Topology optimization methods with gradient-free perimeter perimeter approximation

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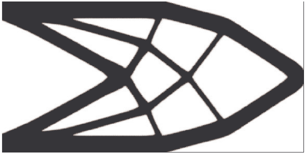
CLASSICAL EXAMPLE: THE CANTILEVER

Class of shapes $\mathcal{A}_D := \{\Omega \subset D \text{ open}\}$ with $D \subset \mathbb{R}^N$ bounded.



Seek optimal Ω: minimizing the cost function:

$$J(u) = \int_D Ae(u) \cdot e(u) dV + \ell Vol(\Omega)$$

where $u = \chi_\Omega \in \mathcal{E} := L^\infty(D, \{0, 1\})$.

The problem is ill posed. However

$$\inf\{J(u) + Per_D(\Omega)\} = \inf\{J(u) + TV(u)\}$$

is well posed.

NUMERICAL ISSUE

- In topology optimization (in particular, to create holes in a structure) one needs to approximate $Per_D(\Omega) = TV(u)$.
- Other issue: $u = \chi_\Omega$ is nonsmooth and $TV(u) = |Du|(D)$ is non-differentiable.
- Heuristically: let us consider a parabolic regularization

$$\partial_t v - \Delta v = 0 \quad \text{with} \quad v(0) = u$$

- Let $\delta t = \epsilon^2$: short-time approximation of the time derivative at the first order:

$$\frac{v_\epsilon - u}{\epsilon^2} - \Delta v_\epsilon = 0 \Leftrightarrow \text{(PDE): } -\epsilon^2 \Delta v_\epsilon + v_\epsilon = u \quad (\star)$$

- Claim A: $\|u - v_\epsilon\|^2_{L^2(D)} \sim \frac{\epsilon}{4} Per_D(\Omega)$

APPROXIMATION BY THE PDE

- Claim A is proved if $D = \mathbb{R}^N$ and Ω smooth.
- By $(\star)$:

$$\frac{1}{\epsilon}\|u - v_\epsilon\|^2_{L^2(D)} \sim F_\epsilon(u) := \frac{1}{2\epsilon}\|u - v_\epsilon\|^2_{L^2(D)} + \frac{\epsilon}{2}\|\nabla v_\epsilon\|^2_{L^2(D)}$$

- Observation:

$$F_\epsilon(u) = \inf_{v \in H^1(D)} \{\frac{1}{2\epsilon}\|u - v\|^2_{L^2(D)} + \frac{\epsilon}{2}\|\nabla v\|^2_{L^2(D)}\}$$

- Relaxation in the weak$\star$ L^∞ sense:

$$\tilde{F}_\epsilon(u) = \inf_{v \in H^1(D)} \{\frac{1}{2\epsilon}\left(\|v\|^2_{L^2(D)} + \langle u, 1 - 2v\rangle\right) + \frac{\epsilon}{2}\|\nabla v\|^2_{L^2(D)}\}$$

GAMMA CONVERGENCE RESULT

Claim B:

$$\tilde{F}_\epsilon(u) \xrightarrow{\Gamma} \tilde{F}(u) := \begin{cases} \frac{1}{4}|Du|(D) & \text{if} \quad u \in BV(D, \{0,1\}) \\ +\infty & \text{otherwise} \end{cases}$$

strongly in $L^1(D, [0,1])$.

- If $u = \chi_\Omega$: $|Du|(D) = Per(\Omega)$.

EQUICOERCIVITY

- Let $\tilde{J} : L^\infty(D, [0,1]) \to \mathbb{R}$ be any cost function. Let

$$I_\epsilon := \inf_{u \in L^\infty(D,[0,1])} \left\{ \tilde{J}(u) + \alpha \tilde{F}_\epsilon(u) \right\}$$

and

$$I := \inf_{u \in L^\infty(D,[0,1])} \left\{ \tilde{J}(u) + \alpha \tilde{F}(u) \right\}$$

Theorem

Let u_ϵ be an approximate minimizer of I_ϵ, i.e.

$$\tilde{J}(u_\epsilon) + \alpha \tilde{F}_\epsilon(u_\epsilon) \leq I_\epsilon + \lambda_\epsilon,$$

with $\lim_{\epsilon \to 0} \lambda_\epsilon = 0$. Assume that $\tilde{J}$ is continuous on $L^1(D, [0,1])$. Then we have $\tilde{J}(u_\epsilon) + \alpha \tilde{F}_\epsilon(u_\epsilon) \to I$. Moreover, (u_ϵ) admits cluster points, and each of these cluster points is a minimizer of I.

NUMERICAL APPROACH

$$I_\epsilon = \inf_{u \in L^\infty(D,[0,1])} \inf_{v \in H^1(D)} \left\{ \tilde{J}(u) + \alpha \left[\frac{\epsilon}{2} \|\nabla v\|^2_{L^2(D)} + \frac{1}{2\epsilon} \left(\|v\|^2_{L^2(D)} + \langle u, 1 - 2v \rangle \right) \right] \right\}.$$

• In the compliance case with homogenization $\tilde{J}(u)$ is also an inf (by the complementary energy).

• One can use an explicit alternate minimization algorithm to solve the problem.

NUMERICAL RESULTS

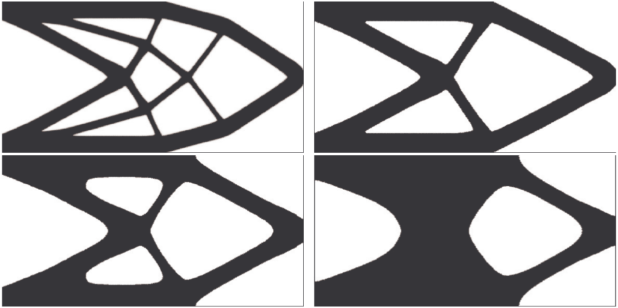


Figure: Cantilever for $\alpha = 0.1, 2, 20, 50,$ respectively.