# *Entropy Conditions for some Flux Limited Diffusion Equations*

*Benasque, August 29-September 9, 2011*

V. Caselles

vicent.caselles@upf.edu

Universitat Pompeu Fabra

Departament de Tecnologia

## Introduction: RHE

$$u_t = \nu \operatorname{div} \left( \frac{u Du}{\sqrt{u^2 + \frac{\nu^2}{c^2}|Du|^2}} \right). \qquad (1)$$

- $\nu > 0$ represents a mean free path, $c$ the speed of light

This type of equations was introduced by J.R. Wilson (circa 1960) as a phenomenological model to control the speed of diffusion:

- when $c \to \infty$ (mean free path small with respect to $c$): the solution goes to

$$u_t = \nu \, \Delta u. \qquad (2)$$

- when $\nu \to \infty$ (free streaming in transparent regions): the solution goes to

$$u_t = c \operatorname{div} \left( u \frac{Du}{|Du|} \right) \qquad (3)$$

# Introduction: General case

We are interested in equations of the type

$$\begin{cases} \dfrac{\partial u}{\partial t} = \text{div}\, \mathbf{a}(u, Du) & \text{in} \quad Q_T = (0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x) & x \in \mathbb{R}^N \end{cases} \tag{4}$$

• $0 \le u_0 \in L^1(\mathbb{R}^N)$
• $\mathbf{a}(z,\xi) = \nabla_\xi f(z,\xi)$ is continuous where $f(z,\xi)$ is convex and differentiable in $\xi$.
• $f$ is coercive and satisfies the linear growth condition

$$C_0(z)\|\xi\| - D_0(z) \le f(z,\xi) \le M_0(z)(\|\xi\| + 1) \tag{5}$$

$\forall (z,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and some positive and continuous functions $C_0$, $D_0$, $M_0$, such that $C_0(z) > 0\ \forall z \neq 0$.

For the RHE: $f(z,\xi) = \frac{c^2}{\nu}|z|\sqrt{z^2 + \frac{\nu^2}{c^2}|\xi|^2}$.

# Brenier's derivation of RHE

The RHE can be derived using a 'gradient descent' with

$$\rho_n^h := \arg \min_{\rho \in \mathcal{P}(\mathbb{R}^N)} \left\{ h W_k^h \left( \rho_{n-1}^h, \rho \right) + E(\rho) \right\}.$$

$$E(\rho) := \int_{\mathbb{R}^N} F(\rho(x))\, dx, \quad F(\rho) := \nu \rho (\ln \rho - 1)$$

$$W_k^h(\rho_0, \rho_1) := \inf_{\gamma} \left\{ \int_{\mathbb{R}^N \times \mathbb{R}^N} k\left( \frac{x-y}{h} \right) d\gamma(x,y) \right\},$$

where $\gamma$ is a probability measure in $\mathbb{R}^N \times \mathbb{R}^N$ with marginals $\rho_0, \rho_1$, and

$$k(z) := \begin{cases} c^2 \left( 1 - \sqrt{1 - \frac{|z|^2}{c^2}} \right) & \text{if } |z| \le c \\ +\infty & \text{if } |z| > c. \end{cases} \tag{6}$$

Formally, we have

$$\frac{\rho_n^h - \rho_{n-1}^h}{h} = \mathrm{div}\left\{\rho_n^h \nabla k^* \left[\nabla\left(F'(\rho_n^h)\right)\right]\right\} + A_n(h)$$

where $A_n(h) \to 0$ as $h \to 0+$ (in the dual of $W^{2,\infty}$).

Recently transformed into a rigorous approach by R. McCann and M. Puel.
Other phenomenological derivations by Ph. Rosenau (1990).

# Plan of the talk

- Recall the notion of entropy solution to have existence and uniqueness result.
- Recall that for the RHE the support of solutions moves with speed $c > 0$
- There are discontinuity fronts propagating at speed $c$
- Interpretation of the notion of entropy solution in terms of the evolution of the discontinuity fronts.
- Final comments

# Entropy solutions: motivation

Start with the RHHE model (with $c = 1$)

$$u_t = \mathrm{div}\left(u\frac{Du}{|Du|}\right) = u\mathrm{div}\left(\frac{Du}{|Du|}\right) + |Du|.$$

Let $u_0(x) = \chi_C(x)$, $C$ convex, for instance $C = B(0, R)$.
Look for solutions of the form

$$u(t,x) = \alpha(t)\chi_{C(t)}, \quad C(t) = C \oplus B(0,t).$$

Then

$$u_t = \alpha'(t)\chi_{C(t)} + \alpha(t)\mathcal{H}^{N-1} \llcorner \partial C(t)$$

$$\mathrm{div}\left(u\frac{Du}{|Du|}\right) = \alpha(t)\chi_{C(t)}\,\mathrm{div}\left(\frac{Du}{|Du|}\right) + \alpha(t)\mathcal{H}^{N-1} \llcorner \partial C(t).$$

$$\Longrightarrow \quad \alpha'(t) = \alpha(t)\,\mathrm{div}\left(\frac{Du}{|Du|}\right) \quad \text{in } C(t)$$

Assuming that $C(t)$ is calibrable

$$\operatorname{div}\left(\frac{Du}{|Du|}\right) = -\frac{P(C(t))}{|C(t)|}\,\chi_{C(t)}.$$

Hence

$$\alpha'(t) = -\alpha(t)\frac{P(C(t))}{|C(t)|}$$

$$\alpha(t) = \alpha\frac{|C|}{|C(t)|}, \qquad u(t,x) = \alpha\frac{|C|}{|C(t)|}\chi_{C(t)}.$$

Observe that

$$u \in C([0,T], L^1(\mathbb{R}^N))$$

$$u \in L^1_{\mathrm{loc}}((0,T), BV(\mathbb{R}^N))$$ (weakly measurable)

$$u_t \in \mathcal{M}((0,T)\times\mathbb{R}^N)$$ (in this case).

# Basic formal estimates:

- Mass preservation:

$$\frac{d}{dt}\int u = \int \operatorname{div} \mathbf{a}(u, Du) = 0$$

- $L^p$-estimates:

$$\frac{d}{dt}\int u^{p+1} = -(p+1)\int \mathbf{a}(u, Du)\cdot Du^p \leq 0$$

- $BV$-estimate (case RHE)

$$\frac{d}{dt}\int u^2 + 2\int \frac{uDu}{\sqrt{u^2+|Du|^2}}\cdot Du \leq 0$$

Use that $\frac{u|\xi|^2}{\sqrt{u^2+|\xi|^2}} \geq u|\xi| - u^2$ to obtain $\frac{d}{dt}\int u^2 + \int |Du^2| \leq C(\|u_0\|_2)$

$$\Longrightarrow T_{a,b}(u(t)) - a \in BV(\mathbb{R}^N) \quad t > 0,\ 0 < a < b.$$

# Entropy solutions: motivation

Recall the usual FORMAL uniqueness proof of solutions of:

$$\frac{\partial u}{\partial t} = \text{div } \mathbf{a}(u, Du) \qquad (7)$$

If $u, v$ are two solutions with values $u(t), v(t) \in BV(\mathbb{R}^N)$, then

$$\frac{1}{2}\frac{d}{dt}\int |u - v| dx = \frac{1}{2}\int (u - v)_t \text{sign}(u - v) =$$

$$-\int (\mathbf{a}(u, Du) - \mathbf{a}(v, Dv)) \cdot (Du - Dv)\delta_0(u - v) =$$

$$-\int (\mathbf{a}(u, Du) - \mathbf{a}(u, Dv)) \cdot (Du - Dv)\delta_0(u - v) \leq 0.$$

Thinking in a Kruzkov's type of proof:

- we need test functions of the form $T(u - l) \in L^1_{\text{loc}}((0, T), BV(\mathbb{R}^N))$.
- we have to give sense to the above integrals.

# Kruzkov's definition of solution:

**Theorem.** (Andreu-C-Mazón)
$u \in C([0,T]; L^1(\mathbb{R}^N))$
$T_{a,b}(u(\cdot)) \in L^1_{loc,w}(0,T,BV(\mathbb{R}^N))\ \forall 0 < a < b$, where

$$T_{a,b}(r) = \max\left(\min(r,b), a\right)$$

(i) $u_t = \operatorname{div} \mathbf{a}(u, \nabla u)$ in $\mathcal{D}'(Q_T)$

(ii) Kruzkov inequalities hold.

$(i) + (ii) \Longrightarrow$ uniqueness

Let us explain how to write Kruzkov's inequalities.

# The Kruzkov's inequalities

We need truncatures $S(u-l)$ to approximate the $\operatorname{sign}(u-l)$, $l \in \mathbb{R}$.
Write them as $S(u)$.
We need truncatures $T(u)$ so that $T(u(t)) \in BV(\mathbb{R}^N)$ ($S = T_{a,b} - a$)
Then formally multiply $u_t = \operatorname{div} \mathbf{a}(u, Du)$ by $S(u)T(u)\phi(t,x)$

$$-\int_0^T \int_{\mathbb{R}^N} J_{TS}(u(t))\phi_t(t)\, dxdt$$

$$\int_0^T \int_{\mathbb{R}^N} \phi\, \underbrace{S(u)\mathbf{a}(u, Du) \cdot DT(u)}_{S(u)h(u, DT(u))}\, dt + \int_0^T \int_{\mathbb{R}^N} \phi\, \underbrace{T(u)\mathbf{a}(u, Du) \cdot DS(u)}_{T(u)h(u, DS(u))}\, dt +$$

$$\int_0^T \int_{\mathbb{R}^N} \mathbf{a}(u, Du) \cdot \nabla\phi\, T(u)S(u)\, dxdt \leq 0$$

$\forall \phi$ test function, $J_{TS}$ the primitive of $T(r)S(r)$.

## The term $S(u)\mathbf{a}(u, Du) \cdot DT(u)$

- Let $F(z, \xi)$ be one of the functions

$$f(z,\xi) \quad \text{or} \quad h(z,\xi)$$

Then if $w \in BV(\mathbb{R}^N)$ we know how to define the measure $S(w)F(w, Dw)$:

$$\langle S(w)F(w,Dw),\phi\rangle := \int \phi(x)S(w)F(w,\nabla w)\,dx$$

$$+\int \phi(x)(SF)^{\circ}(\tilde{w}(x),\frac{D^c w}{|D^c w|}(x))\,d|D^c w|$$

$$+\int\int_{w^-(x)}^{w^+(x)} \phi(x)(SF)^{\circ}(s,\nu_w(x))\,d\mathcal{H}^{N-1}$$

where $(SF)^{\circ}(z,\xi) := \lim_{t\to 0+} tS(u)F(u,\frac{\xi}{t})$ is the recession function. $\tilde{w}(x)$ is the approximate limit of $w$ at $x$

## Functional calculus:

If $T = T_{a,b} + c$, $c \in \mathbb{R}$,

$$\langle S(u)F(u, DT(u)), \phi\rangle := \langle S(T_{a,b}(u))F(T_{a,b}(u), DT_{a,b}(u)), \phi\rangle +$$

$$\int_{[u<a]} \phi(x)[S(u)F(u,0) - S(a)F(a,0)] +$$

$$\int_{[u>b]} \phi(x)[S(u)F(u,0) - S(b)F(b,0)]$$

When need the lower semicontinuity results for $F(w, Dw)$ when $F(z, \xi)$ is convex in $\xi$ (G.Buttazzo, G. Dal Masso, V. De Cicco- N. Fusco- A. Verde).

# Entropy inequalities

Entropy inequalities:

$$S(u)h(u, DT(u)) + T(u)h(u, DS(u)) \leq -(J_{TS}(u(t)))_t + \mathrm{div}\,(\mathbf{z}T(u)S(u))$$

where $\mathbf{z} = \mathbf{a}(u, \nabla u)$

$\forall S, T \in \mathcal{TSUB} \cup \mathcal{TSUPER}$ (positive truncatures)

$\forall \phi$ test funtion.

$(S, T) \in \mathcal{TSUB}$ truncatures: $S \geq 0, S' \geq 0$ and $T \geq 0, T' \geq 0$.

$(S, T) \in \mathcal{TSUPER}$ truncatures: $S \leq 0, S' \geq 0$ and $T \geq 0, T' \leq 0$.

Analysis of the entropy conditions requires test functions in $\mathcal{TSUB}$ and $\mathcal{TSUPER}$.

# Existence and Uniqueness

Theorem: Existence and Uniqueness (Andreu-C-Mazón)

Under assumptions on $f(z,\xi)$ and $\mathbf{a}(z,\xi)$:

For any $0 \le u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ there exists a unique entropy solution $u$ of

$$\begin{aligned} &u_t = \operatorname{div} \mathbf{a}(u, Du) \quad \text{in } Q_T = (0,T) \times \mathbb{R}^N \\ &u(0) = u_0 \end{aligned}$$

If $u(t)$, $\overline{u}(t)$ are the entropy solutions corresponding to initial data $u_0$, $\overline{u}_0 \in \left(L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)\right)^+$, then

$$\|(u(t) - \overline{u}(t))^+\|_1 \le \|(u_0 - \overline{u}_0)^+\|_1 \qquad \text{for all } \ t \ge 0. \tag{8}$$

Use Crandall-Liggett's scheme and Kruzkov's technique.

# Evolution of the support of RHE

**Theorem** (Andreu-C-Mazón-Moll)

Let $C$ be an open bounded set in $\mathbb{R}^N$.

Let $u_0 \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$ with support equal to $\overline{C}$.

Let $u(t)$ be the entropy solution of RHE with $u(0) = u_0$. If we assume that

(*) $u_0 >> 0$ inside its support,

then

$$\mathrm{supp}(u(t)) = \overline{\mathrm{supp}(u_0) \oplus B(0, ct)} \quad \text{for all } t \geq 0.$$

## The Rankine-Hugoniot condition

Let $u \in BV_{\rm loc}((0,T)\times\mathbb{R}^N)$ and let $\mathbf{z} \in L^\infty([0,T]\times\mathbb{R}^N,\mathbb{R}^N)$ be such that $u_t = \operatorname{div}\mathbf{z}$.
Let $(\nu_t, \nu_x)$ be the normal to $J_u$.

We define the speed of the discontinuity set of $u$ as $v(t,x) = \frac{\nu_t(t,x)}{|\nu_x(t,x)|}$ $\mathcal{H}^N$-a.e. on $J_u$.

Observe that $H^N(\{(t,x)\in J_u : \nu_x(t,x) = 0\}) = 0$.

**Proposition**
For $\mathcal{L}^1$ almost any $t>0$ we have

$$[u(t)](x)v(t,x) = [[\mathbf{z}\cdot\nu^{J_{u(t)}}]]_{+-} \qquad \mathcal{H}^{N-1}\text{-a.e. in } J_{u(t)}, \tag{9}$$

where $[[\mathbf{z}\cdot\nu^{J_{u(t)}}]]_{+-}$ denotes the difference of traces from both sides of $J_{u(t)}$.

# The Rankine-Hugoniot condition

Let $B$ be a Borel set of $J_u$ contained in the boundary of an open Lipschitz set. Then by integration by parts:

$$[u]\nu_t \mathcal{H}^N|_B = [[\mathbf{z} \cdot \nu^{J_{u(t)}}]]_{+-} \mathcal{H}^{N-1}|_{J_{u(t)} \cap B}\, dt.$$

We have by disintegration:

$$\nu_x \mathcal{H}^N|_{J_u} = \nu^{J_{u(t)}} \mathcal{H}^{N-1}|_{J_{u(t)}}\, dt$$

Hence (by the definition of $v$):

$$\nu_t \mathcal{H}^N|_{J_u} = \frac{\nu_t}{|\nu_x|} |\nu_x| \mathcal{H}^N|_{J_u} = v \mathcal{H}^{N-1}|_{J_{u(t)}}\, dt.$$

We obtain

$$[u] v \mathcal{H}^{N-1}|_{J_{u(t)} \cap B}\, dt = [[\mathbf{z} \cdot \nu^{J_{u(t)}}]]_{+-} \mathcal{H}^{N-1}|_{J_{u(t)} \cap B}\, dt.$$

This implies the conclusion.

# Is $u_t$ a Radon measure ?

**Proposition**
For the RHE: $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $u_0(x) \geq 0$.
Let $\Gamma_i$, $i = 0, \ldots, \ell$, be the boundaries of bounded open sets of class $C^{1,1}$. Assume that

(i) $\mathrm{dist}\,(\Gamma_i, \Gamma_j) > 0$ for any $i \neq j$,

(ii) $u_0 \in W^{2,1}(\mathbb{R}^N \setminus \cup_{i=0}^{\ell} \Gamma_i)$ and $\nabla u_0 \in L^\infty(\mathbb{R}^N \setminus \cup_{i=0}^{\ell} \Gamma_i)$,

(iii) $u_0$ is discontinuous in $\Gamma_i$.

(iv) $u_0$ is either $0$ or is bounded away from zero in any connected component of $\mathbb{R}^N \setminus \cup_{i=0}^{\ell} \Gamma_i$.

(v) When the trace $u_0^-|_{\Gamma_i}$ is bounded away from zero, and we are on the side corresponding to the upper (resp. lower) trace of $u_0$, the direction of the gradient of $u_0$ and the normal to $\Gamma_i$ are not aligned near the points where $u_0$ is increasing (resp. decreasing) towards $\Gamma_i$.

If $u(t)$ is the solution of the RHE with $u(0) = u_0$, then $u_t$ is a Radon measure.

# Characterization of entropy conditions

Let $u \in C([0,T]; L^1(\mathbb{R}^N)) \cap BV_{\text{loc}}((0,T) \times \mathbb{R}^N)$.

Assume that $u$ is the entropy solution of

$$(RHEm) \qquad u_t = \nu \operatorname{div} \left( \frac{u^m \nabla u}{\sqrt{u^2 + \frac{\nu^2}{c^2} |\nabla u|^2}} \right). \tag{10}$$

$$u(0) = u_0 \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+ \cap BV(\mathbb{R}^N),$$

where $m \geq 1$.

# Characterization of entropy conditions

Recall $\mathbf{z}(t,x) = \mathbf{a}(u, Du)$.

Then $u$ is an entropy solution of (RHEm) if and only if for any $(T,S) \in \mathcal{TSUB}$ (for any $(T,S) \in \mathcal{TSUB} \cup \mathcal{TSUPER}$) we have

$$(S(u)h(u, DT(u))^c + (T(u)h(u, DS(u))^c \leq (\mathbf{z}(t,x) \cdot D(T(u)S(u)))^c$$

and for $\mathcal{L}^1$-almost any $t > 0$ the inequality

$$[STu(t)^m]_{+-} - [J_{TS(u^m)'}(u(t))]_{+-}$$

$$\leq -v[J_{TS}(u(t))]_{+-} + [[\mathbf{z}(t) \cdot \nu^{J_{u(t)}}]T(u(t))S(u(t))]_{+-}$$

holds $\mathcal{H}^{N-1}$ a.e. on $J_{u(t)}$.

(Write the EC on the jump set.)

# Vertical contact angle at the boundary of the evolving front

If $m = 1$, we assume that $u \in BV_{\rm loc}((0,T) \times \mathbb{R}^N)$.
If $m > 1$ and $u_0 \in BV(\mathbb{R}^N)$, then $u \in BV_{\rm loc}((0,T) \times \mathbb{R}^N)$.
Then the entropy conditions hold on the jump set if and only if

$$\left[ \frac{u^m \nabla u}{\sqrt{u^2 + \frac{\nu^2}{c^2}|\nabla u|^2}} \cdot \nu^{J_{u(t)}} \right]_+ = (u^+)^m$$

$$\left[ \frac{u^m \nabla u}{\sqrt{u^2 + \frac{\nu^2}{c^2}|\nabla u|^2}} \cdot \nu^{J_{u(t)}} \right]_- = (u^-)^m.$$

Moreover the velocity of the discontinuity fronts is

$$v = \frac{(u^+)^m - (u^-)^m}{u^+ - u^-}. \tag{11}$$

In case $m = 1$, we have $v = 1$.

## Asymptotic limit as $c \longrightarrow \infty$

**Theorem**

Let $u_c$ be the entropy solution of (RHE) with $u(0,x) = u_0(x) \in (L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))^+$. As $c \to \infty$, $u_c$ converges in $C([0,T], L^1(\mathbb{R}^N))$ to the solution of $U$ the heat equation

$$u_t = \nu \Delta u$$

with $u(0,x) = u_0(x)$.

**Basic estimate:**

Assume that $u_0 > 0$ satisfies $\left\| \frac{\nabla u_0}{u_0} \right\|_\infty < \infty$.

Let $u$ be the entropy solution of RHE. Then for any $t > 0$, $u(t)$ satisfies

$$\sup_{[0,T]\times\mathbb{R}^N} \frac{|\nabla u(t,x)|}{|u(t,x)|} \leq \left\| \frac{\nabla u_0}{u_0} \right\|_\infty . \qquad (12)$$

# Asymptotic limit as $c \longrightarrow \infty$

If $u(x) > 0$ for all $x \in \mathbb{R}^N$, define $u = e^v$. If $u$ is an entropy solution of

$$u_t = \operatorname{div}\left(\frac{uDu}{\sqrt{u^2 + |Du|^2}}\right). \tag{13}$$

Then $v(t,x)$ is a solution of

$$v_t = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}}\right) + \frac{|\nabla v|^2}{\sqrt{1 + |\nabla v|^2}}. \tag{14}$$

1. V. Caselles, “An existence and uniqueness result for flux limited diffusion equations”. To appear in Communications on Pure and Applied Analysis, special volume in honor to Ennio de Giorgi and Guido Stampacchia.
2. V. Caselles, On the entropy conditions for some flux limited diffusion equations. Journal Differential Equations 250, pp. 3311-3348, 2011.