Thin domains with oscillatory boundaries

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We are interested in studying the problem

$$\begin{cases} -\Delta w^\epsilon + w^\epsilon = f_\epsilon & R_\epsilon \\ \dfrac{\partial w^\epsilon}{\partial N^\epsilon} = 0 & \partial R_\epsilon \end{cases}$$

where R_ϵ is a thin domain

$$R_\epsilon = \{(x, y) : 0 < x < 1; 0 < y < \epsilon G_\epsilon(x)\}$$

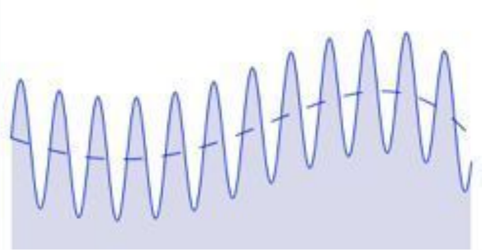
$$0 < G_0 \leq G_\epsilon(x) \leq G_1$$

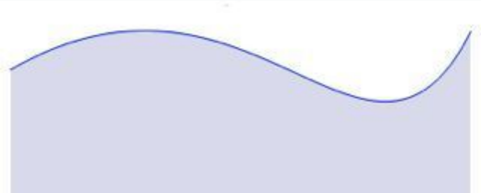
$$G_\epsilon(x) = G(x, x/\epsilon^\alpha)$$

for certain function $G(x, y)$ where $x \in (0, 1)$, $y \in \mathbb{R}$ and $G(x, \cdot)$ is periodic of period L.

For instance:

$$G_\epsilon(x) = a(x) + g(x/\epsilon^\alpha)$$





$$G_\epsilon(x) = a(x) + g(x) \equiv \hat{g}(x)$$

The limit problem is

$$\begin{cases} -\dfrac{1}{\hat{g}(x)}(\hat{g}(x)w_x(x))_x + w(x) = f \quad x \in (0,1) \\ w_x(0) = w_x(1) = 0. \end{cases}$$

J. K. Hale and G. Raugel *Reaction-diffusion equation on thin domains*, J. Math. Pures et Appl. (9) 71, (1992).

G. Raugel *Dynamics of partial differential equations on thin domains*. L.N.M 1609, Springer Verlag (1996)

Weak oscillations: $0 < \alpha < 1$

$$G_\epsilon(x) = a(x) + g(x/\epsilon^\alpha)$$

The limit problem is

$$\begin{cases} -\dfrac{1}{r(x)}\left(\dfrac{1}{s(x)}w_x(x)\right)_x + w(x) = f \quad x \in (0,1) \\ w_x(0) = w_x(1) = 0. \end{cases}$$

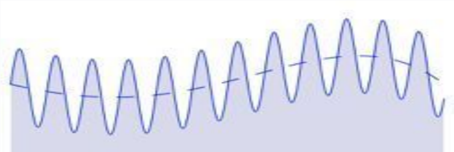
where

i) $a(x) + g(x/\epsilon^\alpha) \rightharpoonup r(x)$, $w - L^2(0,1)$

ii) $\frac{1}{a(x)+g(x/\epsilon^\alpha)} \rightharpoonup s(x)$, $w - L^2(0,1)$.

J. A. *Spectral properties of Schrödinger operators under perturbations of the domain*, Ph.D. Th. 1991.

$$\begin{cases} -\Delta w^\epsilon + w^\epsilon = f_\epsilon & R_\epsilon \\ \dfrac{\partial w^\epsilon}{\partial N^\epsilon} = 0 & \partial R_\epsilon \end{cases}$$



$$G_\epsilon(x) = a(x) + g(x/\epsilon)$$

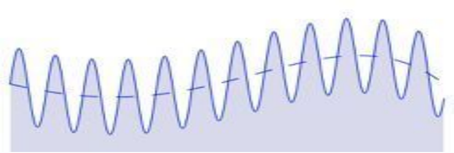
J.A., M.Pereira, “Homogenization in a thin domain with an oscillatory boundary”, *J. de Math. Pures et Apl.* 96, pp: 29-57 (2011)

- Our thin domain:

$$R_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ 0 < x_2 < \epsilon(a(x_1)+g(x_1/\epsilon))\}.$$

- $a : (0,1) \to \mathbb{R}$, $\mathcal{C}^1$ with $0 < \alpha_0 \leq a(x) \leq \alpha_1$
- $g : \mathbb{R} \mapsto \mathbb{R}$, L-periodic $\mathcal{C}^1$, $g_0 \leq g(x) \leq g_1$ with

$$0 < \underbrace{\alpha_0 + g_0}_{G_0} < \underbrace{\alpha_1 + g_1}_{G_1}$$



CASE I: Purely periodic oscillatory boundary.

- $a(x) = a_0$ a constant, so that $a_0 + g(x/\epsilon)$ is periodic.
- We identify the limit equation by the Multiple Scale method.
- We prove the convergence with the oscillatory test function method.

Multiple Scale Method

$$w^{\epsilon}(x_1, x_2) = w_0(x_1, \frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) + \epsilon w_1(x_1, \frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) + \epsilon^2 w_2(x_1, \frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) + \dots$$

Where (x_1, x_2) are the macroscopic variables and $(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon})$ are the microscopic variables.
Hence, if we denote $x = x_1$, $y = x_1/\epsilon$, $z = x_2/\epsilon$.

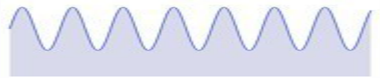
$$\frac{\partial}{\partial x_1} = \partial_x + \frac{1}{\epsilon}\partial_y \qquad \frac{\partial}{\partial x_2} = \frac{1}{\epsilon}\partial_z$$

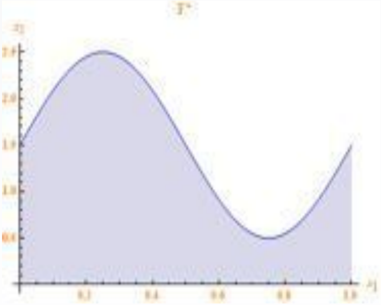
$$\frac{\partial^2}{\partial {x_1}^2} = \partial_{xx} + \frac{2}{\epsilon}\partial_{xy} + \frac{1}{\epsilon^2}\partial_{yy} \qquad \frac{\partial^2}{\partial {x_2}^2} = \frac{1}{\epsilon^2}\partial_{zz}.$$

Functions $w_i(x, y, z)$ are defined in $x \in (0, 1)$ and $(y, z) \in Y^*$, the basic cell:

$$Y^* = \{(y, z) \in \mathbb{R}^2 \; : \; 0 < y < L, \, 0 < z < a_0 + g(y)\},$$

We denote by B_0,B_1 and B_2 the lateral, inferior and superior boundary, respectively.





With some computations w_0 satisfies

$$\begin{cases} -q\dfrac{d^2 w_0}{dx^2}(x) + w_0(x) = f(x), \quad x \in (0,1) \\ w_0'(0) = w_0'(1) = 0 \end{cases}$$

where

$$q = \frac{1}{|Y^*|}\int_{Y^*}\left\{1 - \frac{\partial X}{\partial y}(y,z)\right\}dydz$$

and $X(y,z)$ is the unique solution (up to an additive constant) of

$$\begin{cases} -\Delta_{y,z}X(y,z) = 0 \quad \text{in } Y^* \\ \dfrac{\partial X}{\partial N}(y,g(y)) = -\dfrac{g'(y)}{\sqrt{1+(g'(y))^2}} \quad \text{on } B_1 \\ \dfrac{\partial X}{\partial N}(y,0) = 0 \quad \text{on } B_2 \\ X(y,z) \quad L-\text{periodic in } y. \end{cases}$$

Convergence result

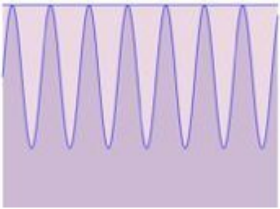
We transform the original domain and problem with the change of variables $(x, y) \to (x, \epsilon y)$ so

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f \quad \text{in } \Omega^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial x_1} N_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2} N_2^\epsilon = 0 \quad \text{on } \partial\Omega^\epsilon \end{cases}$$

$$\Omega^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ 0 < x_2 < a_0 + g(x_1/\epsilon)\}$$

$$\Omega^\epsilon \subset \Omega = (0, 1) \times (0, G_1)$$

$$f \in L^2(\Omega) \text{ with } f(x_1, x_2) = f(x_1).$$



The weak formulation of the problem: $\forall \varphi \in H^1(\Omega^\epsilon)$

$$\int_{\Omega^\epsilon} \left\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \right\} dx_1 dx_2 = \int_{\Omega^\epsilon} f \varphi dx_1 dx_2$$

which, taking $\varphi = u_\epsilon$, implies

$$\left\| \frac{\partial u^\epsilon}{\partial x_1} \right\|^2_{L^2(\Omega_\epsilon)} + \frac{1}{\epsilon^2} \left\| \frac{\partial u^\epsilon}{\partial x_2} \right\|^2_{L^2(\Omega_\epsilon)} + \|u^\epsilon\|^2_{L^2(\Omega_\epsilon)} \leq \|f\|_{L^2} \|u^\epsilon\|_{L^2(\Omega_\epsilon)}.$$

and this shows,

$$\|u^\epsilon\|_{L^2(\Omega_\epsilon)}, \left\| \frac{\partial u^\epsilon}{\partial x_1} \right\|_{L^2(\Omega_\epsilon)}, \frac{1}{\epsilon} \left\| \frac{\partial u^\epsilon}{\partial x_2} \right\|_{L^2(\Omega_\epsilon)} \leq M \; \forall \epsilon > 0.$$

Observe that there is a natural H^1-norm associated to this problem (that we will denote by $\|u_\epsilon\|_{H^1_\epsilon(U)}$):

$$\|u^\epsilon\|^2_{H^1_\epsilon(U)} = \left\|\frac{\partial u^\epsilon}{\partial x_1}\right\|^2_{L^2(U)} + \frac{1}{\epsilon^2}\left\|\frac{\partial u^\epsilon}{\partial x_2}\right\|^2_{L^2(U)} + \|u^\epsilon\|^2_{L^2(U)}$$

We have an extension operator

$$P_\epsilon : H^1(\Omega_\epsilon) \to H^1(\Omega)$$

satisfying

$$\|P_\epsilon \varphi\|_{L^2(\Omega)} \le K \|\varphi\|_{L^2(\Omega_\epsilon)}$$
$$\left\|\frac{\partial P_\epsilon \varphi}{\partial x_1}\right\|_{L^2(\Omega)} \le K\left(\left\|\frac{\partial \varphi}{\partial x_1}\right\|_{L^2(\Omega_\epsilon)} + \frac{1}{\epsilon}\left\|\frac{\partial \varphi}{\partial x_2}\right\|_{L^2(\Omega_\epsilon)}\right)$$
$$\left\|\frac{\partial P_\epsilon \varphi}{\partial x_2}\right\|_{L^2(\Omega)} \le K\left\|\frac{\partial \varphi}{\partial x_2}\right\|_{L^2(\Omega_\epsilon)}$$

That is, $P_\epsilon : H^1_\epsilon(\Omega_\epsilon) \to H^1_\epsilon(\Omega)$ is a bounded operator with

$$\|P_\epsilon \varphi\|_{H^1_\epsilon(\Omega)} \leq K \|\varphi\|_{H^1_\epsilon(\Omega_\epsilon)}.$$

With this extension operator,

$$\|P_\epsilon u^\epsilon\|_{L^2(\Omega)}, \left\|\frac{\partial P_\epsilon u^\epsilon}{\partial x_1}\right\|_{L^2(\Omega)}, \frac{1}{\epsilon}\left\|\frac{\partial P_\epsilon u^\epsilon}{\partial x_2}\right\|_{L^2(\Omega)} \leq \tilde{M}$$

where $\tilde{M} > 0$ independent of $\epsilon > 0$.

We can take a subsequence $P_\epsilon u^\epsilon$ so that

- $P_\epsilon u^\epsilon \rightharpoonup u_0 \quad w - H^1(\Omega)$ and $s - L^2(\Omega)$
- $P_\epsilon u^\epsilon \to u_0 \quad s - L^2(\Omega)$
- $\dfrac{\partial P_\epsilon u^\epsilon}{\partial x_2} \to 0 \quad s - L^2(\Omega)$.

  Hence, $u_0(x_1, x_2) = u_0(x_1)$, that is

$$\frac{\partial u_0}{\partial x_2}(x_1, x_2) = 0 \text{ a.e. } \Omega.$$

Going back to the weak formulation of the problem:

$$\int_{\Omega^\epsilon} \left\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \right\} dx_1 dx_2 = \int_{\Omega^\epsilon} f_\epsilon \varphi dx_1 dx_2$$

denoting by χ_ϵ the characteristic function of Ω_ϵ and considering test functions $\varphi(x_1, x_2) = \varphi(x_1)$ we can write

$$\int_{\Omega} \left\{ \chi_\epsilon \frac{\partial P_\epsilon u^\epsilon}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \chi_\epsilon P_\epsilon u^\epsilon \varphi \right\} dx_1 dx_2 = \int_{\Omega} \chi_\epsilon f_\epsilon \varphi dx_1 dx_2$$

In order to pass to the limit appropriety, we choose an "oscillatory test function", and show that the limit is given by the one obtained formally with the multiple scale method.

$$\int_0^1 \left\{ q|Y^*|\frac{\partial u_0}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + |Y^*|u_0\varphi \right\} dx_1 = \int_0^1 |Y^*| f \varphi dx_1, \ \ \forall \varphi \in H^1(0,1)$$

or equivalenly,

$$\int_0^1 (q\frac{\partial u_0}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + u_0\varphi) dx_1 = \int_0^1 f \varphi dx_1, \ \ \forall \varphi \in H^1(0,1)$$

where

$$q = \frac{1}{|Y^*|} \int_{Y^*} \left\{ 1 - \frac{\partial X}{\partial y}(y,z) \right\} dydz$$

and $X(y,z)$ is the unique solution (up to an additive constant) of

$$\begin{cases} -\Delta_{y,z} X(y,z) = 0 \quad \text{in } Y^* \\ \dfrac{\partial X}{\partial N}(y, g(y)) = -\dfrac{g'(y)}{\sqrt{1 + (g'(y))^2}} \quad \text{on } B_1 \\ \dfrac{\partial X}{\partial N}(y, 0) = 0 \quad \text{on } B_2 \\ X(y,z) \quad L - \text{periodic in } y. \end{cases}$$

Hence, the limit problem is:

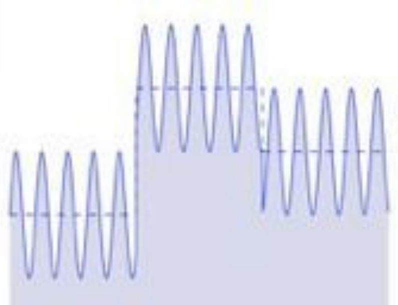
Homogenized problem

$$\begin{cases} -qu_0''(x) + u_0(x) = f(x), \quad x \in (0,1) \\ u_0'(0) = u_0'(1) = 0 \end{cases}$$

CASE II: Piecewise periodic oscillatory boundary.

$$\Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1, 0 < x_2 < a(x_1) + g(x_1/\epsilon)\}.$$

where $a(x) = a_0^i$ for $x \in I_i$ and $(0,1) = I_1 \cup \cdots \cup I_K$ and $\alpha_0 \leq a_0^i \leq \alpha_1$.



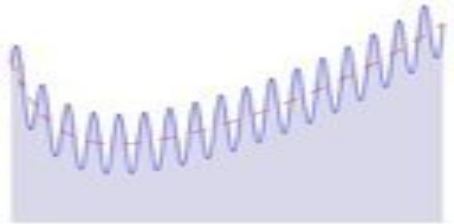
With similar arguments we may find

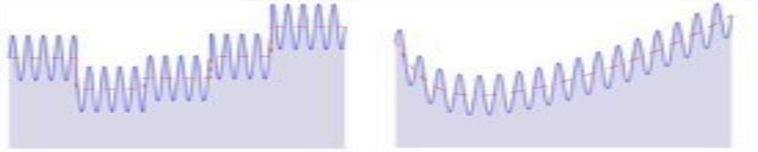
$$\sum_{i=1}^{K} \int_{I_i} q_i |Y_i^*| \frac{\partial u_0}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + |Y_i^*| u_0(x_1)\varphi(x_1) \Big\} dx_1$$

$$= \sum_{i=1}^{K} \int_{I_i} |Y_i^*| f(x_1)\varphi(x_1) dx_1, \ \ \forall \varphi \in H^1(0,1)$$

CASE III: Locally periodic oscillatory boundary.

$$\Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\, 0 < x_2 < a(x_1) + g(x_1/\epsilon)\}.$$



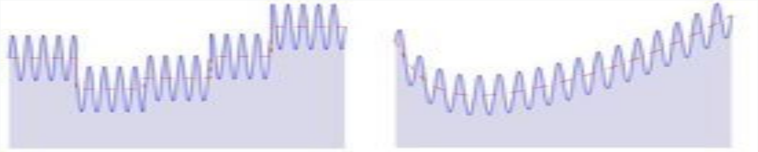


Ω_ϵ^δ $\qquad\qquad$ Ω_ϵ

$$\Omega_\epsilon^\delta = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ 0 < x_2 < a_\delta(x_1) + g(x_1/\epsilon)\}.$$
$$\Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ 0 < x_2 < a(x_1) + g(x_1/\epsilon)\}.$$

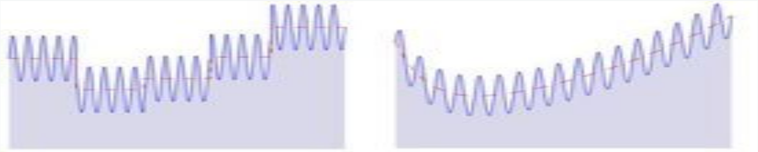
where $a_\delta(x_1)$ is a piecewise constant function satisfying

$$\|a_\delta - a\|_{L^\infty(0,1)} \le \delta.$$



$\downarrow (\epsilon \to 0)$
$\downarrow$
$(\text{Equation})_\delta$

$$(\text{Eq})_\delta : \quad \int_0^1 \left(q_\delta |Y_\delta^*| u_0' \varphi' + |Y_\delta^*| u_0 \varphi \right) dx_1 = \int_0^1 |Y_\delta^*| f \varphi dx_1$$



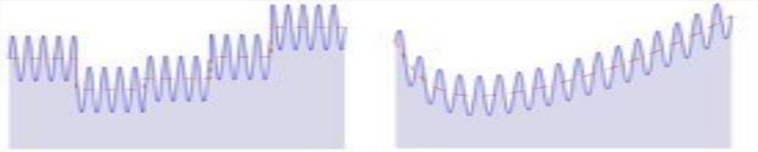
$$\downarrow (\epsilon \to 0)$$

$$\downarrow$$

$$(\text{Equation})_\delta \quad \xrightarrow{\delta \to 0} \quad (\text{Equation})_0$$

$$(\text{Eq})_\delta : \quad \int_0^1 \left(q_\delta |Y_\delta^*| u_0' \varphi' + |Y_\delta^*| u_0 \varphi \right) dx_1 = \int_0^1 |Y_\delta^*| f \varphi dx_1$$

$$(\text{Eq})_0 : \quad \int_0^1 \left(q |Y^*| u_0' \varphi' + |Y^*| u_0 \varphi \right) dx_1 = \int_0^1 |Y^*| f \varphi dx_1$$



$$\downarrow (\epsilon \to 0) \qquad \qquad \downarrow ?$$

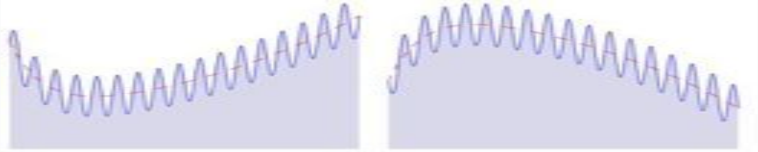
$$\downarrow \qquad \qquad \downarrow ?$$

$$(\text{Equation})_\delta \quad \xrightarrow{\delta \to 0} \quad (\text{Equation})_0$$

$$(\text{Eq})_\delta : \quad \int_0^1 \left(q_\delta |Y_\delta^*| u_0' \varphi' + |Y_\delta^*| u_0 \varphi \right) dx_1 = \int_0^1 |Y_\delta^*| f \varphi dx_1$$

$$(\text{Eq})_0 : \quad \int_0^1 \left(q |Y^*| u_0' \varphi' + |Y^*| u_0 \varphi \right) dx_1 = \int_0^1 |Y^*| f \varphi dx_1$$

We are going to show that the solutions depend continuously on the function $a(x)$ uniformly in ϵ:



Ω_ϵ $\hat{\Omega}_\epsilon$

$$\Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ 0 < x_2 < a(x_1) + g(x_1/\epsilon)\}.$$
$$\hat{\Omega}_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ 0 < x_2 < \hat{a}(x_1) + g(x_1/\epsilon)\}.$$

with $\alpha_0 \le a(x), \hat{a}(x) \le \beta_0$.

Denote by u_ϵ and $\hat{u}_\epsilon$ the solutions of

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f_\epsilon & \text{in } \Omega^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial x_1} N_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2} N_2^\epsilon = 0 & \text{on } \partial\Omega^\epsilon \end{cases}$$

$$\begin{cases} -\dfrac{\partial^2 \hat{u}^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 \hat{u}^\epsilon}{\partial {x_2}^2} + \hat{u}^\epsilon = f_\epsilon & \text{in } \hat{\Omega}^\epsilon \\ \dfrac{\partial \hat{u}^\epsilon}{\partial x_1} \hat{N}_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial \hat{u}^\epsilon}{\partial x_2} \hat{N}_2^\epsilon = 0 & \text{on } \partial\hat{\Omega}^\epsilon \end{cases}$$

with $f_\epsilon \in L^2(\mathbb{R}^2)$.

Theorem

There exists a function $\rho(\delta) \to 0$ as $\delta \to 0$ such that

$$\|u_\epsilon - \hat{u}_\epsilon\|^2_{H^1_\epsilon(\Omega_\epsilon \cap \hat{\Omega}_\epsilon)} + \|u_\epsilon\|^2_{H^1_\epsilon(\Omega_\epsilon \setminus \hat{\Omega}_\epsilon)} + \|\hat{u}_\epsilon\|^2_{H^1_\epsilon(\hat{\Omega}_\epsilon \setminus \Omega_\epsilon)} \le \rho(\delta)$$

uniformly for all

- $\epsilon \in (0, \epsilon_0)$
- pieciwise C^1 functions a, $\hat{a}$ with $\|a - \hat{a}\|_{L^\infty(0,1)} \le \delta$, $\alpha_0 \le a(x), \hat{a}(x) \le \alpha_1$
- $f_\epsilon \in L^2(\mathbb{R}^2)$, $\|f_\epsilon\|_{L^2(\mathbb{R}^2)} \le 1$

Recall: The space, $H^1_\epsilon(U) = H^1(U)$ with the norm

$$\|u\|^2_{H^1_\epsilon(U)} = \|u\|^2_{L^2(U)} + \|u_{x_1}\|^2_{L^2(U)} + \frac{1}{\epsilon^2}\|u_{x_2}\|^2_{L^2(U)}$$

To prove the Theorem we observe:

Characterization of u_ϵ and $\hat{u}_\epsilon$

u_ϵ and $\hat{u}_\epsilon$ are minima of

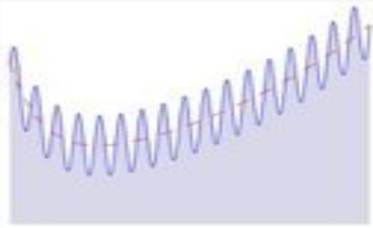
$$V(\varphi) = \frac{1}{2}\int_{\Omega_\epsilon} \left(|\varphi_{x_1}|^2 + \frac{1}{\epsilon^2}|\varphi_{x_2}|^2 + |\varphi|^2 \right) - \int_{\Omega_\epsilon} f_\epsilon \varphi$$

$$\hat{V}(\hat{\varphi}) = \frac{1}{2}\int_{\hat{\Omega}_\epsilon} \left(|\hat{\varphi}_{x_1}|^2 + \frac{1}{\epsilon^2}|\hat{\varphi}_{x_2}|^2 + |\hat{\varphi}|^2 \right) - \int_{\hat{\Omega}_\epsilon} f_\epsilon \hat{\varphi}$$

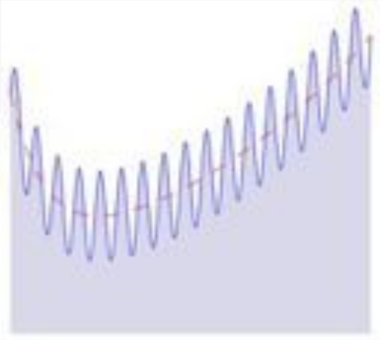
That is

$$V(u_\epsilon) = \min_{\varphi \in H^1(\Omega_\epsilon)} V(\varphi)$$

$$\hat{V}(\hat{u}_\epsilon) = \min_{\hat{\varphi} \in H^1(\hat{\Omega}_\epsilon)} \hat{V}(\hat{\varphi})$$



Ω_ϵ



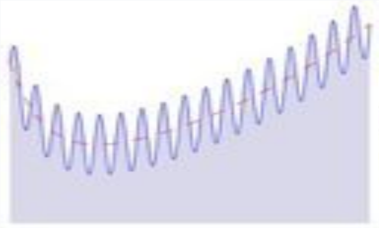
$\Omega_\epsilon(1+\eta)$

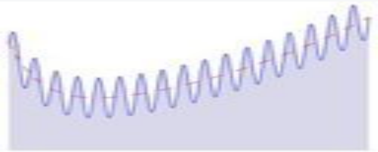
where

$$\Omega_\epsilon(1+\eta) = \{(x,(1+\eta)y) : (x,y) \in \Omega_\epsilon\}$$

We define the operator $P_{1+\eta} : H^1(\Omega_\epsilon) \to H^1(\Omega_\epsilon(1+\eta))$

$$(P_{1+\eta}\varphi)(x,y) = \varphi(x, \frac{y}{1+\eta})$$





Ω_ϵ $\qquad\qquad$ $\Omega_\epsilon\Big(\dfrac{1}{1+\eta}\Big)$

where

$$\Omega_\epsilon(\frac{1}{1+\eta}) = \{(x, \frac{y}{1+\eta}) : (x,y) \in \Omega_\epsilon\}$$

With this analysis, we obtain that the limit is:

$$\int_0^1 q(x_1)|Y^*(x_1)|\frac{\partial u_0}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + |Y^*(x_1)|u_0(x_1)\varphi(x_1)\Big\} dx_1$$

$$= \int_0^1 |Y^*(x_1)|f(x_1)\varphi(x_1)dx_1, \ \ \forall \varphi \in H^1(0,1)$$

Where

$$Y^*(x_1) = \{(y,z) \in \mathbb{R}^2 \ : \ 0 < y < L, 0 < z < a(x_1) + g(y)\},$$

and

$$q(x_1) = \frac{1}{|Y^*(x_1)|}\int_{Y^*(x_1)}\left(1 - \frac{\partial X}{\partial y_1}(x_1,y,z)\right) dydz$$

Where,

$$\begin{cases} -\Delta_{y,z} X(x_1, y, z) = 0 \quad \text{in } Y^*(x_1) \\ \dfrac{\partial X}{\partial N}(x_1, y, g(y)) = -\dfrac{g'(y)}{\sqrt{1 + (g'(y))^2}} \quad \text{on } B_1(x_1) \\ \dfrac{\partial X}{\partial N}(x_1, y, 0) = 0 \quad \text{on } B_2(x_1) \\ X(x_1, y, z) \quad L - \text{periodic in } y. \end{cases}$$

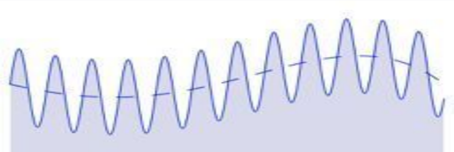
• M.L. Mascarenhas, D. Polisevski, *M2AN* 28 37-57 (1994)

• D. Chenais, M.L. Mascharenhas, L. Trabucho, *M2AN* 31 559-597 (1997)

• G. Chechkin, A. Piatnitski, *Applicable Analysis* 71, pp. 215-235 (1999)

Very highly oscillatory behavior: $\alpha > 1$.

$$\begin{cases} -\Delta w^\epsilon + w^\epsilon = f_\epsilon & R_\epsilon \\ \dfrac{\partial w^\epsilon}{\partial N^\epsilon} = 0 & \partial R_\epsilon \end{cases}$$



$$G_\epsilon(x) = a(x) + g(x/\epsilon^\alpha), \quad \alpha > 1$$

J.A., M. Pereira, "Thin domains with extremely high oscillatory boundaries", *Submitted*

We also transform the original domain and problem with the change of variables $(x, y) \to (x, \epsilon y)$ so

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f \quad \text{in } \Omega^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial x_1} N_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2} N_2^\epsilon = 0 \quad \text{on } \partial\Omega^\epsilon \end{cases}$$

$$\Omega^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ 0 < x_2 < a(x_1) + g(x_1/\epsilon^\alpha)\}$$

- R. Brizzi, J.P. Chalot, *Ricerche di Matematica* XLVI (1997)

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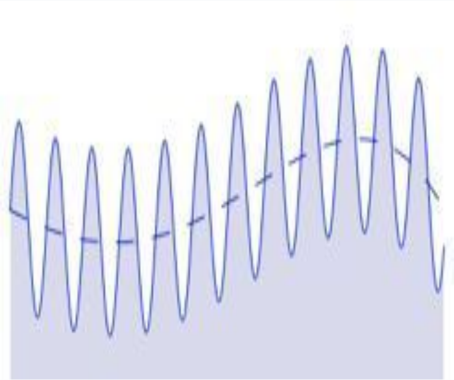
- R. Brizzi, J.P. Chalot, *Ricerche di Matematica* XLVI (1997)



$$g_0 = \min_{x\in[0,1]} g(x)$$

Weak formulation of the problem: $\forall \varphi \in H^1(\Omega^\epsilon)$

$$\int_{\Omega^\epsilon} \left\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \right\} dx_1 dx_2 = \int_{\Omega^\epsilon} f \varphi dx_1 dx_2.$$

Taking $\varphi = u_\epsilon$:

$$\|u^\epsilon\|_{L^2(\Omega_\epsilon)}, \left\| \frac{\partial u^\epsilon}{\partial x_1} \right\|_{L^2(\Omega_\epsilon)}, \frac{1}{\epsilon} \left\| \frac{\partial u^\epsilon}{\partial x_2} \right\|_{L^2(\Omega_\epsilon)} \leq M \; \forall \epsilon > 0.$$

Considering the restriction of u_ϵ to Ω_0, we have the existence of a function $u_0 \in H^1(\Omega_0)$ such that (via subsequences)

- $u^\epsilon \rightharpoonup u_0 \quad w - H^1(\Omega_0)$ and $s - L^2(\Omega_0)$
- $\dfrac{\partial u^\epsilon}{\partial x_2} \to 0 \quad s - L^2(\Omega_0)$.

Hence, $u_0(x_1, x_2) = u_0(x_1)$.

It also can be proved (without using any kind of extension operator):

- $\|u_\epsilon - u_0\|_{L^2(\Omega_\epsilon)} \to 0$ as $\epsilon \to 0$.
- $\|\frac{\partial u_\epsilon}{\partial x_1}\|_{L^2(\Omega_\epsilon \setminus \Omega_0)} + \frac{1}{\epsilon}\|\frac{\partial u_\epsilon}{\partial x_2}\|_{L^2(\Omega_\epsilon \setminus \Omega_0)} \to 0$ as $\epsilon \to 0$.

Choosing $f_\epsilon = f(x)$ and with appropriate test functions now we can show that the function $u_0 = u_0(x)$ satisfies the following weak formulation:

$$\int_0^1 \{(a(x) + g_0)u_x\phi_x + (a(x) + \bar{g})u\phi\}dx = \int_0^1 (a(x) + \bar{g})f\phi$$

where $\bar{g} = \frac{1}{L}\int_0^L g(x)dx$, which is the weak formulation of:

$$\frac{1}{a(x) + \bar{g}}\left((a(x) + g_0)u_x\right)_x + u = f, \quad x \in (0,1)$$

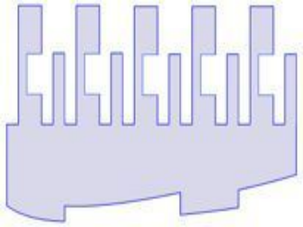
with Neumann BC's.

For instance, if $a(x) \equiv a_0$ then the limit problem is: $-du_{xx} + u = f$ where the diffusion coefficient

$$d = \frac{a_0 + g_0}{a_0 + \bar{g}} = \frac{L(a_0 + g_0)}{La_0 + \int_0^L g(x)}$$

$$= \frac{\text{area of the nonoscillating part of unit cell}}{\text{area of unit cell}}$$

This same technique can be extended to other more complicated geometries:



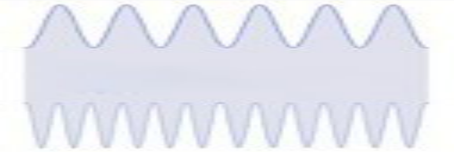
and the limiting equation is

$$\int_0^1 \{(A_{NonOsc}(x)u_x\phi_x + A(x)u\phi\}dx = \int_0^1 A(x)f\phi$$

$$\begin{cases} -\Delta w^\epsilon + w^\epsilon = f_\epsilon & R_\epsilon \\ \dfrac{\partial w^\epsilon}{\partial N^\epsilon} = 0 & \partial R_\epsilon \end{cases}$$

$$R_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2, -\epsilon h(x/\epsilon^\alpha) < x_2 < \epsilon g(x/\epsilon)\}$$

$g(\cdot)$ is L_1 periodic and h is L_2 periodic (purely periodic case).



J.A. and M. Villanueva, *work in progress*.

The limit problem is:

$$-\frac{q}{\frac{|Y^*|}{L_1}+p}u_{xx}+u=f$$

where

$$p=\frac{1}{L_2}\int_0^{L_2}h(s)ds$$

$$q=\frac{1}{L_1}\int_{Y^*}\{1-\frac{\partial X}{\partial y_1}(y_1,y_2)\}dy_1dy_2$$