Wave Equations with non Lipschitz coefficients

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Joint works

- **(C.DG.S.'79)** F.C.-E.De Giorgi-S.Spagnolo ('79)
- **(C.L.'95)** F.C.-N.Lerner ('95)
- **(C.DS.K.'02)** F.C.-D.Del Santo-T.Kinoshita ('02)
- **(C.DS.R.'03)** F.C.-D.Del Santo-M.Reissig ('03)

- **(C.C.'06)** M.Cicognani-F.C. ('06)

- **(C.M.'08)** F.C.-G.Métivier ('08)

- **(C.DS.'09)** F.C.-D.Del Santo ('09)

- **(C.F.'10)** F.C.-F.Fanelli ('10)

- **(C.DS.F.M.)** F.C.-D.Del Santo-F.Fanelli-G.Métivier (in progress)

Let us consider the Cauchy problem in $Q := [0,T] \times \mathbb{R}^n_x$

$$\begin{cases} \partial_t^2 u - \displaystyle\sum_{i,j=1}^{n} \partial_{x_i}\Big(a_{ij}(t,x)\partial_{x_j} u\Big) = 0 \\ \\ u(0,x) = u_0, \quad \partial_t u(0,x) = u_1, \end{cases} \tag{$\mathcal{CP}$}$$

under the **strict hyperbolicity** assumption

$$\lambda_0 |\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \le \Lambda_0 |\xi|^2$$

$$0 < \lambda_0 \ , \qquad a_{ij} = a_{ji}$$

Classical assumptions:

$$t \mapsto a_{ij}(t,x) \quad \text{Lipschitz continuous uniformly in } x$$

$$x \mapsto a_{ij}(t,x) \quad \text{smooth.}$$

Then $(\mathcal{CP})$ is C^∞ and, $\forall s \in \mathbb{R}$, H^s well-posed. Moreover

$$\|u(t)\|^2_{H^s} + \|\partial_t u(t)\|^2_{H^{s-1}} \leq C_s \left(\|u_0\|^2_{H^s} + \|u_1\|^2_{H^{s-1}} \right), \ \forall t \in [0,T]$$

$\forall\, u_0 \in H^s$, $\forall\, u_1 \in H^{s-1}$

In **(C.DG.S.'79)** one considers $a_{ij} = a_{ij}(t) \in LL([0,T])$, where $f : I \to \mathbf{R}$ is said **Log-Lipschitz continuous** if

$$\|f\|_{LL(I)} := \sup_{\substack{t,s\in I \\ 0<|t-s|<1/2}} \frac{|f(t)-f(s)|}{|t-s|\,|\log|t-s||} < +\infty,$$

and one proves that $(\mathcal{CP})$ is still C^∞ well-posed, but the phenomenon of the **loss of derivatives** arises:

$$\|u(t)\|^2_{H^{s-\beta t}} + \|\partial_t u(t)\|^2_{H^{s-1-\beta t}} \le C_s \left(\|u_0\|^2_{H^s} + \|u_1\|^2_{H^{s-1}}\right)$$

for some $\beta > 0$ depending on λ_0, Λ_0 and $\|a_{ij}\|_{LL([0,T])}$.

Idea of the proof: case $n = 1$, i.e. $\partial_t^2 u - a(t)\partial_x^2 u = 0$

- Fourier transform in x

$$v(t,\xi) := \widehat{u}^x(t,\xi) \Longrightarrow v \text{ solves } v'' + a(t)|\xi|^2 v = 0$$

- Introduce $a_\varepsilon = a * \varrho_\varepsilon$

$$a \in \mathrm{LL} \Longrightarrow \begin{cases} \int_0^T |a - a_\varepsilon|\, dt \le C\varepsilon \log\left(\frac{1}{\varepsilon} + 1\right) \\ \\ \int_0^T |a'_\varepsilon|\, dt \le C \log\left(\frac{1}{\varepsilon} + 1\right) \end{cases}$$

-**Classical energy** associate to the equation

$$E(t,\xi) = |v'(t)|^2 + a(t)|\xi|^2|v(t)|^2$$

- We define **Approximate energy**

$$E_\varepsilon(t,\xi) = |v'(t)|^2 + a_\varepsilon(t)|\xi|^2|v(t)|^2$$

and we obtain

$$\begin{aligned} E_\varepsilon(t,\xi) &\leq E_\varepsilon(0,\xi)\exp\left[c\left(\int \frac{|a'_\varepsilon|}{a_\varepsilon}\,dt + |\xi|\int \frac{|a-a_\varepsilon|}{\sqrt{a_\varepsilon}}\,dt\right)\right] \\ &\leq E_\varepsilon(0,\xi)\exp\left[c(t|\log\varepsilon| + t|\xi|\varepsilon|\log\varepsilon|)\right] \end{aligned}$$

- Now we choose $\varepsilon = |\xi|^{-1}$:

$$E_\varepsilon(t,\xi) \leq E_\varepsilon(0,\xi)\exp(c\,t\log|\xi|)$$

$\Longrightarrow$ well-posedness in C^∞:

loss of derivatives proportional to t

Log-Lipschitz regularity is optimal

Theorem (C.DG.S.'79). *There exists* $a(t)$, $1/2 \le a(t) \le 3/2$, $a \in \bigcap_{\alpha<1} C^{\alpha}([0,T]) \cap C^{\infty}(]0,T])$ *and there exist* $u_0, u_1 \in C^{\infty}$ *s.t.* $(\mathcal{CP})$ *has no solution* $u \in C([0,T], \mathcal{D}')$

More precisely:

Theorem (C.L.'95). $\forall \Omega(\tau)$ *s.t.* $\lim_{\tau \to 0^+} \Omega(\tau) = +\infty$ *there exists* $a(t)$, $1/2 \le a(t) \le 3/2$

$$|a(t+\tau) - a(t)| \le C|\tau| \Big| \log |\tau| \Big| \Omega(|\tau|)$$

and there exist $u_0, u_1 \in C^{\infty}$ *such that* $(\mathcal{CP})$ *has no solution in* $\mathcal{D}'$

Two natural questions:

(i) What well-posedness results for coefficients with regularity between Lip and $\mathrm{Log}-\mathrm{Lip}$?

(ii) Loss of derivatives really occurs?

Answer to (i):

Theorem (C.C.'06). *Let* $a_{ij}(t)$ *verify*

$$|a_{ij}(t+\tau) - a_{ij}(t)| \leq C|\tau| \Big| \log|\tau| \Big| \omega(|\tau|)$$

with $\omega(\tau) \searrow 0$ *as* $\tau \to 0^+$. *Then:*

$\forall \delta > 0$, $\exists\, C_\delta > 0$ *s.t.* $\forall\, t \in [0, T]$

$$\|u(t)\|_{H^{s+1-\delta}} + \|\partial_t u(t)\|_{H^{s-\delta}} \leq C_\delta \Big(\|u_0\|_{H^{s+1}} + \|u_1\|_{H^s} \Big)$$

i.e. loss of derivatives **arbitrary small**

Answer to (ii):

Theorem (C.C.'06). *For* $M > 0$ *let us set*

$$\mathcal{A}(M) = \left\{ a(t) \,:\, \frac{1}{2} \leq a(t) \leq \frac{3}{2}, \, \|a\|_{LL} \leq M \right\}$$

Then $\forall\, M$, $\exists\, \{a_k\}_{k \in \mathbb{N}} \subset \mathcal{A}(M)$ *and* $\exists\, u_k$ *s.t.*

- $\partial_t^2 u_k - a_k(t) \partial_x^2 u_k = 0$,
- $\|u_k(0)\|_{H^1} + \|\partial_t u_k(0)\|_{H^0} = 1$

but $\forall\, t > 0$, $\forall\, s_0 < \frac{1}{10} M\, t$ *it results*

$$\sup_{k \in \mathbb{N}} \Big(\|u_k(t)\|_{H^{1-s_0}} + \|\partial_t u_k(t)\|_{H^{-s_0}} \Big) = +\infty.$$

Remark. Real loss of derivatives proportional to $M\,t$

Coefficients a_{ij} depending also on x:

$$\lambda_0|\xi|^2 \le \sum_{i,j} a_{ij}(t,x)\xi_i\xi_j \le \Lambda_0|\xi|^2, \quad a_{ij} \in LL(\mathbb{R}_t \times \mathbb{R}^n_x)$$

Theorem (C.L.'95). *Let u solution of $(\mathcal{CP})$. Then $\exists\, T^*$, $\exists\, C$, $\exists\, \beta$: $\forall\, t \in [0, T^*]$ we have*

$$\sup_{0\le s\le t}\left(\|u(s)\|_{H^{1-\beta s}} + \|\partial_t u(s)\|_{H^{-\beta s}}\right) \le C\left(\|u_0\|_{H^1} + \|u_1\|_{H^0}\right).$$

So $(\mathcal{CP})$ is well-posed for $t \le T^*$.

Remark. T^* depends on $\|a_{ij}\|_{LL(\mathbb{R}^{n+1})}$

Let us now consider the case

$$\begin{cases} t & \mapsto & a_{ij}(t,x) & \text{Log} - \text{Lip uniformly in } x \\ x & \mapsto & a_{ij}(t,x) & C^\infty \end{cases}$$

with $D_x^\alpha a_{ij} \in L^\infty, |\alpha| \leq 2$.

Theorem (C.L.'95). *There exists* $\beta > 0$, *with* $\beta = \beta(\lambda_0, \Lambda_0, \|a_{ij}\|_{LL}, \|D_x^\alpha a_{ij}\|_{L^\infty})$, $|\alpha| \leq 2$, *s.t. if* u *is solution of* $(\mathcal{CP})$, $\forall\, m \geq 0$ *there exists* C_m

$$\sup_{0 \leq t \leq T^*} \Big(\|u(t)\|_{H^{m+1-\beta t}} + \|\partial_t u(t)\|_{H^{m-\beta t}} \Big) \leq C_m \Big(\|u_0\|_{H^{m+1}} + \|u_1\|_{H^m} \Big)$$

In order to prove these theorems we use **approximate energy**, **Littlewood-Paley decomposition** and **Bony paradifferential calculus**.

Idea of the proof: again for simplicity the case $n = 1$

We set

$$a_\varepsilon(t,x) := \iint \rho_\varepsilon(t-s)\rho_\varepsilon(x-y)a(s,y)\,ds\,dy$$

and we obtain

$$\sup_{(t,x)} |a_\varepsilon(t,x) - a(t,x)| \leq C\,\varepsilon \log\left(\frac{1}{\varepsilon} + 1\right)$$

$$\sup_{(t,x)} |\partial_x a_\varepsilon(t,x)| \leq C \log\left(\frac{1}{\varepsilon} + 1\right)$$

$$\sup_{(t,x)} |\partial_t a_\varepsilon(t,x)| \leq C \log\left(\frac{1}{\varepsilon} + 1\right)$$

Now we use Littlewood-Paley decomposition.

Let $\varphi_0 \in C_0^\infty(\mathbb{R}_\xi)$, $0 \leq \varphi_0(\xi) \leq 1$, $\varphi_0(\xi) = 1$ if $|\xi| \leq 1$, $\varphi_0(\xi) = 0$ if $|\xi| \geq 2$, φ_0 even and φ_0 decreasing on $[0, +\infty)$.

We set $\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi)$ and, if $\nu \geq 1$, $\varphi_\nu(\xi) = \varphi(2^{-\nu}\xi)$.

Let w be a tempered distribution; we define

$$w_\nu(x) = \varphi_\nu(D_x)w(x) = \frac{1}{2\pi}\int e^{ix\xi}\varphi_\nu(\xi)\widehat{w}(\xi)\,d\xi$$
$$= \frac{1}{2\pi}\int \widehat{\varphi}_\nu(y)w(x-y)\,dy$$

For all ν, w_ν is an entire analytic function belonging to L^2 and for all $m \in \mathbb{R}$ there exists $K_m > 0$ such that

$$\frac{1}{K_m}\sum_{\nu=0}^{\infty} \|w_\nu\|_{L^2}^2 2^{2m\nu} \le \|w\|_{H^m}^2 \le K_m \sum_{\nu=0}^{\infty} \|w_\nu\|_{L^2}^2 2^{2m\nu}.$$

Let now $u(t,x)$ be a solution of $L(u) = 0$ in $C^2([0,T], H^\infty(\mathbb{R}))$.

We set $u_\nu(t,x) = \varphi_\nu(D)u(t,x)$.

We obtain

$$\partial_t^2 u_\nu = \partial_x(a(t,x)\partial_x u_\nu) + \partial_x([\varphi_\nu, a]\partial_x u).$$

We introduce the approximate energy of u_ν, setting

$$e_{\nu,\varepsilon}(t) := \int_{\mathbb{R}} \Big(|\partial_t u_\nu|^2 + a_\varepsilon |\partial_x u_\nu|^2 + |u_\nu|^2 \Big)\, dx,$$

and we estimate $\frac{d}{dt} e_{\nu,\varepsilon}(t)$ by using the equation.

Finally we choose $\varepsilon = 2^{-\nu}$.

We define the total energy for the function u setting

$$E(t) := \sum_{\nu=0}^{\infty} \exp(-2\beta(\nu+1)t)\, 2^{-2\nu\theta} e_{\nu,2^{-\nu}}(t),$$

with $\beta > 0$ and $0 < \theta < 1/2$ to be suitably chosen and we estimate $\frac{d}{dt}E(t)$:

$$\frac{d}{dt}E(t) \leq (C-2\beta) \sum_{\nu=0}^{\infty} (\nu+1) \exp(-2\beta(\nu+1)t) 2^{-2\nu\theta} e_{\nu,2^{-\nu}}(t) + R$$

for some constant C, and with the reminder R due to the commutator $\partial_x([\varphi_\nu, a]\partial_x u)$.

In **(C.L.'95)** homogeneous assumptions in (t, x), but only global in x results

<u>Questions:</u>

- local existence results?

- local uniqueness results?

Local results **(C.M.'08)**

$$Lu := \sum_{i,j=0}^{n} \partial_{y_i}\left(a_{ij}\partial_{y_j}u\right) + \sum_{j=0}^{n}\left[b_j\partial_{y_j}u + \partial_{y_j}(c_j u)\right] + du, \;\; y \in \mathbb{R}^{n+1}$$

Assumptions: L defined in a neighborhood of $\overline{y} \in \Omega$;
$a_{ij} \in LL(\Omega)$, $b_j, c_j \in C^{\alpha}(\Omega)$ with $\frac{1}{2} < \alpha < 1$; $d \in L^{\infty}$

Σ smooth surface, $\overline{y} \in \Sigma$, L strictly hyperbolic in the direction conormal to Σ. We can suppose $\Sigma = \{\varphi = 0\}$ near $\overline{y}$

Lemma. *(i)* $\forall\, s \in]1-\alpha, \alpha[$ *and* $u \in H^s_{\mathrm{loc}}(\Omega \cap \{\varphi > 0\})$ *all terms in* L *are well-defined as elements of* $H^{s-2}_{\mathrm{loc}}(\Omega \cap \{\varphi > 0\})$

(ii) If $u \in H^s_{\mathrm{loc}}(\Omega \cap \{\varphi > 0\})$ *and* $Lu \in L^1_{\mathrm{loc}}(\Omega \cap \{\varphi > 0\})$ $\Longrightarrow u_{|\Sigma}$ *and* $X_\Sigma u$ *are well-defined*

Here X_Σ is a first order operator, depending on the second order part of L

We can now state the local results **(C.M.'08)**

Local existence:

Theorem. *Let* $s > 1-\alpha$ *and* ω *neighborhood of* $\overline{y}$ *in* Σ. *Then* $\exists\, s' \in]1-\alpha, \alpha[$, Ω' *neighborhood of* $\overline{y}$ *in* $\mathbb{R}^{n+1}$ *s.t.*

$$\forall \begin{cases} (u_0, u_1) \in H^s(\omega) \times H^{s-1}(\omega) \\ \\ f \in L^2(\Omega' \cap \{\varphi > 0\}) \end{cases}$$

the Cauchy problem

$$Lu = f, \quad u_{|\Sigma} = u_0, \quad X_\Sigma u = u_1$$

has a solution $u \in H^{s'}(\Omega' \cap \{\varphi > 0\})$

Local uniqueness:

Theorem. *If* $s > 1 - \alpha$ *and* $u \in H^s(\Omega \cap \{\varphi > 0\})$ *satisfies*

$$Lu = 0, \quad u_{|\Sigma} = 0, \quad X_\Sigma u = 0$$

then $u \equiv 0$ *near* $\overline{y}$

Application: a blow up criterion for nonlinear wave equations

Let us consider

$$\begin{aligned}\partial_t(a_0(u)\partial_t u) &+ \sum_{j=1}^{n} [\partial_t(a_j(u)\partial_{x_j})u + \partial_{x_j}(a_j(u)\partial_t u)] \\ &- \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(u)\partial_{x_j} u) + \partial_t(b_0(u)) \\ &+ \sum_{j=1}^{n} \partial_{x_j}(b_j(u)) = F(u)\end{aligned}$$

Assumptions:

- strict hyperbolicity

- all coefficients $\in C^\infty$

Let $s > \frac{n}{2} + 1$ and $u_0 \in H^s(\mathbb{R}^n)$, $u_1 \in H^{s-1}(\mathbb{R}^n)$

It is well-known that $(\mathcal{CP})$ is well-posed, i.e. for some $T > 0$

$$\exists!\, u \in C^0([0,T], H^s) \cap C^1([0,T], H^{s-1})$$

Moreover by uniqueness there exists T^* s.t. T^* is the maximal time of existence

Classical blow-up criterion:

If $T^* < +\infty$, then

$$\sup_{0<t<T^*} \Big[\|u(t)\|_{L^\infty} + \|Du(t)\|_{L^\infty} \Big] = +\infty \qquad (\star)$$

where $D := (\partial_t, \partial_{x_1}, \ldots, \partial_{x_n})$

Theorem. *Thanks to the previous local results, one can replace in* $(\star)$ Lip *norms with* Log – Lip *ones, i.e.*

$$\sup_{0<t<T^*} \Big[\|u(t)\|_{L^\infty(\mathbb{R}^n_x)} + \|u(t)\|_{LL([0,t]\times\mathbb{R}^n_x))} \Big] = +\infty \quad (\star\star)$$

Until now, regularity of coefficients of $\mathrm{Log}-\mathrm{Lip}$ type

In **(C.DS.K.'02)** another type of regularity, like

$$a_{ij}(t) \in C^1(]0,T]); \quad |\partial_t a_{ij}(t)| \leq \frac{C}{t}$$

$\Longrightarrow$ $(\mathcal{CP})$ well-posed in C^∞, again with loss of derivatives

Proof is based on approximate energy, by using again **first** derivatives of $a_{ij}(t)$

In **(C.DS.R.'03)** we use **first and second** derivatives:

$$a_{ij}(t) \in C^2(]0,T]); \quad |\partial_t^k a_{ij}(t)| \le C\left(\frac{1}{t}\log\frac{1}{t}\right)^k, \ k = 1,2$$

$\Longrightarrow$ $(\mathcal{CP})$ well-posed in C^∞, with loss of derivatives

Problem. If the previous assumption holds only for $k = 1$, do we have well-posedness?

Some energy estimates obtained by using

$$\partial_t^k a_{ij}(t),\ k = 1, 2, \quad \text{in S. Tarama '07}$$

Assumption: **Log–Zygmund** type regularity

$$|a_{ij}(t+\tau) + a_{ij}(t-\tau) - 2a_{ij}(t)| \le C|\tau| \Big| \log|\tau| \Big|,\ |\tau| < \frac{1}{2} \tag{LZ}$$

Remark. $a_{ij} \in LL \Longrightarrow (LZ)$ satisfied. The converse is **not true**

Example. The Weierstrass function

$$w(t) = \sum_{n=1}^{\infty} 2^{-n} n \sin(2^n t)$$

satisfies (LZ) but is not LL

Tarama uses approximate energy $\widetilde{E}(t,\xi)$ defined as

$$\widetilde{E}(t,\xi) = \frac{1}{a(t)}\left|\partial_t v(t,\xi) + \frac{a'(t)}{2a(t)}v(t,\xi)\right|^2 + a(t)|v(t,\xi)|^2$$

where again $v(t,\xi) := \widehat{u}^x(t,\xi)$ is the Fourier transform with respect to x of the solution $u(t,x)$ (here for simplicity $n=1$) and he proves that

$$(LZ) \Longrightarrow (\mathcal{CP}) \text{ is well-posed}$$

Log–Zygmund hyperbolic equations with coefficients depending on time and space - case $n = 1$.

In **(C.DS.'09)** one considers the case $n = 1$

$$Lu := \partial_t^2 u - \partial_x(a(t,x)\partial_x u), \quad \text{with } 0 < \lambda_0 \le a(t,x) \le \Lambda_0$$

Assumptions:

$$|a(t+\tau,x) + a(t-\tau,x) - 2a(t,x)| \le C|\tau| \Big| \log|\tau| \Big| \qquad (A)$$

$$|a(t,x+y) - a(t,x)| \le C|y| \Big| \log|y| \Big| \qquad (B)$$

Theorem. *(A) and (B) $\Longrightarrow$ $(\mathcal{CP})$ for L is C^∞ well-posed with loss of derivatives*

Idea of the proof. We set again

$$a_\varepsilon(t,x) := \iint \rho_\varepsilon(t-s)\rho_\varepsilon(x-y)a(s,y)\,ds\,dy,$$

and we obtain

$$\sup_{(t,x)} |a_\varepsilon(t,x) - a(t,x)| \le C\,\varepsilon \log\left(\frac{1}{\varepsilon}+1\right)$$

$$\sup_{(t,x)} |\partial_x a_\varepsilon(t,x)| \le C \log\left(\frac{1}{\varepsilon}+1\right)$$

$$\sup_{(t,x)} |\partial_t a_\varepsilon(t,x)| \le C \left(\log\left(\frac{1}{\varepsilon}+1\right)\right)^2$$

$$\sup_{(t,x)} |\partial_t^2 a_\varepsilon(t,x)|| \le C\frac{1}{\varepsilon} \log\left(\frac{1}{\varepsilon}+1\right)$$

$$\sup_{(t,x)} |\partial_t\partial_x a_\varepsilon(t,x)| \le C\frac{1}{\varepsilon} \log\left(\frac{1}{\varepsilon}+1\right)$$

The total energy is

$$E(t) := \sum_{\nu=0}^{\infty} \exp(-2\beta(\nu+1)t)\, 2^{-2\nu\theta} e_{\nu,2^{-\nu}}(t),$$

where $0 < \theta < \frac{1}{2}$ and $\beta > 0$ to be suitably chosen, while

$$e_{\nu,\varepsilon}(t) := \int_{\mathbb{R}} \left(\frac{1}{\sqrt{a_\varepsilon}} \left| \partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu \right|^2 + \sqrt{a_\varepsilon} |\partial_x u_\nu|^2 + |u_\nu|^2 \right) dx,$$

and u_ν is the standard **Littlewood-Paley decomposition** of u. Finally again we choose $\varepsilon = 2^{-\nu}$.

Remark that in the case of Colombini and Lerner the approximate energy of the ν component was simply

$$e_{\nu,\varepsilon}(t) := \int_{\mathbb{R}} \left(|\partial_t u_\nu|^2 + a_\varepsilon |\partial_x u_\nu|^2 + |u_\nu|^2 \right) dx.$$

Again we have to estimate $\frac{d}{dt}E(t)$:

$$\frac{d}{dt}E(t) \leq (C-2\beta)\sum_{\nu=0}^{\infty}(\nu+1)\exp(-2\beta(\nu+1)t)2^{-2\nu\theta}e_{\nu,2^{-\nu}}(t)+R$$

and the term R from the commutator $\partial_x([\varphi_\nu, a]\partial_x u)$ is now much more complicate to estimate.

The term to estimate is

$$\sum_{\nu=0}^{\infty} \exp(-2\beta(\nu+1)t)2^{-2\nu\theta}$$

$$\cdot \int \frac{2}{\sqrt{a_{2^{-\nu}}}} \mathrm{Re}\Big(\partial_x([\varphi_\nu, a]\partial_x u) \cdot \overline{\Big(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\Big)}\Big)\, dx$$

We set $\varphi_{-1} := 0$ and we define, for $\mu \geq 0$, $\psi_\mu := \varphi_{\mu-1} + \varphi_\mu + \varphi_{\mu+1}$. Then

$$\psi_\mu(D_x)(\varphi_\mu(D_x)\partial_x u) = \varphi_\mu(D_x)\partial_x u = \partial_x u_\mu,$$

and, consequently,

$$[\varphi_\nu, a]\partial_x u = [\varphi_\nu, a]\Big(\sum_\mu \partial_x u_\mu\Big) = \sum_\mu ([\varphi_\nu, a]\psi_\mu)\partial_x u_\mu.$$

The term to estimate will be dominated by

$$C \sum_{\nu,\mu} k_{\nu,\mu}(\nu+1)^{1/2} \exp(-\beta(\nu+1)t) 2^{-\nu\theta}(e_{\nu,2^{-\nu}}(t))^{1/2}$$
$$\cdot (\mu+1)^{1/2} \exp(-\beta(\mu+1)t) 2^{-\mu\theta}(e_{\mu,2^{-\mu}}(t))^{1/2},$$

where

$$k_{\nu,\mu} = e^{-(\nu-\mu)\beta t}\, 2^{-(\nu-\mu)\theta}\, 2^{\nu}(\nu+1)^{-1/2}(\mu+1)^{-1/2}\|([\varphi_\nu, a]\psi_\mu)\|_{L(L^2)}$$

and

$$\|([\varphi_\nu, a]\psi_\mu)\|_{L(L^2)} \le \begin{cases} C2^{-\nu}(\nu+1) & \text{if } |\nu-\mu| \le 2 \\ C2^{-\max\{\nu,\mu\}} \max\{\nu+1, \mu+1\} & \text{if } |\nu-\mu| \ge 3 \end{cases}$$

Our aim is to use Schur's Lemma, so we have to estimate

$$\sup_{\mu} \sum_{\nu} |k_{\nu,\mu}| + \sup_{\nu} \sum_{\mu} |k_{\nu,\mu}|.$$

Fixing suitably the value of βT^*, it is possible to prove that there exists a positive constant Γ_θ such that

$$\sup_{\mu} \sum_{\nu=0}^{+\infty} |k_{\nu,\mu}| + \sup_{\nu} \sum_{\mu=0}^{+\infty} |k_{\nu,\mu}| \leq \Gamma_\theta.$$

We finally obtain

$$\Big| \sum_{\nu=0}^{\infty} \exp(-2\beta(\nu+1)t)2^{-2\nu\theta}$$
$$\cdot \int \frac{2}{\sqrt{a_{2^{-\nu}}}} \mathrm{Re} \left(\partial_x([\varphi_\nu, a]\partial_x u) \cdot \overline{\left(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu \right)} \right) dx \Big|$$
$$\leq \Gamma_\theta \sum_{\nu=0}^{\infty} (\nu+1) \exp(-2\beta(\nu+1)t)2^{-2\nu\theta} e_{\nu,2^{-\nu}}(t)$$

The conclusion follows.

In **(C.F.'10)** one considers the complete second order operator (again $n = 1$)

$$Lu := \partial_t^2 u - \partial_x(a(t,x)\partial_x u) + b_0(t,x)\partial_t u + b_1(t,x)\partial_x u + c(t,x)u \ ,$$

$$0 < \lambda_0 \leq a(t,x) \leq \Lambda_0$$

with coefficient $a(t,x)$ again satisfying conditions

(A) (**Log-Zygmund** continuity in t)

(B) (**Log-Lipschitz** continuity in x)

while $b_0\,,\, b_1 \in L^\infty(\mathbb{R}_t; C^\alpha(\mathbb{R}_x))$, $\alpha > 0$, $\quad c \in L^\infty(\mathbb{R}_t \times \mathbb{R}_x)$.

Theorem. $(\mathcal{CP})$ *for* L *is* C^∞ *well-posed with loss of derivatives*

Log–Zygmund hyperbolic equations with coefficients depending on time and space - case $n > 1$.

In **(C.DS.F.M. in progress)** one considers the case of more space variables: $n \geq 1$.

We consider

$$L = \partial_t^2 - \sum_{j,k=1}^{n} \partial_{x_j}(a_{jk}(t,x)\partial_{x_k})$$

strictly hyperbolic on the strip $[0,T]\times\mathbb{R}^n$. Assumptions:

$$\sup_{(t,x)} |a_{j,k}(t+\tau,x)+a_{j,k}(t-\tau,x)-2a_{j,k}(t,x)| \leq C|\tau|\Big|\log|\tau|\Big|,$$

$$\sup_{(t,x)} |a_{j,k}(t,x+y)-a_{j,k}(t,x)| \leq C|y|\Big|\log|y|\Big|.$$

Theorem. $(\mathcal{CP})$ *for* L *is* C^∞ *well-posed with loss of derivatives*

Idea of the proof. We have now a difficulty in the definition of the approximate energy. In the case of Colombini and Lerner it is possible to pass from

$$e_{\nu,\varepsilon}(t) := \int_{\mathbb{R}} \left(|\partial_t u_\nu|^2 + a_\varepsilon |\partial_x u_\nu|^2 + |u_\nu|^2\right) dx$$

to

$$e_{\nu,\varepsilon}(t) := \int_{\mathbb{R}^n} \left(|\partial_t u_\nu|^2 + \sum_{j,k}^{n} a^\varepsilon_{j,k} \partial_{x_j} u_\nu \overline{\partial_{x_k} u_\nu} + |u_\nu|^2\right) dx,$$

where

$$a^\varepsilon_{j,k}(t,x) := \iint \rho_\varepsilon(t-s)\rho_\varepsilon(x-y) a_{j,k}(s,y)\, ds\, dy.$$

Now the presence of a_ε to the denominator is the cause of the introduction of paradifferential operators.

We define

$$a_\varepsilon(t,x,\xi) := \int \sum_{j,k}^{n} \rho_\varepsilon(t-s) a_{j,k}(s,x)\xi_j \xi_k \, ds$$

(remark that there is the regularization only w.r.t. t).

We denote by T_{a_ε} the paradifferential operator associated to the symbol a_ε and by σ_{a_ε} its classical symbol and the same for $\partial_t a_\varepsilon$, $\partial_t^2 a_\varepsilon$ and the powers of a_ε. It is possible to show that

$$|\partial_\xi^\alpha \sigma_{a_\varepsilon}(t,x,\xi)| \leq C_\alpha (1+|\xi|)^{2-|\alpha|},$$

$$|\partial_\xi^\alpha \partial_x^\beta \sigma_{a_\varepsilon}(t,x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{2-|\alpha|+|\beta|-1} \log(2+|\xi|).$$

Moreover for σ_{a_ε} the usual paradifferential spectral conditions hold.

Moreover

$$|\partial_\xi^\alpha \sigma_{\partial_t a_\varepsilon}(t,x,\xi)| \le C_\alpha \log(2+\frac{1}{\varepsilon})(1+|\xi|)^{2-|\alpha|},$$

$$|\partial_\xi^\alpha \partial_x^\beta \sigma_{\partial_t a_\varepsilon}(t,x,\xi)| \le C_{\alpha,\beta}\frac{1}{\varepsilon}(1+|\xi|)^{2-|\alpha|+|\beta|-1}\log(2+|\xi|),$$

and

$$|\partial_\xi^\alpha \sigma_{\partial_t^2 a_\varepsilon}(t,x,\xi)| \le C_\alpha \frac{1}{\varepsilon}\log(2+\frac{1}{\varepsilon})(1+|\xi|)^{2-|\alpha|},$$

$$|\partial_\xi^\alpha \partial_x^\beta \sigma_{\partial_t^2 a_\varepsilon}(t,x,\xi)| \le C_{\alpha,\beta}\frac{1}{\varepsilon^2}(1+|\xi|)^{2-|\alpha|+|\beta|-1}\log(2+|\xi|),$$

and for both of them the usual spectral conditions hold.

Similarly we define and check the behaviour of the paradifferential operators associated to the powers of a_ε.

Finally we set

$$e_{\nu,\varepsilon}(t) := \|T_{a_\varepsilon^{-1/4}}\partial_t u_\nu - T_{\partial_t(a_\varepsilon^{-1/4})}u_\nu\|_{L^2}^2 + \|T_{a_\varepsilon^{1/4}}u_\nu\|_{L^2}^2,$$

and the computations follow similarly to the previous case.

Open Problem. The case of coefficients $a_{ij}(t,x)$, $x \in \mathbb{R}^n$, with Log–Zygmund regularity in all the variables t and x, even for $n = 1$.

Partial Answer (C.DS.F.M. in progress). The case of coefficients $a_{ij}(t,x)$, $x \in \mathbb{R}^n$, with Zygmund regularity (not only Log–Zygmund) in all the variables t and x.

In this case the coefficients are Log–Lipschitz continuous, and so (**C.L.'95**) the Cauchy problem is well posed, **but with** a loss of regularity.

Really, it is possible to prove, again by using the para-differential calculus, that, thanks to the Zygmund regularity, the Cauchy problem is well posed **without** a loss of regularity for initial data u_0, u_1 in **precise** Sobolev spaces: $H^{1/2}$, $H^{-1/2}$.

More precisely, one can prove for the solution of the Cauchy problem the following energy estimate:

$$\|u(t)\|^2_{H^{1/2}} + \|\partial_t u(t)\|^2_{H^{-1/2}} \leq C \left(\|u_0\|^2_{H^{1/2}} + \|u_1\|^2_{H^{-1/2}} \right).$$