg}, y_{ge}, y_{eg}, y_{ee}),$$

we obtain

Approximate model

$$i\frac{\partial y_{gg}}{\partial t} = (v_0 + v_r\mathbf{a}^\dagger + v_b\mathbf{a})y_{eg} + (w_0 + w_r\mathbf{a}^\dagger + w_b\mathbf{a})y_{ge}$$

$$i\frac{\partial y_{ge}}{\partial t} = (v_0 + v_r\mathbf{a}^\dagger + v_b\mathbf{a})y_{ee} + (w_0^* + w_r^*\mathbf{a} + w_b^*\mathbf{a}^\dagger)y_{gg}$$

$$i\frac{\partial y_{eg}}{\partial t} = (v_0^* + v_r^*\mathbf{a} + v_b^*\mathbf{a}^\dagger)y_{gg} + (w_0 + w_r\mathbf{a}^\dagger + w_b\mathbf{a})y_{ee}$$

$$i\frac{\partial y_{ee}}{\partial t} = (v_0^* + v_r^*\mathbf{a} + v_b^*\mathbf{a}^\dagger)y_{ge} + (w_0^* + w_r^*\mathbf{a} + w_b^*\mathbf{a}^\dagger)y_{eg}$$

$$y(0) = y^0.$$

Strategy

- Find a control (exact if possible) for the approximate system which drives an initial configuration to a desired one in time T. We would like to have only one of the controls (v_0, v_r, v_b or w_0, w_r, w_b) being active at each time these controls being piecewise constant (not mandatory...).
- Take this control in the original system. This will provide an approximate control for the real system in time T. This can be proved due to approximation properties for the Lamb-Dicke and the averaging approximations mentionned above.
- Approximate control is relevant here because when we switch off control we keep close to the target (property of Schrödinger system).
- Both the original and the approximate systems are reversible and preserve the $(L^2)^4$-norm.

Control of the approximate system

It remains to study the control properties for the approximate system. Here we have only partial results at the moment and a (strong) conjecture for obtaining the global result.
We use the spectral decomposition of operator A. Its eigenfunctions ϕ_n are the Hermite functions associated with eigenvalues $n+\dfrac{1}{2}$ that, for convenience, we may write $\phi_n = |n>$. We then have

$$A|n> = (n+\frac{1}{2})|n>,$$

and

$$\mathbf{a}|0> = |0>,\ \mathbf{a}|n+1> = \sqrt{n+1}|n>, \mathbf{a}^{\dagger}|n> = \sqrt{n+1}|n+1> .$$

Control of the approximate system

For instance, if we write $|gg, n\rangle = \langle n|y_{gg}\rangle$ and similar notations, and if only v_r is active, $|gg, n >$ and $|eg, n-1\rangle$ form an independent system which solves

$$i\partial_t|gg, n\rangle = v_r\sqrt{n}|eg, n-1\rangle, \quad i\partial_t|eg, n-1\rangle = v_r^*\sqrt{n}|eg, n\rangle.$$

Of course, similar computations can also be done when the other controls are active.
We can represent these decompositions and their dynamics as follows:

Control of the approximate system

$$v_0 \begin{cases} |gg,n\rangle \overset{|v_0|}{\longleftrightarrow} |eg,n\rangle, \\ |ge,n\rangle \overset{|v_0|}{\longleftrightarrow} |ee,n\rangle, \end{cases} \qquad w_0 \begin{cases} |gg,n\rangle \overset{|w_0|}{\longleftrightarrow} |ge,n\rangle, \\ |eg,n\rangle \overset{|w_0|}{\longleftrightarrow} |ee,n\rangle, \end{cases}$$

$$v_r \begin{cases} |gg,n+1\rangle \overset{\sqrt{n+1}|v_r|}{\longleftrightarrow} |eg,n\rangle, \\ |ge,n+1\rangle \overset{\sqrt{n+1}|v_r|}{\longleftrightarrow} |ee,n\rangle, \end{cases} \qquad w_r \begin{cases} |gg,n+1\rangle \overset{\sqrt{n+1}|w_r|}{\longleftrightarrow} |ge,n\rangle, \\ |eg,n+1\rangle \overset{\sqrt{n+1}|w_r|}{\longleftrightarrow} |ee,n\rangle, \end{cases}$$

$$v_b \begin{cases} |gg,n\rangle \overset{\sqrt{n+1}|v_b|}{\longleftrightarrow} |eg,n+1\rangle, \\ |ge,n\rangle \overset{\sqrt{n+1}|v_b|}{\longleftrightarrow} |ee,n+1\rangle, \end{cases} \qquad w_b \begin{cases} |gg,n\rangle \overset{\sqrt{n+1}|w_b|}{\longleftrightarrow} |ge,n+1\rangle, \\ |eg,n\rangle \overset{\sqrt{n+1}|w_b|}{\longleftrightarrow} |ee,n+1\rangle. \end{cases}$$

Easy examples I

One can go from any pure state $|ee, n>$ to any pure state $|gg, m>$. Let us take the case $m < n$.

$$|ee, n> \stackrel{\sqrt{n}|v_b|}{\longrightarrow} |ge, n-1> \stackrel{|v_0|}{\longrightarrow} |ee, n-1> \cdots |ee, m+1>$$

$$|ee, m+1> \stackrel{\sqrt{m+1}|v_b|}{\longrightarrow} |ge, m> \stackrel{|w_0|}{\longrightarrow} |gg, m> .$$

Easy examples II

To go from $|gg,0>$ to $(|gg,0> + |ee,0>)/\sqrt{2}$, we use 4 steps: v_b, w_0, w_b, w_0:

$$|gg,0> \xrightarrow{v_b} \frac{1}{\sqrt{2}}(|gg,0> + |eg,1>) \xrightarrow{w_0} \frac{1}{\sqrt{2}}(|ge,0> + |ee,1>)$$

$$\xrightarrow{w_b} \frac{1}{\sqrt{2}}(|gg,0> - |eg,0>) \xrightarrow{w_0} \frac{1}{\sqrt{2}}(|gg,0> + |ee,0>).$$

Easy examples III

To go from $a_0|gg,0> + b_0|ge,0> + c_0|eg,0> + d_0|ee,0>$ with $|a_0|^2 + |b_0|^2 + |c_0|^2 + |d_0|^2 = 1$ to $|gg,0>$.

- Turn on w_0 to kill term in $|ee,0>$.
- Turn on v_r during t_1 with $|v_r|t_1 = \frac{\pi}{2}$ to obtain $a_1|gg,0> + b_1|ge,0> + c_1|gg,1>$.
- Turn on w_r during time t_2 to kill term in $|gg,1>$. We obtain $a_2|gg,0> + b_2|ge,0>$.
- Turn on w_0 to obtain $|gg,0>$.

Invariant spaces

Let us introduce the spaces

$$X_n^0 = \text{Span}\left\{\, |gg,n\rangle, |ge,n\rangle, |eg,n\rangle, |ee,n\rangle \right\}/_{\mathbb{C}}, \;\; n \in \mathbb{N},$$
$$X_{n+1}^b = \text{Span}\left\{\, |gg,n\rangle, |ge,n+1\rangle, |eg,n+1\rangle, |ee,n+2\rangle \right\}/_{\mathbb{C}}, \;\; n \in \mathbb{N},$$
$$X_{n+1}^r = \text{Span}\left\{\, |ee,n\rangle, |ge,n+1\rangle, |eg,n+1\rangle, |gg,n+2\rangle \right\}/_{\mathbb{C}}, \;\; n \in \mathbb{N},$$

and

$$X_0^b = \text{Span}\left\{\, |ge,0\rangle, |eg,0\rangle, |ee,1\rangle \right\}/_{\mathbb{C}}.$$
$$X_0^r = \text{Span}\left\{\, |ge,0\rangle, |eg,0\rangle, |gg,1\rangle \right\}/_{\mathbb{C}}.$$

- X_n^0 is invariant under the action of the controls v_0, w_0;
- X_n^b is invariant under the action of the controls v_b, w_b;
- X_n^r is invariant under the action of the controls v_r, w_r.

Invariant spaces

We also define the spaces

$$Y_n^0 = \mathrm{Span}\,\{\ |gg,k\rangle, |ge,k\rangle, |eg,k\rangle, |ee,k\rangle,\ \ k \leq n\}\,/_{\mathbf{C}},$$
$$Y_n^r = \mathrm{Span}\,\{\ |ee,k\rangle, |ge,k+1\rangle, |eg,k+1\rangle, |gg,k+2\rangle,\ \ k+1 \leq n\}\,/_{\mathbf{C}}.$$

We have

$$Y_n^0 = \bigcup_{k \leq n} X_0^k,$$

$$Y_n^r = \bigcup_{k+1 \leq n} X_{k+1}^r.$$

Toward a general result?

It can be shown, as for the "easy examples", that :

- X_0^r is controllable with controls v_r and w_r.
- Y_0^r is controllable with controls v_r, w_r and w_0.
- Y_0^0 is controllable with controls v_r, w_r and w_0.

How to obtain a general result?

In order to obtain a general result, it would be enough to show that we can drive Y_n^r to Y_{n-1}^r in a controlled time.
We have

$$Y_n^r = Y_{n-1}^r \cup X_n^r,$$

and we know that both Y_{n-1}^r and X_n^r are invariant under the action of v_r and w_r.
Therefore we would like to use only the controls v_r and w_r. Then we want to show that with these controls, any element of X_n^r can be driven to $|ee, n>$ for example in a controlled time.
This question is still open at the moment. We are trying (without success until now) to give an explicit construction and this is a problem in X_n^r only and therefore in finite dimension (4) !! .....