LAMA Université de Savoie

Partial regularity for general free discontinuity problems

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The first problem : the Mumford-Shah functional

$$\mathcal{D} \subseteq \mathbb{R}^2, g \in L^\infty(\mathcal{D}),$$

$$\min_{K \subseteq D} \min_{u \in H^1(D \setminus K)} \int_{D \setminus K} |\nabla u|^2 dx + \int_D (u-g)^2 dx + \mathcal{H}^1(K) := E(u,K).$$

= Shape optimization problem =

- the shape is K (closed) or $D \setminus K$ (open)
- if K is known, u solves

$$\begin{cases} -\Delta u + u = & g \text{ in } D \setminus K \\ \frac{\partial u}{\partial n} = & 0 \;\; \partial D \cup K \end{cases}$$

The Mumford-Shah functional

- very difficult to prove directly existence of a solution (K, u);
- no shape optimization arguments known!
- easy in 2D, if constraints on the number of connected components.

The Mumford-Shah functional

Idea of De Giorgi - Ambrosio $'88$: relax the functional in the BV framework

- if (K, u) is a test couple, K closed and smooth, $u \in H^1(D \setminus K) \cap L^\infty(D)$ and $\mathcal{H}^1(K) < +\infty$, then

$$Du(B) = \int_B \nabla u dx + \int_{K \cap B} (u^+ - u^-)\nu_u d\mathcal{H}^1,$$

  is a finite Radon measure on D.

- consequently $u \in BV(D)$ and more specifically $u \in SBV(D)$;

The Mumford-Shah functional

- $$BV(D) = \{v \in L^1(D) : \; Dv \text{ is a finite Radon measure}\}.$$

  There exists J_v is a $(N-1)$-rectifiable set such that Dv admits the following representation for every Borel set $B \subseteq \mathbb{R}^N$:

  $$Dv(B) = \int_B \nabla^a v \, dx + \int_{J_v \cap B} (v^+ - v^-)\nu_v \, d\mathcal{H}^{N-1} + D^c v(B),$$

- $$SBV(D) = \{v \in BV(D) : D^c v = 0\}$$

The Mumford-Shah functional

- relaxed form :

$$\min_{u\in SBV(D)} \int_D |\nabla^a u|^2 dx + \int_D (u-g)^2 dx + \mathcal{H}^1(J_u)$$

- Ambrosio's compactness theorem in SBV gives the existence of a minimizer $u \in SBV(D) \Longrightarrow$ Free discontinuity problem

The Mumford-Shah functional

- First fundamental question : does $u \in SBV(D)$ provide a classical solution ?
- Equivalently : is J_u a closed set and $u \in H^1(D \setminus J_u)$?
- Second question : if yes, what is the structure and regularity of J_u ?

The Mumford-Shah functional

- De Giorgi, Carriero, Leaci '89 prove by a blow up technique in $\mathbb{R}^N$ that
  - J_u is closed
  - $u \in H^1(D \setminus J_u)$ is smooth outside J_u
  - J_u satisfies density properties : $\exists C > 0$, $\forall x \in J_u$

$$Cr^{N-1} \leq \mathcal{H}^{N-1}(J_u \cap B_r(x)) \leq \frac{1}{C} r^{N-1}.$$

- Dal Maso, Morel, Solimini '89 prove in $\mathbb{R}^2$ the same kind of result, by a different technique, working with prescribed, but arbitrarily large number, of connected components.
- Regularity and structure of J_u : Bonnet, David, Ambrosio, Fusco, Pallara, $'94 -' 98$, ... still open questions.

Crack propagation : Francfort and Marigo, '97

Quasi static movement, solving at each discrete time step :

$$\min_{\tilde{K}\subseteq K} \min_{u\in H^1(D\setminus K), u=g\partial D} \int_{D\setminus K} |\nabla u|^2 dx + \mathcal{H}^1(K).$$

- $\mathcal{H}^1$ is the cracking energy (dissipation distance in the Mielke framework) !
- relaxed solution in SBV (Francfort, Larsen '03) ;
- various recent result on more complex models ;
- need to work with more realistic energies

$$\int_{J_u} \Phi(n) d\mathcal{H}^N, \int_{J_u} \Phi(n, u^+, u^-) d\mathcal{H}^N.$$

- no more finiteness of the Hausdorff measure ! ! !

Isoperimetric inequalities with Robin boundary conditions

For $\beta > 0$

$$\begin{cases} -\Delta u = & \nu u \text{ in } \Omega \\ \frac{\partial u}{\partial n} + \beta u = & 0 \ \partial\Omega \end{cases}$$

Rayleigh quotioent :

$$\nu_1(\Omega) = \min_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx + \beta \int_{\partial\Omega} |u|^2 d\mathcal{H}^{N-1}}{\int_\Omega u^2 dx}$$

Robin boundary conditions

The ball minimizes ν_1 : Bossel ($\mathbb{R}^2$) 1986, Daners ($\mathbb{R}^N$) 2006 among Lip-domains.

Proof : "dearrangement" technique using C^1-regularity of the solution up to the boundary and density of C^2 domains in Lip-domains.

$$H_\Omega(U,\phi) = \frac{1}{|U|}\Big(\int_{\partial_i U} \phi d\mathcal{H}^{N-1} + \int_{\partial_e U} \beta d\mathcal{H}^{N-1} - \int_U |\phi|^2 dx \Big).$$

$U := \{u > t\}, \phi := \frac{|\nabla u|}{u}$

Question : What is the most general class where the isoperimetric inequality is valid ? Uniqueness ?

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Robin eigenvalues on non smooth domains

Daners 1995, Daners-Dancer 1996, Arendt-Warma 2003-2011 use the Mazya space :
completion of $C^1(\overline{\Omega})$ in the norm $\|\cdot\|_{H^1(\Omega)} + \|\cdot\|_{L^2(\partial\Omega)}$: subspace in $H^1(\Omega) \times L^2(\partial\Omega, \mathcal{H}^{N-1})$.

- is well defined for every domain
- compact embedding in $L^2(\Omega)$, so spectrum of eigenvalues
- no cracks
- the trace operator is not well defined. The zero function may have the trace equal to 1 !

Variational approach

Existence + regularity $\Longrightarrow$ ball

B. Giacomini : take the first eigenfunction of the Robin problem in Lipschitz set and extend it by zero.
The (square of) the new function seen in $\mathbb{R}^N$ has a distributional derivative

$$Du^2 = \nabla u^2 dx|_\Omega + u^2 \nu \mathcal{H}^{N-1}|_{\partial\Omega}.$$

So $u^2 \in SBV(\mathbb{R}^N)$!

Variational approach

$$\min_{u \in SBV^{\frac{1}{2}}(\mathbb{R}^N), |\{u \neq 0\} = m|} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx + \beta \int_{J_u} (|u^+|^2 + |u^-|^2) d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2 dx}$$

Theorem (B. Giacomini 2009)

$$\nu_1(ball_m) \leq \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx + \beta \int_{J_u} (|u^+|^2 + |u^-|^2) d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2 dx}$$

and is the unique minimizer.

Robin boundary conditions : open questions

- optimization of the eigenvalue in a box not containing the ball !
- Other isoperimetric inequalities are still open : the maximization of the torsional rigidity !
- The reflection argument could work for isoperimetric inequalities provided there exists a classical solution (smoothness is not required).

Bernoulli like free discontinuity problem

$D_1 \subset D_2$ bounded smooth open sets, $g \in H^1(D_1) \cap L^\infty(D)$, $g \geq \alpha > 0$, $\beta > 0$. We solve

$$\min_{D_1 \subset \Omega \subset D_2} \min_{u \in H^1(\Omega), u=g \text{ on } D_1} \int_\Omega |\nabla u|^2 dx + \beta \int_{\partial\Omega} u^2 d\mathcal{H}^{N-1} + |\Omega|.$$

Relaxed formulation

$$\min_{u \in SBV^{\frac{1}{2}}(D_2), u=g \text{ on } D_1} \int_{D_2} |\nabla u|^2 dx + \beta \int_{J_u} |u^+|^2 + |u^-|^2 d\mathcal{H}^{N-1} + |\{u > 0\}|.$$

Bernoulli like free discontinuity problem

Are the $SBV^{\frac{1}{2}}$ classical solutions ? What is the smoothness of J_u ?

Main differences with Mumford-Shah :

- A priori the Hausdorff measure of J_u is not finite ?
- Is J_u closed ? The blow up technique of De Giorgi-Carriero-Leaci is not working on this particular term : multiplying u can not equilibrate the two energy terms...
- the boundary energy involves the trace of u.

Almost-quasi minimizers of a free discontinuity problems

Quasi-minimizer for the Mumford-Shah functional ($\alpha > 0$) :

$$\forall\ 0 < \rho < \rho_0, \forall v \in SBV(\mathbb{R}^N) \text{ such that } \{u \neq v\} \subseteq B_\rho(x) \Longrightarrow$$

$$\int_{B_\rho(x)} |\nabla u|^2 dx + \mathcal{H}^{N-1}(J_u \cap \overline{B}_\rho(x))$$

$$\leq \int_{B_\rho(x)} |\nabla v|^2 dx + \mathcal{H}^{N-1}(J_v \cap \overline{B}_\rho(x)) + c_\alpha \rho^{N-1+\alpha}.$$

$\Longrightarrow$ Partial regularity

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Almost-quasi minimizer ($\Lambda \geq 1$) :

$$\int_{B_\rho(x)} |\nabla u|^2 dx + \mathcal{H}^{N-1}(J_u \cap \overline{B}_\rho(x))$$

$$\leq \int_{B_\rho(x)} |\nabla v|^2 dx + \Lambda \mathcal{H}^{N-1}(J_v \cap \overline{B}_\rho(x)) + c_\alpha \rho^{N-1+\alpha}.$$

Monotoncity formula

B.-Luckhaus 2011 (work in progress)
Theorem Let u be an almost-quasi minimizer. Then

$$r \longmapsto E(r)$$

$$= \Big[\frac{1}{r^{N-1}}\Big(\int_{B_r} |\nabla u|^2 dx + \mathcal{H}^{N-1}(J_u \cap \overline{B}_r)\Big)\Big] \wedge \frac{c_d \Lambda^{2-N}}{N-1} + (N-1)\frac{c_\alpha}{\alpha} r^\alpha$$

is non decreasing on $(0, \rho_0)$.

Consequence : partial regularity

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Theorem Let u be a minimizer of the Bernoulli like free discontinuity problem :

$$\min_{u \in SBV^{\frac{1}{2}}(D_2), u=g \text{ on } D_1} \int_{D_2} |\nabla u|^2 dx + \beta \int_{J_u} |u^+|^2 + |u^-|^2 d\mathcal{H}^{N-1} + |\{u > 0\}|.$$

Then u is an almost-quasi minimizer.

Crucial step : u minorated on its positivity set ! Caffarelli type argument : reverse asymptotic method !

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