Parabolic-elliptic Keller-Segel system with nonlinear diffusion

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The Patlak-Keller-Segel (PKS) equation in $\mathbb{R}^d$, $d \geq 3$

$$\partial_t u = \text{ div } \left(\nabla u - u \left(\nabla E_d * u\right)\right) , \quad (t,x) \in (0,\infty) \times \mathbb{R}^d ,$$

where E_d is the Poisson kernel $E_d(x) := c_d \, |x|^{-(d-2)}$.

For any $M > 0$, there is u_0 with $\|u_0\|_1 = M$ such that u blows up in finite time: the diffusion is too weak to prevent concentration (unlike when $d = 2$).

Nonlinear diffusion prevents crowding

The generalized Keller-Segel (GKS) equation in $\mathbb{R}^d$, $d \geq 3$

$$\partial_t u = \text{ div } \left(\nabla u^m - u \left(\nabla E_d * u\right)\right) , \quad (t,x) \in (0,\infty) \times \mathbb{R}^d ,$$

with

- $d \geq 3$,
- $m > 1$,
- E_d is the Poisson kernel $E_d(x) := c_d \, |x|^{-(d-2)}$.

Global existence/blowup

Define

$$m_d := \frac{2(d-1)}{d}.$$

- if $m > m_d$, global existence.
- if $m \in (1, m_d]$, global existence for small initial data and convergence to zero as the diffusion equation.
- if $m \in (1, m_d)$, finite time blowup for some initial data.

[Sugiyama & Kunii (2006), Luckhaus & Sugiyama (2006), Sugiyama (2007)]

A Liapunov functional

The functional

$$\mathcal{F}[u(t)] := \int_{\mathbb{R}^d} \frac{u(t,x)^m}{m-1}\, dx - \frac{1}{2} \int_{\mathbb{R}^d} (E_d * u)(t,x)\, u(t,x)\, dx\,,$$

is a decreasing function of time: two competing terms in $\mathcal{F}$

Virial identity

$$\frac{d}{dt}\int_{\mathbb{R}^d} |x|^2\, u(t,x)\, dx = 2(d-2)\, \mathcal{F}[u(t)] + \frac{2d}{m-1}\, (m - m_d)\, \|u(t)\|_m^m\,.$$

$\longrightarrow$ if $m \leq m_d$ and $\mathcal{F}[u_0] < 0$ we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} |x|^2\, u(t,x)\, dx \leq 2(d-2)\, \mathcal{F}[u_0] < 0\,,$$

and thus non-existence of global solutions.

Study of $\mathcal{F}$

Study of $\mathcal{F}$: $m < m_d$

For $M > 0$, define

$$\mathcal{Y}_M := \left\{ h \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)\ ,\ \|h\|_1 = M \right\}$$

and

$$\mu_M := \inf_{h \in \mathcal{Y}_M} \mathcal{F}[h]\,.$$

If $m < m_d$ and $M > 0$, then $\mu_M = -\infty$ (scaling argument).

Study of $\mathcal{F}$: $m = m_d$

For $M > 0$, define

$$\mathcal{Y}_M := \left\{ h \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d) \,,\; \|h\|_1 = M \right\}$$

and

$$\mu_M := \inf_{h \in \mathcal{Y}_M} \mathcal{F}[h]\,.$$

There is $M_c > 0$ such that

- If $M \in (0, M_c)$, then $\mu_M = 0$ and there is no minimiser.
- If $M = M_c$, then $\mu_M = 0$ and there are minimisers.
- If $M > M_c$, then $\mu_M = -\infty$.

Modified Hardy-Littlewood-Sobolev (HLS) inequality
[Blanchet, Carrillo & L. (2009), Suzuki & Takahashi (2009)]

Global existence and blowup: $m = m_d$

- Global existence if $\|u_0\|_1 < M_c$ by the modified HLS inequality.
- Global existence if $\|u_0\|_1 = M_c$ by a concentrated-compactness argument.
- If $\|u_0\|_1 > M_c$ and $\mathcal{F}[u_0] < 0$, finite time blowup.

[Blanchet, Carrillo & L. (2009), Suzuki & Takahashi (2009)]

Question: what happens if $\|u_0\|_1 > M_c$ and $\mathcal{F}[u_0] \geq 0$?

Stationary solutions: $m = m_d$

There is a two-parameter family $\{V_{z,R}\}$ of non-negative and compactly supported stationary solutions such that

$$\|V_{z,R}\|_1 = M_c\,, \quad z \in \mathbb{R}^d\,,\ R > 0\,.$$

[Chavanis & Sire (2008), Blanchet, Carrillo & L. (2009)]

Remark: if $m = 2d/(d+2) \leq m_d$, existence of a two-parameter family of stationary solutions:

$$2^{(d+2)/4}\, d^{(d+2)/2} \left(\frac{b}{b^2 + |x-z|^2} \right)^{(d+2)/2}, \quad z \in \mathbb{R}^d\,,\ b > 0\,.$$

[Chen, Liu & Wang (2011)]

Question: other values of $m < m_d$ for which there are stationary solutions?

Self-similar blowing-up/backward solutions: $m = m_d$

Given $T > 0$ and $a > 0$, there is a solution of the form

$$u_a(t,x) = \frac{1}{s(t)^d}\, U_a\left(\frac{|x|}{s(t)}\right) \quad \text{for} \quad (t,x) \in [0,T) \times \mathbb{R}^d$$

with $s(t) := [d(T-t)]^{1/d} \longrightarrow$ blowup at time T.
Furthermore, there is $a_c > 0$ such that

- if $a \in (0, a_c)$, then $\|u_a(t)\|_1 = \infty$.
- if $a \geq a_c$, then U_a is compactly supported and $\|u_a(t)\|_1 = \|u_a(0)\|_1 < \infty$.
- $$\lim_{a\to\infty} \|u_a(0)\|_1 = M_c \quad \text{and} \quad \sup_{a\in[a_c,\infty)} \|u_a(0)\|_1 \leq M_2 < \infty\,.$$

ODE approach [Blanchet & L. (2009)]

Questions

$m = m_d$:

- If $\|u_0\|_1 \in (0, M_c)$, convergence to a unique self-similar solution?
- If $\|u_0\|_1 = M_c$, convergence to a steady state or blowup in infinite time and concentration to a Dirac mass?
- Behaviour if $\|u_0\|_1 > M_c$ and $\mathcal{F}[u_0] \geq 0$?
- Stability of the self-similar blowing-up solutions?
- Behaviour if $\|u_0\|_1 > M_2$?

$m < m_d$:

- Existence of stationary solutions?
- Existence of backward self-similar solutions?

Extension to more general diffusions and drift: [Bedrossian, Bertozzi & Rodriguez (2011)]