Controllability results for degenerate parabolic equations

Piermarco Cannarsa

University of Rome "Tor Vergata"

PDE's, Optimal Design and Numerics IV

CENTRO DE CIENCIAS DE BENASQUE PEDRO PASCUAL

August 28 – September 9, 2011

Outline

Outline

Examples of degenerate parabolic operators (DPO)

- Budyko-Sellers one-dimensional model
- linearized Crocco equation
- Kolmogorov operators & degenerate diffusions

Control of DPO with boundary degeneracy

- null controllability in dimension 1
- approximate controllability in dimension 1
- null controllability in dimension 2

Control of DPO with interior degeneracy

- parabolic operators with interior degeneracy
- control of Grushin-type operators in dimension 2

Outline

Examples of degenerate parabolic operators (DPO)

- Budyko-Sellers one-dimensional model
- linearized Crocco equation
- Kolmogorov operators & degenerate diffusions

Control of DPO with boundary degeneracy

- null controllability in dimension 1
- approximate controllability in dimension 1
- null controllability in dimension 2

Control of DPO with interior degeneracy

- parabolic operators with interior degeneracy
- control of Grushin-type operators in dimension 2

Outline

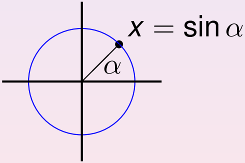
Budyko-Sellers model

$$\begin{cases} u_t - \left((1-x^2)u_x\right)_x = f(x)\, g(u) - h(u) & x \in (-1,1) \\ (1-x^2)u_{x|_{x=\pm 1}} = 0 \end{cases}$$

- $u(t,x)$= sea-level zonally averaged temperature
- $f(x)$= solar input
- $g(u)$= co-albedo
- $h(u)$= outgoing infrared radiation

Budyko-Sellers model

$$\begin{cases} u_t - \left((1-x^2)u_x\right)_x = f(x)\, g(u) - h(u) & x \in (-1,1) \\ (1-x^2)u_{x|_{x=\pm 1}} = 0 \end{cases}$$



- $u(t,x)$= sea-level zonally averaged temperature
- $f(x)$= solar input
- $g(u)$= co-albedo
- $h(u)$= outgoing infrared radiation

Outline

laminar flow

boundary layer model

$\downarrow$

Prandtl's equations $\longleftarrow$ nonlinear degenerate

$\downarrow$ $\longleftarrow$ $\begin{cases} \text{Crocco's transormation} \\ \text{linearization} \end{cases}$

Crocco's equation

$$\begin{cases} u_t + b(t,y)u_x - \big(a(y)u_y\big)_y = f \\ + \text{ boundary and initial conditions} \end{cases}$$

double degeneracy $\quad A(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & a(0) \end{pmatrix}$ with $a(0) = 0$

laminar flow

boundary layer model

↓

Prandtl's equations ← nonlinear degenerate

↓ ← { Crocco's transormation, linearization }

Crocco's equation

$$\begin{cases} u_t + b(t,y)u_x - \big(a(y)u_y\big)_y = f \\ + \text{ boundary and initial conditions} \end{cases}$$

double degeneracy $A(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & a(0) \end{pmatrix}$ with $a(0) = 0$

laminar flow

boundary layer model

$\downarrow$

Prandtl's equations $\leftarrow$ nonlinear degenerate

$\downarrow$ $\leftarrow$ $\begin{cases} \text{Crocco's transormation} \\ \text{linearization} \end{cases}$

Crocco's equation

$$\begin{cases} u_t + b(t,y)u_x - \big(a(y)u_y\big)_y = f \\ + \text{ boundary and initial conditions} \end{cases}$$

double degeneracy $A(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & a(0) \end{pmatrix}$ with $a(0) = 0$

Outline

stochastic flows

- $X(\cdot, x)$ unique solution

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))\, dW(t) & t \geq 0 \\ X(0) = x \in \mathbb{R}^n \end{cases}$$

  - $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ Lipschitz
  - $W(t)$ m-dimensional Brownian $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$
- transition semigroup $P_t\varphi(x) := \mathbb{E}\Big[\varphi(X(t,x))\Big]$
- $u(t,x) = P_t\varphi(x)$ solution of Kolmogorov equation

$$\begin{cases} u_t = \underbrace{\frac{1}{2}\,\mathrm{Tr}\,[A(x)\nabla^2 u] + \langle b(x), \nabla u\rangle}_{Lu} & \text{in} \quad (0,\infty)\times\mathbb{R}^n \\ u(0,x) = \varphi(x) & x \in \mathbb{R}^n \end{cases}$$

where $A(x) = \sigma(x)\sigma^*(x) \geq 0$ may be degenerate

stochastic flows

- $X(\cdot, x)$ unique solution

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))\, dW(t) & t \geq 0 \\ X(0) = x \in \mathbb{R}^n \end{cases}$$

  - $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ Lipschitz
  - $W(t)$ m-dimensional Brownian $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$

- transition semigroup $P_t\varphi(x) := \mathbb{E}\Big[\varphi(X(t,x))\Big]$

- $u(t,x) = P_t\varphi(x)$ solution of Kolmogorov equation

$$\begin{cases} u_t = \underbrace{\frac{1}{2}\,\mathrm{Tr}\,[A(x)\nabla^2 u] + \langle b(x), \nabla u\rangle}_{Lu} & \text{in} \quad (0,\infty)\times\mathbb{R}^n \\ u(0,x) = \varphi(x) & x \in \mathbb{R}^n \end{cases}$$

where $A(x) = \sigma(x)\sigma^*(x) \geq 0$ may be degenerate

stochastic flows

- $X(\cdot, x)$ unique solution

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))\, dW(t) & t \geq 0 \\ X(0) = x \in \mathbb{R}^n \end{cases}$$

  - $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ Lipschitz
  - $W(t)$ m-dimensional Brownian $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$

- transition semigroup $P_t\varphi(x) := \mathbb{E}\Big[\varphi(X(t,x))\Big]$

- $u(t,x) = P_t\varphi(x)$ solution of Kolmogorov equation

$$\begin{cases} u_t = \underbrace{\frac{1}{2}\,\mathrm{Tr}\,[A(x)\nabla^2 u] + \langle b(x), \nabla u\rangle}_{Lu} & \text{in} \quad (0,\infty) \times \mathbb{R}^n \\ u(0,x) = \varphi(x) & x \in \mathbb{R}^n \end{cases}$$

where $A(x) = \sigma(x)\sigma^*(x) \geq 0$ may be degenerate

stochastic flows

- $X(\cdot, x)$ unique solution

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))\, dW(t) & t \geq 0 \\ X(0) = x \in \mathbb{R}^n \end{cases}$$

  - $b : \mathbb{R}^n \to \mathbb{R}^n, \ \sigma : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ Lipschitz
  - $W(t)$ m-dimensional Brownian $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$

- transition semigroup $P_t\varphi(x) := \mathbb{E}\Big[\varphi(X(t,x))\Big]$
- $u(t,x) = P_t\varphi(x)$ solution of Kolmogorov equation

$$\begin{cases} u_t = \underbrace{\frac{1}{2}\,\mathrm{Tr}\,[A(x)\nabla^2 u] + \langle b(x), \nabla u\rangle}_{Lu} & \text{in} \quad (0,\infty) \times \mathbb{R}^n \\ u(0,x) = \varphi(x) & x \in \mathbb{R}^n \end{cases}$$

where $A(x) = \sigma(x)\sigma^*(x) \geq 0$ may be degenerate

invariant sets for a stochastic flow

$K \subset \mathbb{R}^n$ invariant

$$x \in K \quad \Longrightarrow \quad X(t,x) \in K \qquad \mathbb{P}-\text{a.s.} \qquad \forall t \geq 0$$

$\Omega \subset \mathbb{R}^n$ open set $\partial\Omega = \Gamma$

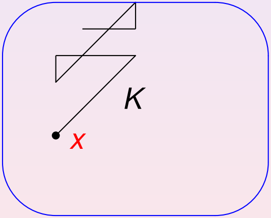
Problem

find (necessary and sufficient) conditions for invariance of $\overline{\Omega}$

invariant sets for a stochastic flow

$K \subset \mathbb{R}^n$ invariant

$$x \in K \quad \Longrightarrow \quad X(t,x) \in K \qquad \mathbb{P}-\text{a.s.} \qquad \forall t \geq 0$$



$\Omega \subset \mathbb{R}^n$ open set $\partial\Omega = \Gamma$

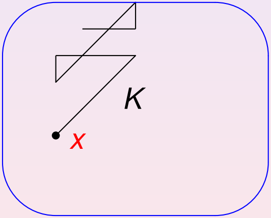
Problem

find (necessary and sufficient) conditions for invariance of $\overline{\Omega}$

invariant sets for a stochastic flow

$K \subset \mathbb{R}^n$ invariant

$$x \in K \quad \Longrightarrow \quad X(t,x) \in K \qquad \mathbb{P} - \text{a.s.} \qquad \forall t \geq 0$$



$\Omega \subset \mathbb{R}^n$ open set $\partial\Omega = \Gamma$

Problem

find (necessary and sufficient) conditions for invariance of $\overline{\Omega}$

conditions for invariance

$\Omega \subset \mathbb{R}^n$ open set $\partial\Omega = \Gamma$

- oriented distance from Γ

$$d_\Gamma(x) = \begin{cases} \mathrm{dist}(x,\Gamma) & \text{if } x \in \Omega \\ -\mathrm{dist}(x,\Gamma) & \text{if } x \in \Omega^c \end{cases}$$

- $\overline{\Omega}$ invariant $\iff \forall x \in \Gamma \begin{cases} Ld_\Gamma(x) \geq 0 \\ \langle A(x)\nabla d_\Gamma(x), \nabla d_\Gamma(x)\rangle = 0 \end{cases}$

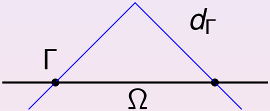
  Friedman and Pinsky (1975), ...
- $\overline{\Omega}$ invariant $\Longrightarrow$ L degenerate on Γ in normal direction

conditions for invariance

$\Omega \subset \mathbb{R}^n$ open set $\partial\Omega = \Gamma$

- oriented distance from Γ

$$d_\Gamma(x) = \begin{cases} \mathrm{dist}(x,\Gamma) & \text{if } x \in \Omega \\ -\mathrm{dist}(x,\Gamma) & \text{if } x \in \Omega^c \end{cases}$$



- $\overline{\Omega}$ invariant $\iff \forall x \in \Gamma \begin{cases} Ld_\Gamma(x) \geq 0 \\ \langle A(x)\nabla d_\Gamma(x), \nabla d_\Gamma(x)\rangle = 0 \end{cases}$

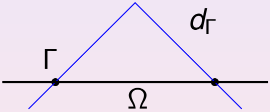
  Friedman and Pinsky (1975), ...
- $\overline{\Omega}$ invariant $\Longrightarrow$ L degenerate on Γ in normal direction

conditions for invariance

$\Omega \subset \mathbb{R}^n$ open set $\partial\Omega = \Gamma$

- oriented distance from Γ

$$d_\Gamma(x) = \begin{cases} \mathrm{dist}(x,\Gamma) & \text{if } x \in \Omega \\ -\mathrm{dist}(x,\Gamma) & \text{if } x \in \Omega^c \end{cases}$$



- $\overline{\Omega}$ invariant $\Longleftrightarrow$ $\forall x \in \Gamma$ $\begin{cases} L d_\Gamma(x) \geq 0 \\ \langle A(x)\nabla d_\Gamma(x), \nabla d_\Gamma(x)\rangle = 0 \end{cases}$

Friedman and Pinsky (1975), ...

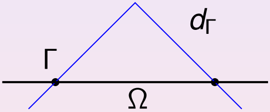
- $\overline{\Omega}$ invariant $\Longrightarrow$ L degenerate on Γ in normal direction

conditions for invariance

$\Omega \subset \mathbb{R}^n$ open set $\partial\Omega = \Gamma$

- oriented distance from Γ

$$d_\Gamma(x) = \begin{cases} \text{dist}(x,\Gamma) & \text{if } x \in \Omega \\ -\text{dist}(x,\Gamma) & \text{if } x \in \Omega^c \end{cases}$$



- $\overline{\Omega}$ invariant $\Longleftrightarrow$ $\forall x \in \Gamma$ $\begin{cases} Ld_\Gamma(x) \geq 0 \\ \langle A(x)\nabla d_\Gamma(x), \nabla d_\Gamma(x)\rangle = 0 \end{cases}$

  Friedman and Pinsky (1975), ...
- $\overline{\Omega}$ invariant $\Longrightarrow$ L degenerate on Γ in normal direction

operators with boundary degeneracy

$\Omega \subset \mathbb{R}^n$ bounded

$$\begin{cases} u_t - Lu = f & \text{in} \quad \Omega \times]0,T[\\ u(x,0) = u_0(x) & x \in \Omega \\ + \text{b. c.} & \text{on} \quad \partial\Omega \times]0,T[\end{cases}$$

where

$$Lu = \begin{cases} \operatorname{div}(A(x)\nabla u) + \text{lower order tms} \\ \quad \text{or} \\ \operatorname{Tr}\,[A(x)\nabla^2 u] + \text{lower order tms} \end{cases}$$

with

$$A(x) > 0 \quad \text{in} \quad \Omega \quad \text{but} \quad A(x) \geq 0 \quad \text{on} \quad \partial\Omega$$

the null controllability problem

want to study null-controllability in time $T > 0$

$$\forall u_0 \in L^2(\Omega) \quad \exists f \in L^2(Q_T) \; : \; \begin{cases} u^f(\cdot, T) = 0 \\ \int_{Q_T} |f|^2 \leq C_T \int_\Omega |u_0|^2 \end{cases}$$

uniformly parabolic equations:

$$\exists\, m > 0 \; : \; A(x) \geq m\,\mathbb{I}_n \quad \Longrightarrow \quad \text{null-controllability } \forall\, T > 0$$

- Fattorini and Russell (1971), Russell (1978)
- Lebeau and Robbiano (1995)
- Fursikov and Emanouilov (1996)
- Tataru (1997)

the null controllability problem

want to study null-controllability in time $T > 0$

$$\forall u_0 \in L^2(\Omega) \quad \exists f \in L^2(Q_T) \; : \; \begin{cases} u^f(\cdot, T) \equiv 0 \\ \int_{Q_T} |f|^2 \leq C_T \int_\Omega |u_0|^2 \end{cases}$$

uniformly parabolic equations:

$$\exists\, m > 0 \; : \; A(x) \geq m \mathbb{I}_n \quad \Longrightarrow \quad \text{null-controllability } \forall\, T > 0$$

- Fattorini and Russell (1971), Russell (1978)
- Lebeau and Robbiano (1995)
- Fursikov and Emanouilov (1996)
- Tataru (1997)

the null controllability problem

want to study null-controllability in time $T > 0$

$$\forall u_0 \in L^2(\Omega) \quad \exists f \in L^2(Q_T) \ : \ \begin{cases} u^f(\cdot, T) \equiv 0 \\ \int_{Q_T} |f|^2 \leq C_T \int_\Omega |u_0|^2 \end{cases}$$

uniformly parabolic equations:

$$\exists\ m > 0 \ : \ A(x) \geq m\,\mathbb{I}_n \quad \Longrightarrow \quad \text{null-controllability } \forall\ T > 0$$

- Fattorini and Russell (1971), Russell (1978)
- Lebeau and Robbiano (1995)
- Fursikov and Emanouilov (1996)
- Tataru (1997)

the roadmap to null controllability

- show equivalence with observability inequality

adjoint problem
$$\begin{cases} v_t + \mathrm{div}(A(x)\nabla v) = 0 & \text{in } Q_T \\ v = 0 & \text{on } \Gamma\times]0,T[\end{cases}$$

$$\Longrightarrow \quad \int_\Omega v^2(x,0)\,dx \le C_T \int_0^T\int_\omega v^2(x,t)\,dxdt$$

- prove observability by Carleman estimates

$\tau >> 1$

$$\iint_{Q_T} \underbrace{\tau^3\theta^3(t)v^2}_{+\tau\theta(t)|Dv|^2+\cdots} \; e^{2s\phi(x,t)}\,dxdt \le C\int_0^T\int_\omega v^2\,dxdt$$

$\phi(x,t) = \theta(t)\left[e^{r\psi(x)} - e^{2r\|\psi\|_\infty}\right]$ any $D\psi(x) \neq 0$ in $\Omega\setminus\omega$

the roadmap to null controllability

- show equivalence with observability inequality

adjoint problem
$$\begin{cases} v_t + \mathrm{div}(A(x)\nabla v) = 0 & \textit{in } Q_T \\ v = 0 & \text{on } \Gamma\times]0,T[\end{cases}$$

$$\Longrightarrow \quad \int_\Omega v^2(x,0)\,dx \le C_T \int_0^T\!\!\int_\omega v^2(x,t)\,dxdt$$

- prove observability by Carleman estimates

$\tau >> 1$

$$\iint_{Q_T} \underbrace{\tau^3\theta^3(t)v^2}_{+\tau\theta(t)|Dv|^2+\cdots}\; e^{2s\phi(x,t)}\,dxdt \le C\int_0^T\!\!\int_\omega v^2\,dxdt$$

$\phi(x,t) = \theta(t)\left[e^{r\psi(x)} - e^{2r\|\psi\|_\infty}\right]$ any $D\psi(x) \neq 0$ in $\Omega\setminus\omega$

the roadmap to null controllability

- show equivalence with observability inequality

adjoint problem
$$\begin{cases} v_t + \operatorname{div}(A(x)\nabla v) = 0 & \text{in } Q_T \\ v = 0 & \text{on } \Gamma\times]0,T[\end{cases}$$

$$\Longrightarrow \quad \int_\Omega v^2(x,0)\,dx \le C_T \int_0^T\!\!\int_\omega v^2(x,t)\,dxdt$$

- prove observability by Carleman estimates

$\tau >> 1$

$$\iint_{Q_T} \underbrace{\tau^3\theta^3(t)v^2}_{+\tau\theta(t)|Dv|^2+\cdots}\; e^{2s\phi(x,t)}\,dxdt \le C\int_0^T\!\!\int_\omega v^2\,dxdt$$

$$\phi(x,t) = \theta(t)\left[e^{r\psi(x)} - e^{2r\|\psi\|_\infty}\right]$$ any $D\psi(x)\neq 0$ in $\Omega\setminus\omega$

difficulties in degenerate case

- observability ($\Rightarrow$ null controllability) may fail (for violent degeneracies)
- ϕ in Carleman must be adapted to degeneracy
- Hardy's inequality essential

$$\int_\Omega d_\Gamma^{\alpha-2} w^2 \, dx \leq C_\alpha \int_\Omega d_\Gamma^{\alpha} |\nabla w|^2 \, dx \qquad (\alpha \neq 1)$$

difficulties in degenerate case

- observability ($\Rightarrow$ null controllability) may fail (for violent degeneracies)
- ϕ in Carleman must be adapted to degeneracy
- Hardy's inequality essential

$$\int_\Omega d_\Gamma^{\alpha-2} w^2 \, dx \leq C_\alpha \int_\Omega d_\Gamma^\alpha |\nabla w|^2 \, dx \qquad (\alpha \neq 1)$$

difficulties in degenerate case

- observability ($\Rightarrow$ null controllability) may fail (for violent degeneracies)
- ϕ in Carleman must be adapted to degeneracy
- Hardy's inequality essential

$$\int_\Omega d_\Gamma^{\alpha-2} w^2 \, dx \le C_\alpha \int_\Omega d_\Gamma^\alpha |\nabla w|^2 \, dx \qquad (\alpha \neq 1)$$

difficulties in degenerate case

- observability ($\Rightarrow$ null controllability) may fail (for violent degeneracies)
- ϕ in Carleman must be adapted to degeneracy
- Hardy's inequality essential

$$\int_\Omega d_\Gamma^{\alpha-2} w^2 \, dx \le C_\alpha \int_\Omega d_\Gamma^\alpha |\nabla w|^2 \, dx \qquad (\alpha \neq 1)$$

Outline

weak and strong degeneracy

$a \in C([0,1]) \cap C^1(]0,1])$ and $a > 0$ on $]0,1]$

$$\begin{cases} u_t - \left(a(x)u_x\right)_x = f & in\ Q_T =]0,1[\times]0,T[\\ u(x,0) = u_0(x) & u(1,t) = 0 \ + \ \text{b.c. at } \ x = 0 \end{cases}$$

$u_0 \in L^2(0,1)\,,\ f \in L^2(Q_T)$

weakly degenerate case: $1/a \in L^1(0,1)$

$$H^1_a(0,1) = \left\{u \in L^2(0,1) \mid \int_0^1 a u_x^2\,dx < \infty \ \&\ u(0) = 0 = u(1)\right\}$$

strongly degenerate case: $1/a \notin L^1(0,1)$

$$H^1_a(0,1) = \left\{u \in L^2(0,1) \mid \int_0^1 a u_x^2\,dx < \infty \ \&\ u(1) = 0\right\}$$

weak and strong degeneracy

$a \in C([0,1]) \cap C^1(]0,1])$ and $a > 0$ on $]0,1]$

$$\begin{cases} u_t - \big(a(x)u_x\big)_x = f & \text{in } Q_T =]0,1[\times]0,T[\\ u(x,0) = u_0(x) & u(1,t) = 0 \; + \text{ b.c. at } x = 0 \end{cases}$$

$u_0 \in L^2(0,1)\,,\; f \in L^2(Q_T)$

weakly degenerate case: $1/a \in L^1(0,1)$

$$H^1_a(0,1) = \left\{ u \in L^2(0,1) \mid \int_0^1 a u_x^2\, dx < \infty \;\&\; u(0) = 0 = u(1) \right\}$$

strongly degenerate case: $1/a \notin L^1(0,1)$

$$H^1_a(0,1) = \left\{ u \in L^2(0,1) \mid \int_0^1 a u_x^2\, dx < \infty \;\&\; u(1) = 0 \right\}$$

weak and strong degeneracy

$a \in C([0,1]) \cap C^1(]0,1])$ and $a > 0$ on $]0,1]$

$$\begin{cases} u_t - \big(a(x)u_x\big)_x = f & in\ Q_T =]0,1[\times]0,T[\\ u(x,0) = u_0(x) & u(1,t) = 0\ +\ \text{b.c. at}\ \ x = 0 \end{cases}$$

$u_0 \in L^2(0,1)\,,\ f \in L^2(Q_T)$

weakly degenerate case: $\boxed{1/a \in L^1(0,1)}$

$$H^1_a(0,1) = \Big\{ u \in L^2(0,1) \mid \int_0^1 a u_x^2\, dx < \infty \ \&\ u(0) = 0 = u(1) \Big\}$$

strongly degenerate case: $\boxed{1/a \notin L^1(0,1)}$

$$H^1_a(0,1) = \Big\{ u \in L^2(0,1) \mid \int_0^1 a u_x^2\, dx < \infty \ \&\ u(1) = 0 \Big\}$$

well-posedness

$$\begin{cases} D(\mathcal{A}) = \{u \in H^1_a(0,1) \mid au_x \in H^1(0,1)\} \\ \mathcal{A}u = (au_x)_x \end{cases}$$

generates analytic semigroup in $L^2(0,1)$

- unique solution $\quad u \in C(0,T;L^2(0,1)) \cap L^2(0,T;H^1_a(0,1))$

  $$\begin{cases} u_t - (a(x)u_x)_x = f \quad \text{in } Q_T =]0,1[\times]0,T[\\ u(x,0) = u_0(x) \end{cases}$$

- maximal regularity

  $$u_0 \in H^1_a(0,1) \quad \Longrightarrow \quad u \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;D(\mathcal{A}))$$

  (needed to justify integration by parts)

- strongly degenerate case incorporates b.c.

  $$u \in D(\mathcal{A}) \quad \Longrightarrow \quad au_x \xrightarrow{(x \to 0)} 0$$

well-posedness

$$\begin{cases} D(\mathcal{A}) = \{u \in H^1_a(0,1) \mid au_x \in H^1(0,1)\} \\ \mathcal{A}u = (au_x)_x \end{cases}$$

generates analytic semigroup in $L^2(0,1)$

- unique solution $\quad u \in C(0,T;L^2(0,1)) \cap L^2(0,T;H^1_a(0,1))$

$$\begin{cases} u_t - (a(x)u_x)_x = f & in\ Q_T =]0,1[\times]0,T[\\ u(x,0) = u_0(x) \end{cases}$$

- maximal regularity

$$u_0 \in H^1_a(0,1) \quad \Longrightarrow \quad u \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;D(\mathcal{A}))$$

(needed to justify integration by parts)

- strongly degenerate case incorporates b.c.

$$u \in D(\mathcal{A}) \quad \Longrightarrow \quad au_x \overset{(x\to 0)}{\longrightarrow} 0$$

well-posedness

$$\begin{cases} D(\mathcal{A}) = \left\{u \in H^1_a(0,1) \mid au_x \in H^1(0,1)\right\} \\ \mathcal{A}u = \left(au_x\right)_x \end{cases}$$

generates analytic semigroup in $L^2(0,1)$

- unique solution $u \in C(0,T;L^2(0,1)) \cap L^2(0,T;H^1_a(0,1))$

$$\begin{cases} u_t - \left(a(x)u_x\right)_x = f \quad in\ Q_T =]0,1[\times]0,T[\\ u(x,0) = u_0(x) \end{cases}$$

- maximal regularity

$$u_0 \in H^1_a(0,1) \quad \Longrightarrow \quad u \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;D(\mathcal{A}))$$

(needed to justify integration by parts)

- strongly degenerate case incorporates b.c.

$$u \in D(\mathcal{A}) \quad \Longrightarrow \quad au_x \stackrel{(x\to 0)}{\longrightarrow} 0$$

well-posedness

$$\begin{cases} D(\mathcal{A}) = \{u \in H^1_a(0,1) \mid au_x \in H^1(0,1)\} \\ \mathcal{A}u = (au_x)_x \end{cases}$$

generates analytic semigroup in $L^2(0,1)$

- unique solution $u \in C(0,T;L^2(0,1)) \cap L^2(0,T;H^1_a(0,1))$

$$\begin{cases} u_t - (a(x)u_x)_x = f \quad in\ Q_T =]0,1[\times]0,T[\\ u(x,0) = u_0(x) \end{cases}$$

- maximal regularity

$$u_0 \in H^1_a(0,1) \quad \Longrightarrow \quad u \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;D(\mathcal{A}))$$

(needed to justify integration by parts)

- strongly degenerate case incorporates b.c.

$$u \in D(\mathcal{A}) \quad \Longrightarrow \quad au_x \stackrel{(x \to 0)}{\longrightarrow} 0$$

well-posedness

$$\begin{cases} D(\mathcal{A}) = \left\{u \in H^1_a(0,1) \mid au_x \in H^1(0,1)\right\} \\ \mathcal{A}u = \left(au_x\right)_x \end{cases}$$

generates analytic semigroup in $L^2(0,1)$

- unique solution $u \in C(0,T;L^2(0,1)) \cap L^2(0,T;H^1_a(0,1))$

  $$\begin{cases} u_t - \left(a(x)u_x\right)_x = f \quad in\ Q_T =]0,1[\times]0,T[\\ u(x,0) = u_0(x) \end{cases}$$

- maximal regularity

  $$u_0 \in H^1_a(0,1) \quad \Longrightarrow \quad u \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;D(\mathcal{A}))$$

  (needed to justify integration by parts)

- strongly degenerate case incorporates b.c.

  $$u \in D(\mathcal{A}) \quad \Longrightarrow \quad au_x \overset{(x\to 0)}{\longrightarrow} 0$$

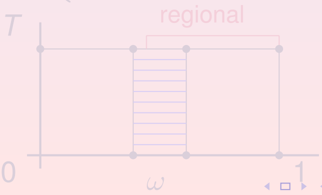
the simplest example of degeneracy

$$\omega =]a,b[\subset\subset]0,1[\qquad \boxed{a(x) = x^\alpha} \qquad (\alpha > 0)$$

$$u_t - \left(x^\alpha u_x\right)_x = \chi_\omega f \,, \qquad u(x,0) = u_0(x)$$

C–, Martinez & Vancostenoble (2008):

$$\textbf{n. c.} \begin{cases} \text{false} & \alpha \geq 2 \quad (\to \text{regional null controllability}) \\ \text{true} & 0 \leq \alpha < 2 \begin{cases} \text{any b.c.} & 0 \leq \alpha < 1 \quad \text{weak} \\ \text{Neumann b.c.} & 1 \leq \alpha < 2 \quad \text{strong} \end{cases} \end{cases}$$



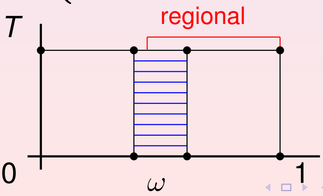
the simplest example of degeneracy

$$\omega =]a, b[\subset\subset]0, 1[\qquad \boxed{a(x) = x^\alpha} \qquad (\alpha > 0)$$

$$u_t - \left(x^\alpha u_x\right)_x = \chi_\omega f \ , \qquad u(x, 0) = u_0(x)$$

C–, Martinez & Vancostenoble (2008):

$$\textbf{n. c.} \begin{cases} \text{false} & \alpha \geq 2 \quad (\to \text{regional null controllability}) \\ \text{true} & 0 \leq \alpha < 2 \begin{cases} \text{any b.c.} & 0 \leq \alpha < 1 \quad \text{weak} \\ \text{Neumann b.c.} & 1 \leq \alpha < 2 \quad \text{strong} \end{cases} \end{cases}$$



lack of null controllability for $\alpha \geq 2$

- classical change of variable (Courant-Hilbert)

$$y(x) = \int_x^1 \frac{ds}{s^{\alpha/2}} \qquad U(y(x), t) = x^{\alpha/4} u(x, t)$$

transforms equation into $\widetilde{\omega} =]\widetilde{b}, \widetilde{a}[$ bounded

$$U_t - U_{yy} + c_\alpha(y) U = \chi_{\widetilde{\omega}} F \qquad 0 < y < \infty$$

$c(y) = \frac{\alpha(3\alpha-4)}{4[2+(\alpha-2)y]^2}$ bounded for $\alpha \geq 2$

- Micu, Zuazua (2001)
  Escauriaza, Seregin, Sverak (2003, 2004)

$$U(\cdot, T) = 0 \iff \operatorname{spt}(U(\cdot, 0)) \subset [0, \widetilde{a}[$$

- $u_t - (x^\alpha u_x)_x = \chi_\omega f \quad \begin{cases} \text{no} & [0,1] \\ \text{yes} & \text{if } \operatorname{spt}(u_0) \subset]a, 1] \end{cases}$

lack of null controllability for $\alpha \geq 2$

- classical change of variable (Courant-Hilbert)

$$y(x) = \int_x^1 \frac{ds}{s^{\alpha/2}} \qquad U(y(x),t) = x^{\alpha/4} u(x,t)$$

transforms equation into $\widetilde{\omega} =]\widetilde{b}, \widetilde{a}[$ bounded

$$U_t - U_{yy} + c_\alpha(y)U = \chi_{\widetilde{\omega}} F \qquad 0 < y < \infty$$

$c(y) = \frac{\alpha(3\alpha-4)}{4[2+(\alpha-2)y]^2}$ bounded for $\alpha \geq 2$

- Micu, Zuazua (2001)
  Escauriaza, Seregin, Sverak (2003, 2004)

$$U(\cdot, T) = 0 \iff \text{spt}(U(\cdot,0)) \subset [0, \widetilde{a}[$$

- $u_t - (x^\alpha u_x)_x = \chi_\omega f \quad \begin{cases} \text{no} & [0,1] \\ \text{yes} & \text{if } \text{spt}(u_0) \subset]a,1] \end{cases}$

lack of null controllability for $\alpha \geq 2$

- classical change of variable (Courant-Hilbert)

$$y(x) = \int_x^1 \frac{ds}{s^{\alpha/2}} \qquad U(y(x), t) = x^{\alpha/4} u(x, t)$$

transforms equation into $\widetilde{\omega} =]\widetilde{b}, \widetilde{a}[$ bounded

$$U_t - U_{yy} + c_\alpha(y) U = \chi_{\widetilde{\omega}} F \qquad 0 < y < \infty$$

$c(y) = \frac{\alpha(3\alpha-4)}{4[2+(\alpha-2)y]^2}$ bounded for $\alpha \geq 2$

- Micu, Zuazua (2001)
  Escauriaza, Seregin, Sverak (2003, 2004)

$$U(\cdot, T) = 0 \iff \operatorname{spt}\big(U(\cdot, 0)\big) \subset [0, \widetilde{a}[$$

- $u_t - (x^\alpha u_x)_x = \chi_\omega f$ $\begin{cases} \text{no} & [0,1] \\ \text{yes} & \text{if } \operatorname{spt}(u_0) \subset]a,1] \end{cases}$

lack of null controllability for $\alpha \geq 2$

- classical change of variable (Courant-Hilbert)

$$y(x) = \int_x^1 \frac{ds}{s^{\alpha/2}} \qquad U(y(x),t) = x^{\alpha/4} u(x,t)$$

transforms equation into $\widetilde{\omega} =]\widetilde{b}, \widetilde{a}[$ bounded

$$U_t - U_{yy} + c_\alpha(y) U = \chi_{\widetilde{\omega}} F \qquad 0 < y < \infty$$

$c(y) = \frac{\alpha(3\alpha-4)}{4[2+(\alpha-2)y]^2}$ bounded for $\alpha \geq 2$

- Micu, Zuazua (2001)
  Escauriaza, Seregin, Sverak (2003, 2004)

$$U(\cdot, T) = 0 \iff \mathrm{spt}\big(U(\cdot, 0)\big) \subset [0, \widetilde{a}[$$

- $u_t - \left(x^\alpha u_x\right)_x = \chi_\omega f \quad \begin{cases} \text{no} & [0,1] \\ \text{yes} & \text{if} \quad \mathrm{spt}(u_0) \subset]a, 1] \end{cases}$

Carleman estimate $0 < \alpha < 2$

$$w_t + \left(x^\alpha w_x\right)_x = f \quad in \quad]0,1[\times]0,T[\qquad + \quad \text{b. c.}$$

let $\boxed{\varphi(t,x) = \theta(t)\psi(x)}$ where

$$\theta(t) = \left(\frac{1}{t(T-t)}\right)^4 \qquad \psi(x) = \frac{x^{2-\alpha} - 2}{(2-\alpha)^2}$$

Theorem (C–, Martinez & Vancostenoble 2008)

There exists $\tau_0, C > 0$ *such that* $\forall \tau \geq \tau_0$

$$\iint_{Q_T} \left(\frac{w_t^2}{\tau\theta} + \tau\theta x^\alpha w_x^2 + \tau^3\theta^3 x^{2-\alpha} w^2\right) e^{2\tau\varphi}\, dxdt$$

$$\leq C \iint_{Q_T} |f|^2 e^{2\tau\varphi}\, dxdt + C\int_0^T \left\{\tau\theta w_x^2 e^{2\tau\varphi}\right\}_{|x=1} dt$$

Carleman estimate $0 < \alpha < 2$

$$w_t + \left(x^\alpha w_x\right)_x = f \quad in \quad]0,1[\times]0,T[\qquad + \quad \text{b. c.}$$

let $\boxed{\varphi(t,x) = \theta(t)\psi(x)}$ where

$$\theta(t) = \left(\frac{1}{t(T-t)}\right)^4 \qquad \psi(x) = \frac{x^{2-\alpha} - 2}{(2-\alpha)^2}$$

Theorem (C–, Martinez & Vancostenoble 2008)

There exists $\tau_0, C > 0$ *such that* $\forall \tau \geq \tau_0$

$$\iint_{Q_T} \left(\frac{w_t^2}{\tau\theta} + \tau\theta x^\alpha w_x^2 + \tau^3\theta^3 x^{2-\alpha} w^2\right) e^{2\tau\varphi}\, dxdt$$

$$\leq C \iint_{Q_T} |f|^2 e^{2\tau\varphi}\, dxdt + C\int_0^T \left\{\tau\theta w_x^2 e^{2\tau\varphi}\right\}_{|x=1} dt$$

Carleman estimate $0 < \alpha < 2$

$$w_t + (x^\alpha w_x)_x = f \quad in \quad]0,1[\times]0,T[\qquad + \quad \text{b. c.}$$

let $\boxed{\varphi(t,x) = \theta(t)\psi(x)}$ where

$$\theta(t) = \left(\frac{1}{t(T-t)}\right)^4 \qquad \psi(x) = \frac{x^{2-\alpha} - 2}{(2-\alpha)^2}$$

Theorem (C–, Martinez & Vancostenoble 2008)

There exists $\tau_0, C > 0$ *such that* $\forall \tau \geq \tau_0$

$$\iint_{Q_T} \left(\frac{w_t^2}{\tau\theta} + \tau\theta x^\alpha w_x^2 + \tau^3\theta^3 x^{2-\alpha} w^2\right) e^{2\tau\varphi}\, dxdt$$

$$\leq C \iint_{Q_T} |f|^2 e^{2\tau\varphi}\, dxdt + C\int_0^T \left\{\tau\theta w_x^2 e^{2\tau\varphi}\right\}_{|x=1} dt$$

application to adjoint problem

$$\begin{cases} v_t + \left(x^\alpha v_x\right)_x = 0 \quad in \quad]0,1[\times]0,T[\\ + \quad \text{b. c.} \end{cases}$$

by Carleman estimate

$$\iint_{Q_T} \left(\tau\theta x^\alpha v_x^2 + \tau^3\theta^3 \underbrace{x^{2-\alpha}}_{\to 0} v^2\right) e^{2\tau\varphi}\, dxdt \le C_T \int_0^T \int_\omega |v|^2\, dxdt$$

- zero order estimate too weak
- need Hardy's inequality: $(\alpha \neq 1) \quad \forall w \in H^1_\alpha(0,1)$

$$\int_0^1 x^{\alpha-2} w^2\, dx \le \frac{4}{(\alpha-1)^2} \int_0^1 x^\alpha w_x^2\, dx$$

application to adjoint problem

$$\begin{cases} v_t + \left(x^\alpha v_x\right)_x = 0 \quad in \quad]0,1[\times]0,T[\\ + \quad \text{b. c.} \end{cases}$$

by Carleman estimate

$$\iint_{Q_T} \Big(\tau\theta x^\alpha v_x^2 + \tau^3\theta^3 \underbrace{x^{2-\alpha}}_{\to 0} v^2\Big) e^{2\tau\varphi}\, dxdt \le C_T \int_0^T \int_\omega |v|^2\, dxdt$$

- zero order estimate too weak
- need Hardy's inequality: $(\alpha \neq 1) \quad \forall w \in H^1_\alpha(0,1)$

$$\int_0^1 x^{\alpha-2} w^2\, dx \le \frac{4}{(\alpha-1)^2} \int_0^1 x^\alpha w_x^2\, dx$$

application to adjoint problem

$$\begin{cases} v_t + \left(x^\alpha v_x\right)_x = 0 \quad in \quad]0,1[\times]0,T[\\ + \quad \text{b. c.} \end{cases}$$

by Carleman estimate

$$\iint_{Q_T} \left(\tau\theta x^\alpha v_x^2 + \tau^3\theta^3 \underbrace{x^{2-\alpha}}_{\to 0} v^2\right) e^{2\tau\varphi}\, dxdt \le C_T \int_0^T \int_\omega |v|^2\, dxdt$$

- zero order estimate too weak
- need Hardy's inequality: $(\alpha \neq 1) \quad \forall w \in H^1_\alpha(0,1)$

$$\int_0^1 x^{\alpha-2} w^2\, dx \le \frac{4}{(\alpha-1)^2} \int_0^1 x^\alpha w_x^2\, dx$$

application to adjoint problem

$$\begin{cases} v_t + \left(x^\alpha v_x\right)_x = 0 \quad in \quad]0,1[\times]0,T[\\ + \quad \text{b. c.} \end{cases}$$

by Carleman estimate

$$\iint_{Q_T} \left(\tau\theta x^\alpha v_x^2 + \tau^3\theta^3 \underbrace{x^{2-\alpha}}_{\to 0} v^2\right) e^{2\tau\varphi}\, dxdt \leq C_T \int_0^T \int_\omega |v|^2\, dxdt$$

- zero order estimate too weak
- need Hardy's inequality: $(\alpha \neq 1) \quad \forall w \in H^1_\alpha(0,1)$

$$\int_0^1 x^{\alpha-2} w^2\, dx \leq \frac{4}{(\alpha-1)^2} \int_0^1 x^\alpha w_x^2\, dx$$

application to adjoint problem

$$\begin{cases} v_t + \left(x^\alpha v_x\right)_x = 0 \quad in \quad]0,1[\times]0,T[\\ + \quad \text{b. c.} \end{cases}$$

by Carleman estimate

$$\iint_{Q_T} \left(\tau\theta x^\alpha v_x^2 + \tau^3\theta^3 \underbrace{x^{2-\alpha}}_{\to 0} v^2\right) e^{2\tau\varphi}\, dxdt \leq C_T \int_0^T \int_\omega |v|^2\, dxdt$$

- zero order estimate too weak
- need Hardy's inequality: $(\alpha \neq 1) \quad \forall w \in H^1_\alpha(0,1)$

$$\int_0^1 x^{\alpha-2} w^2\, dx \leq \frac{4}{(\alpha-1)^2} \int_0^1 x^\alpha w_x^2\, dx$$

CE $\Rightarrow$ observability for $v_t + \left(x^\alpha v_x\right)_x = 0$

- $t \mapsto \int_0^1 x^\alpha v_x^2 \, dx$ increasing
- integrate & use Carleman's estimate

$$\int_0^1 x^\alpha v_x^2(x,0)\,dx \le \frac{2}{T}\int_{T/4}^{3T/4}\int_0^1 x^\alpha v_x^2(x,t)\,dxdt$$

$$\le C_T \int\int_{Q_T} \theta(t) x^\alpha v_x^2(x,t) e^{2s\phi(x,t)}\,dxdt \le C_T \int_0^T\int_\omega v^2(x,t)\,dxdt$$

- use Hardy's inequality

$$\int_0^1 x^{\alpha-2} v^2(x,0)\,dx \le \int_0^1 x^\alpha v_x^2(0,x)\,dx \le C\int_0^T\int_\omega v^2(x,t)dxdt$$

CE $\Rightarrow$ observability for $v_t + \left(x^\alpha v_x\right)_x = 0$

- $t \mapsto \int_0^1 x^\alpha v_x^2 \, dx$ increasing
- integrate & use Carleman's estimate

$$\int_0^1 x^\alpha v_x^2(x,0)\, dx \le \frac{2}{T} \int_{T/4}^{3T/4} \int_0^1 x^\alpha v_x^2(x,t)\, dxdt$$

$$\le C_T \int\int_{Q_T} \theta(t) x^\alpha v_x^2(x,t) e^{2s\phi(x,t)}\, dxdt \le C_T \int_0^T \int_\omega v^2(x,t)\, dxdt$$

- use Hardy's inequality

$$\int_0^1 x^{\alpha-2} v^2(x,0)\, dx \le \int_0^1 x^\alpha v_x^2(0,x)\, dx \le C \int_0^T \int_\omega v^2(x,t) dxdt$$

CE $\Rightarrow$ observability for $v_t + \left(x^\alpha v_x\right)_x = 0$

- $t \mapsto \int_0^1 x^\alpha v_x^2\,dx$ increasing
- integrate & use Carleman's estimate

$$\int_0^1 x^\alpha v_x^2(x,0)\,dx \le \frac{2}{T}\int_{T/4}^{3T/4}\int_0^1 x^\alpha v_x^2(x,t)\,dxdt$$

$$\le C_T \int\int_{Q_T} \theta(t) x^\alpha v_x^2(x,t) e^{2s\phi(x,t)}\,dxdt \le C_T \int_0^T \int_\omega v^2(x,t)\,dxdt$$

- use Hardy's inequality

$$\int_0^1 x^{\alpha-2} v^2(x,0)\,dx \le \int_0^1 x^\alpha v_x^2(0,x)\,dx \le C\int_0^T\int_\omega v^2(x,t)dxdt$$

CE $\Rightarrow$ observability for $v_t + \left(x^\alpha v_x\right)_x = 0$

- $t \mapsto \int_0^1 x^\alpha v_x^2 \, dx$ increasing
- integrate & use Carleman's estimate

$$\int_0^1 x^\alpha v_x^2(x,0)\,dx \le \frac{2}{T}\int_{T/4}^{3T/4}\int_0^1 x^\alpha v_x^2(x,t)\,dxdt$$

$$\le C_T \int\int_{Q_T} \theta(t) x^\alpha v_x^2(x,t) e^{2s\phi(x,t)}\,dxdt \le C_T \int_0^T \int_\omega v^2(x,t)\,dxdt$$

- use Hardy's inequality

$$\int_0^1 x^{\alpha-2} v^2(x,0)\,dx \le \int_0^1 x^\alpha v_x^2(0,x)\,dx \le C \int_0^T \int_\omega v^2(x,t) dxdt$$

more general 1-d problems

- divergence form
  - Martinez, Vancostenoble (2006) $u_t - (a(x)u_x)_x = \chi_\omega f$
  - Alabau, C, Fragnelli (2006) $u_t - (a(x)u_x)_x + g(u) = \chi_\omega f$
  - Flores, de Teresa (2010) $u_t - (x^\alpha u_x)_x + x^\beta b(x,t)u_x = \chi_\omega f$
- non-divergence form C, Fragnelli, Rocchetti (2007, 2008)

$$u_t - a(x)u_{xx} - b(x)u_x = \chi_\omega f$$

  - (invariant) measure $\rho(x)dx = \frac{1}{a(x)} e^{\int_{1/2}^x \frac{b(s)}{a(s)}\,ds} dx$

    symmetric structure $a(x)u_{xx} + b(x)u_x = \frac{1}{\rho(x)}\left[\rho(x)a(x)u_x\right]_x$

    weighted space $L^2_\rho(0,1) := \left\{u \in L^2(0,1) \mid \int_0^1 u^2\rho\,dx < \infty\right\}$
- degenerate/singular problems
  - Vancostenoble, Zuazua (2008), Vancostenoble (2009)

$$u_t - (x^\alpha u_x)_x - \frac{\lambda}{x^\beta}u = \chi_\omega f$$

more general 1-d problems

- divergence form
  - Martinez, Vancostenoble (2006) $u_t - (a(x)u_x)_x = \chi_\omega f$
  - Alabau, C, Fragnelli (2006) $u_t - (a(x)u_x)_x + g(u) = \chi_\omega f$
  - Flores, de Teresa (2010) $u_t - (x^\alpha u_x)_x + x^\beta b(x,t)u_x = \chi_\omega f$
- non-divergence form C, Fragnelli, Rocchetti (2007, 2008)

$$u_t - a(x)u_{xx} - b(x)u_x = \chi_\omega f$$

  - (invariant) measure $\rho(x)dx = \frac{1}{a(x)} e^{\int_{1/2}^x \frac{b(s)}{a(s)}ds} dx$

    symmetric structure $a(x)u_{xx} + b(x)u_x = \frac{1}{\rho(x)}\left[\rho(x)a(x)u_x\right]_x$

    weighted space $L^2_\rho(0,1) := \left\{ u \in L^2(0,1) \mid \int_0^1 u^2 \rho \, dx < \infty \right\}$
- degenerate/singular problems
  - Vancostenoble, Zuazua (2008), Vancostenoble (2009)

$$u_t - (x^\alpha u_x)_x - \frac{\lambda}{x^\beta} u = \chi_\omega f$$

more general 1-d problems

- divergence form
  - Martinez, Vancostenoble (2006) $u_t - \big(a(x)u_x\big)_x = \chi_\omega f$
  - Alabau, C, Fragnelli (2006) $u_t - \big(a(x)u_x\big)_x + g(u) = \chi_\omega f$
  - Flores, de Teresa (2010) $u_t - \big(x^\alpha u_x\big)_x + x^\beta b(x,t)u_x = \chi_\omega f$
- non-divergence form C, Fragnelli, Rocchetti (2007, 2008)

$$u_t - a(x)u_{xx} - b(x)u_x = \chi_\omega f$$

  - (invariant) measure $\rho(x)dx = \frac{1}{a(x)} e^{\int_{1/2}^{x} \frac{b(s)}{a(s)}\, ds} dx$

    symmetric structure $a(x)u_{xx} + b(x)u_x = \frac{1}{\rho(x)}\big[\rho(x)a(x)u_x\big]_x$

    weighted space $L^2_\rho(0,1) := \left\{ u \in L^2(0,1) \;\middle|\; \int_0^1 u^2 \rho\, dx < \infty \right\}$
- degenerate/singular problems
  - Vancostenoble, Zuazua (2008), Vancostenoble (2009)

$$u_t - \big(x^\alpha u_x\big)_x - \frac{\lambda}{x^\beta} u = \chi_\omega f$$

more general 1-d problems

- divergence form
  - Martinez, Vancostenoble (2006) $u_t - \big(a(x)u_x\big)_x = \chi_\omega f$
  - Alabau, C, Fragnelli (2006) $u_t - \big(a(x)u_x\big)_x + g(u) = \chi_\omega f$
  - Flores, de Teresa (2010) $u_t - \big(x^\alpha u_x\big)_x + x^\beta b(x,t)u_x = \chi_\omega f$
- non-divergence form C, Fragnelli, Rocchetti (2007, 2008)

$$u_t - a(x)u_{xx} - b(x)u_x = \chi_\omega f$$

  - (invariant) measure $\rho(x)dx = \frac{1}{a(x)} e^{\int_{1/2}^{x} \frac{b(s)}{a(s)}\,ds} dx$

    symmetric structure $a(x)u_{xx} + b(x)u_x = \frac{1}{\rho(x)}\big[\rho(x)a(x)u_x\big]_x$

    weighted space $L^2_\rho(0,1) := \Big\{u \in L^2(0,1) \;\Big|\; \int_0^1 u^2\rho\,dx < \infty\Big\}$
- degenerate/singular problems
  - Vancostenoble, Zuazua (2008), Vancostenoble (2009)

$$u_t - \big(x^\alpha u_x\big)_x - \frac{\lambda}{x^\beta}u = \chi_\omega f$$

Outline

- Examples of degenerate parabolic operators (DPO)
  - Budyko-Sellers one-dimensional model
  - linearized Crocco equation
  - Kolmogorov operators & degenerate diffusions
- Control of DPO with boundary degeneracy
  - null controllability in dimension 1
  - approximate controllability in dimension 1
  - null controllability in dimension 2
- Control of DPO with interior degeneracy
  - parabolic operators with interior degeneracy
  - control of Grushin-type operators in dimension 2

a boundary control problem

$0 < \alpha < 1\,,\ T > 0$

Problem

given $u_0, u_T \in L^2(0,1)$ and $\varepsilon > 0$ find g such that the solution of

$$\begin{cases} u_t - \left(x^\alpha u_x\right)_x = 0 & \text{in } Q_T =]0,1[\times]0,T[\\ u(0,t) = g(t) \\ u(1,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

satisfies $\quad \|u(\cdot,T) - u_T\|_{L^2(Q_T)} < \varepsilon$

- reduction to a locally distributed control system by extending the space domain would produce interior degeneracy
- need unique continuation obtained via local Carleman estimate

  C – Tort – Yamamoto (2011)

a boundary control problem

$0 < \alpha < 1\,,\ T > 0$

Problem

given $u_0, u_T \in L^2(0,1)$ *and* $\varepsilon > 0$ *find* g *such that the solution of*

$$\begin{cases} u_t - \left(x^\alpha u_x\right)_x = 0 & \text{in } Q_T =]0,1[\times]0,T[\\ u(0,t) = g(t) \\ u(1,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

satisfies $\|u(\cdot,T) - u_T\|_{L^2(Q_T)} < \varepsilon$

- reduction to a locally distributed control system by extending the space domain would produce interior degeneracy
- need unique continuation obtained via local Carleman estimate
  C – Tort – Yamamoto (2011)

a boundary control problem

$0 < \alpha < 1\,,\ T > 0$

Problem

given $u_0, u_T \in L^2(0,1)$ *and* $\varepsilon > 0$ *find* g *such that the solution of*

$$\begin{cases} u_t - \left(x^\alpha u_x\right)_x = 0 & \text{in } Q_T =]0,1[\times]0,T[\\ u(0,t) = g(t) \\ u(1,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

satisfies $\qquad \|u(\cdot,T) - u_T\|_{L^2(Q_T)} < \varepsilon$

- reduction to a locally distributed control system by extending the space domain would produce interior degeneracy
- need unique continuation obtained via local Carleman estimate
  C – Tort – Yamamoto (2011)

a boundary control problem

$0 < \alpha < 1\,,\ T > 0$

Problem

given $u_0, u_T \in L^2(0,1)$ *and* $\varepsilon > 0$ *find g such that the solution of*

$$\begin{cases} u_t - \left(x^\alpha u_x\right)_x = 0 \quad \text{in } Q_T =]0,1[\times]0,T[\\ u(0,t) = g(t) \\ u(1,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

satisfies $\quad \|u(\cdot,T) - u_T\|_{L^2(Q_T)} < \varepsilon$

- reduction to a locally distributed control system by extending the space domain would produce interior degeneracy
- need unique continuation obtained via local Carleman estimate
  C – Tort – Yamamoto (2011)

a local Carleman estimate

$$\begin{cases} w_t + (x^\alpha w_x)_x = f & in\ Q_T =]0,1[\times]0,T[\\ w(0,t) = 0\,, & (x^\alpha w_x)(0,t) = 0 \\ w(1,t) = 0\,, & (x^\alpha w_x)(1,t) = 0 \end{cases}$$

let $1-\alpha < \beta < 1 - \frac{\alpha}{2}$ and define $\varphi(t,x) = \theta(t)\psi(x)$ where

$$\theta(t) = \frac{1}{t(T-t)}\,, \qquad \psi(x) = -x^\beta$$

Theorem

There exists $\tau_0, C > 0$ *such that* $\forall \tau \geq \tau_0$

$$\iint_{Q_T} \left(\tau\theta x^{2\alpha+\beta-2} w_x^2 + \tau^3\theta^3 w^2\right) e^{2\tau\varphi} \leq C \iint_{Q_T} |f|^2 e^{2\tau\varphi}$$

a local Carleman estimate

$$\begin{cases} w_t + \left(x^\alpha w_x\right)_x = f & in\ Q_T =]0,1[\times]0,T[\\ w(0,t) = 0\,, & (x^\alpha w_x)(0,t) = 0 \\ w(1,t) = 0\,, & (x^\alpha w_x)(1,t) = 0 \end{cases}$$

let $1-\alpha < \beta < 1-\frac{\alpha}{2}$ and define $\varphi(t,x) = \theta(t)\psi(x)$ where

$$\theta(t) = \frac{1}{t(T-t)}\,, \qquad \psi(x) = -x^\beta$$

Theorem

There exists $\tau_0, C > 0$ *such that* $\forall \tau \geq \tau_0$

$$\iint_{Q_T} \left(\tau\theta x^{2\alpha+\beta-2} w_x^2 + \tau^3\theta^3 w^2\right) e^{2\tau\varphi} \leq C \iint_{Q_T} |f|^2 e^{2\tau\varphi}$$

a local Carleman estimate

$$\begin{cases} w_t + \left(x^\alpha w_x\right)_x = f & \text{in } Q_T =]0,1[\times]0,T[\\ w(0,t) = 0\,, & (x^\alpha w_x)(0,t) = 0 \\ w(1,t) = 0\,, & (x^\alpha w_x)(1,t) = 0 \end{cases}$$

let $\boxed{1-\alpha < \beta < 1 - \frac{\alpha}{2}}$ and define $\boxed{\varphi(t,x) = \theta(t)\psi(x)}$ where

$$\theta(t) = \frac{1}{t(T-t)}\,, \qquad \psi(x) = -x^\beta$$

Theorem

There exists $\tau_0, C > 0$ *such that* $\forall \tau \geq \tau_0$

$$\iint_{Q_T} \left(\tau\theta x^{2\alpha+\beta-2} w_x^2 + \tau^3\theta^3 w^2\right) e^{2\tau\varphi} \leq C \iint_{Q_T} |f|^2 e^{2\tau\varphi}$$

a local Carleman estimate

$$\begin{cases} w_t + \left(x^\alpha w_x\right)_x = f & \textit{in } Q_T =]0,1[\times]0,T[\\ w(0,t) = 0\,, & (x^\alpha w_x)(0,t) = 0 \\ w(1,t) = 0\,, & (x^\alpha w_x)(1,t) = 0 \end{cases}$$

let $1 - \alpha < \beta < 1 - \frac{\alpha}{2}$ and define $\varphi(t,x) = \theta(t)\psi(x)$ where

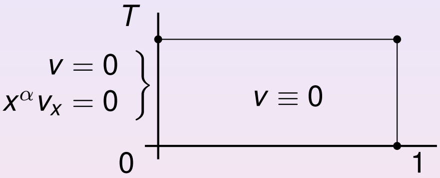
$$\theta(t) = \frac{1}{t(T-t)}\,, \qquad \psi(x) = -x^\beta$$

Theorem

There exists $\tau_0, C > 0$ *such that* $\forall \tau \geq \tau_0$

$$\iint_{Q_T} \left(\tau\theta x^{2\alpha+\beta-2} w_x^2 + \tau^3\theta^3 w^2\right) e^{2\tau\varphi} \leq C \iint_{Q_T} |f|^2 e^{2\tau\varphi}$$

a unique continuation result



Corollary

$$\begin{cases} v_t + (x^\alpha v_x)_x = 0 & in\ Q_T =]0,1[\times]0,T[\\ v(0,t) = 0\,, & (x^\alpha v_x)(0,t) = 0 \end{cases}$$

Then $\boxed{v \equiv 0 \ in\ Q_T}$

approximate controllability

$0 < \alpha < 1\,,\ T > 0\,,\ u_0 \in H^1_\alpha(0,1)$

$$\begin{cases} u_t - \left(x^\alpha u_x\right)_x = 0 & in\ Q_T =]0,1[\times]0,T[\\ u(0,t) = g(t) \\ u(1,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Proposition

For all $u_T \in L^2(0,1)\,,\ \varepsilon > 0$ there exists $g \in H^1_0(0,T)$ such that

$$\|u(\cdot,T) - u_T\|_{L^2(Q_T)} < \varepsilon$$

approximate controllability for all $u_0 \in L^2(0,1)$ follows

approximate controllability

$0 < \alpha < 1\,,\ T > 0\,,\ u_0 \in H^1_\alpha(0,1)$

$$\begin{cases} u_t - \left(x^\alpha u_x\right)_x = 0 & in\ Q_T =]0,1[\times]0,T[\\ u(0,t) = g(t) \\ u(1,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Proposition

For all $u_T \in L^2(0,1)\,,\ \varepsilon > 0$ there exists $g \in H^1_0(0,T)$ such that

$$\|u(\cdot,T) - u_T\|_{L^2(Q_T)} < \varepsilon$$

approximate controllability for all $u_0 \in L^2(0,1)$ follows

approximate controllability

$0 < \alpha < 1\,,\; T > 0\,,\; u_0 \in H^1_\alpha(0,1)$

$$\begin{cases} u_t - \left(x^\alpha u_x\right)_x = 0 & in\ Q_T =]0,1[\times]0,T[\\ u(0,t) = g(t) \\ u(1,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Proposition

For all $u_T \in L^2(0,1)\,,\; \varepsilon > 0$ there exists $g \in H^1_0(0,T)$ such that

$$\|u(\cdot,T) - u_T\|_{L^2(Q_T)} < \varepsilon$$

approximate controllability for all $u_0 \in L^2(0,1)$ follows

approximate controllability

$0 < \alpha < 1\,,\ T > 0\,,\ u_0 \in H^1_\alpha(0,1)$

$$\begin{cases} u_t - \left(x^\alpha u_x\right)_x = 0 \quad in\ Q_T =]0,1[\times]0,T[\\ u(0,t) = g(t) \\ u(1,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Proposition

For all $u_T \in L^2(0,1)\,,\ \varepsilon > 0$ *there exists* $g \in H^1_0(0,T)$ *such that*

$$\|u(\cdot,T) - u_T\|_{L^2(Q_T)} < \varepsilon$$

approximate controllability for all $u_0 \in L^2(0,1)$ follows

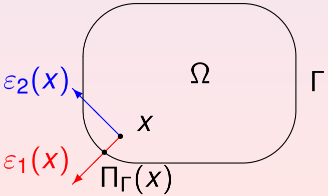
Outline

the simplest problem in 2d

$$n = 2 \quad \begin{cases} u_t - \operatorname{div}(A(x)\nabla u) = \chi_\omega(x) f(t,x) & \text{in} \quad Q_T \\ u(x,0) = u_0(x) & x \in \Omega \\ + \text{ b. c. on } \quad \Gamma & \end{cases}$$

$$\sigma(A(x)) = \{\lambda_1(x), \lambda_2(x)\} \quad \text{eigenvectors} \quad \varepsilon_1(x), \varepsilon_2(x)$$

$$\begin{cases} \lambda_1(x) \sim d_\Gamma(x)^\alpha, \quad \varepsilon_1(x) \sim -Dd_\Gamma(x) = \nu_\Gamma(\Pi_\Gamma(x)) & \text{near} \quad \Gamma \\ \lambda_2(x) \geq m > 0 & \forall x \in \Omega \end{cases}$$



null controllability in 2d

Theorem (C, Martinez, Vancostenoble – CRAS 2009)

Let $0 \leq \alpha < 2$ *Then*

$$\forall u_0 \in L^2(\Omega)\ \exists f \in L^2(Q_T)\ :\ \begin{cases} u^f(\cdot, T) \equiv 0 \\ \int_{Q_T} |f|^2 \leq C_T \int_\Omega |u_0|^2 \end{cases}$$

- $\alpha \geq 2$ null-controllability fails
  - radial case reduces to one-dimensional problem
- proof uses
  - topological lemma to construct adapted weight
  - Carleman's estimate to provide observability inequality
  - Hardy's inequality to control degenerate terms

null controllability in 2d

Theorem (C, Martinez, Vancostenoble – CRAS 2009)

Let $0 \leq \alpha < 2$ *Then*

$$\forall u_0 \in L^2(\Omega)\ \exists f \in L^2(Q_T)\ :\ \begin{cases} u^f(\cdot, T) \equiv 0 \\ \int_{Q_T} |f|^2 \leq C_T \int_\Omega |u_0|^2 \end{cases}$$

- $\alpha \geq 2$ null-controllability fails
  - radial case reduces to one-dimensional problem
- proof uses
  - topological lemma to construct adapted weight
  - Carleman's estimate to provide observability inequality
  - Hardy's inequality to control degenerate terms

null controllability in 2d

Theorem (C, Martinez, Vancostenoble – CRAS 2009)

Let $0 \leq \alpha < 2$ *Then*

$$\forall u_0 \in L^2(\Omega)\ \exists f \in L^2(Q_T)\ :\ \begin{cases} u^f(\cdot, T) \equiv 0 \\ \int_{Q_T} |f|^2 \leq C_T \int_\Omega |u_0|^2 \end{cases}$$

- $\alpha \geq 2$ null-controllability fails
  - radial case reduces to one-dimensional problem
- proof uses
  - topological lemma to construct adapted weight
  - Carleman's estimate to provide observability inequality
  - Hardy's inequality to control degenerate terms

topological lemma

- $\Omega \subset \mathbb{R}^n$ boundary of class C^4
- $\omega \subset\subset \Omega$ open
- $\alpha \in [0, 2)$

Then there exist $\delta > 0$ and $\psi \in C^4(\Omega) \cap C(\overline{\Omega})$ such that

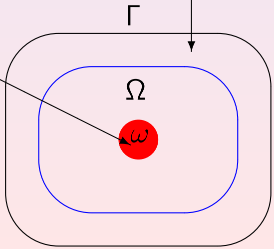
- $\forall x \in \Omega$ with $d_\Gamma(x) < \delta$ $\quad \psi(x) = \frac{1}{2-\alpha} d_\Gamma(x)^{2-\alpha}$
- $\{x \in \Omega \mid \nabla\psi(x) = 0\} \subset \omega$

topological lemma

- $\Omega \subset \mathbb{R}^n$ boundary of class C^4
- $\omega \subset\subset \Omega$ open
- $\alpha \in [0, 2)$

Then there exist $\delta > 0$ and $\psi \in C^4(\Omega) \cap C(\overline{\Omega})$ such that

- $\forall x \in \Omega$ with $d_\Gamma(x) < \delta$ $\boxed{\psi(x) = \frac{1}{2-\alpha} d_\Gamma(x)^{2-\alpha}}$
- $\boxed{\{x \in \Omega \mid \nabla\psi(x) = 0\} \subset \omega}$



Carleman's estimate for adjoint problem

- consider adjoint problem

$$\begin{cases} v_t + \operatorname{div}(A(x)\nabla v) = 0 & \text{in} \quad Q_T \\ v(t,\cdot) = 0 & \text{on} \quad \Gamma \quad (\alpha < 1) \end{cases}$$

- $\boxed{\varphi(t,x) = \theta(t)[e^{\tau\psi(x)} - e^{2\tau\|\psi\|_\infty}]}$ $\begin{cases} \psi & \text{by lemma} \\ \theta(t) = \left(\frac{1}{t(T-t)}\right)^4 \end{cases}$

$$\iint_{Q_T} (A\nabla\psi\cdot\nabla\psi)^2 v^2 e^{2\tau\varphi} + \int_0^T\int_{\{d_\Gamma<\delta\}} d_\Gamma^{2-\alpha} v^2 e^{2\tau\varphi} \le C\int_0^T\int_\omega v^2$$

$$\iint_{Q_T} (A\nabla v\cdot\nabla v) e^{2\tau\varphi} + \int_0^T\int_{\{d_\Gamma<\delta\}} d_\Gamma^{\alpha} (\nabla v\cdot\varepsilon_1)^2 e^{2\tau\varphi} \le C\int_0^T\int_\omega v^2$$

Carleman's estimate for adjoint problem

- consider adjoint problem

$$\begin{cases} v_t + \operatorname{div}(A(x)\nabla v) = 0 & \text{in} \quad Q_T \\ v(t,\cdot) = 0 & \text{on} \quad \Gamma \quad (\alpha < 1) \end{cases}$$

- $\varphi(t,x) = \theta(t)[e^{\tau\psi(x)} - e^{2\tau\|\psi\|_\infty}]$ $\quad \begin{cases} \psi & \text{by lemma} \\ \theta(t) = \left(\frac{1}{t(T-t)}\right)^4 \end{cases}$

$$\iint_{Q_T} (A\nabla\psi\cdot\nabla\psi)^2 v^2 e^{2\tau\varphi} + \int_0^T\int_{\{d_\Gamma<\delta\}} d_\Gamma^{2-\alpha} v^2 e^{2\tau\varphi} \leq C \int_0^T\int_\omega v^2$$

$$\iint_{Q_T} (A\nabla v\cdot\nabla v) e^{2\tau\varphi} + \int_0^T\int_{\{d_\Gamma<\delta\}} d_\Gamma^{\alpha} (\nabla v\cdot\varepsilon_1)^2 e^{2\tau\varphi} \leq C \int_0^T\int_\omega v^2$$

Outline

parabolic operators with interior degeneracy

$\Omega \subset \mathbb{R}^n$ bounded

$$\begin{cases} u_t - Lu = f & \text{in} \quad \Omega\times]0,T[\\ u(x,0) = u_0(x) & x \in \Omega \\ + \text{ b. c.} & \text{on} \quad \partial\Omega\times]0,T[\end{cases}$$

where

$$Lu = \begin{cases} \operatorname{div}(A(x)\nabla u) + \text{lower order tms} \\ \quad \text{or} \\ \operatorname{Tr}\,[A(x)\nabla^2 u] + \text{lower order tms} \end{cases}$$

with

$$A(x) \geq 0 \quad \text{in} \quad \Omega$$

references

- Crocco-type equation

$$u_t + u_x - \left(a(y)u_y\right)_y = \chi_\omega(x,y)f(x,y,t)$$

  - Martinez – Raymond – Vancostenoble (2003) ($a \equiv 1$)
  - C – Martinez – Vancostenoble (2005, 2008) ($a = (1-y)^\alpha$)
- Kolmogorov-type equation

$$u_t + yu_x - u_{yy} = \chi_\omega(x,y)f(x,y,t)$$

Beauchard – Zuazua (2009)

- Grushin-type equation

$$u_t - u_{xx} - |x|^{2\gamma}u_{yy} = \chi_\omega(x,y)f(x,y,t)$$

Beauchard – C – Guglielmi (2011)

references

- Crocco-type equation

$$u_t + u_x - \big(a(y)u_y\big)_y = \chi_\omega(x,y)f(x,y,t)$$

  - Martinez – Raymond – Vancostenoble (2003) ($a \equiv 1$)
  - C – Martinez – Vancostenoble (2005, 2008) ($a = (1-y)^\alpha$)

- Kolmogorov-type equation

$$u_t + yu_x - u_{yy} = \chi_\omega(x,y)f(x,y,t)$$

  Beauchard – Zuazua (2009)

- Grushin-type equation

$$u_t - u_{xx} - |x|^{2\gamma}u_{yy} = \chi_\omega(x,y)f(x,y,t)$$

  Beauchard – C – Guglielmi (2011)

references

- Crocco-type equation

$$u_t + u_x - \big(a(y)u_y\big)_y = \chi_\omega(x,y)f(x,y,t)$$

  - Martinez – Raymond – Vancostenoble (2003) ($a \equiv 1$)
  - C – Martinez – Vancostenoble (2005, 2008) ($a = (1-y)^\alpha$)
- Kolmogorov-type equation

$$u_t + yu_x - u_{yy} = \chi_\omega(x,y)f(x,y,t)$$

Beauchard – Zuazua (2009)

- Grushin-type equation

$$u_t - u_{xx} - |x|^{2\gamma}u_{yy} = \chi_\omega(x,y)f(x,y,t)$$

Beauchard – C – Guglielmi (2011)

references

- Crocco-type equation

$$u_t + u_x - \big(a(y)u_y\big)_y = \chi_\omega(x,y)f(x,y,t)$$

  - Martinez – Raymond – Vancostenoble (2003) ($a \equiv 1$)
  - C – Martinez – Vancostenoble (2005, 2008) ($a = (1-y)^\alpha$)

- Kolmogorov-type equation

$$u_t + yu_x - u_{yy} = \chi_\omega(x,y)f(x,y,t)$$

  Beauchard – Zuazua (2009)

- Grushin-type equation

$$u_t - u_{xx} - |x|^{2\gamma}u_{yy} = \chi_\omega(x,y)f(x,y,t)$$

  Beauchard – C – Guglielmi (2011)

Outline

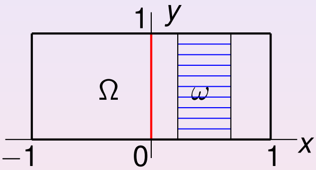
a Grushin-type operator

$$\Omega := (-1,1)\times(0,1), \quad \omega = (a,b)\times(0,1) \quad \text{with } 0<a<b<1$$

$$\gamma > 0 \qquad \begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = \chi_\omega(x,y) f(x,y,t) \\ u(\pm 1, y, t) = 0\,,\ u(x,0,t) = 0 = u(x,1,t) \\ u(x,y,0) = u_0(x,y) \end{cases} \qquad (G)$$

a Grushin-type operator

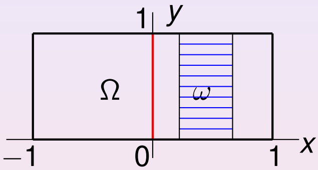
$$\Omega := (-1,1)\times(0,1), \quad \omega = (a,b)\times(0,1) \quad \text{with } 0 < a < b < 1$$



$$\gamma > 0 \qquad \begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = \chi_\omega(x,y) f(x,y,t) \\ u(\pm 1, y, t) = 0\,,\; u(x,0,t) = 0 = u(x,1,t) \\ u(x,y,0) = u_0(x,y) \end{cases} \qquad (G)$$

a Grushin-type operator

$$\Omega := (-1,1)\times(0,1), \quad \omega = (a,b)\times(0,1) \quad \text{with } 0<a<b<1$$



$$\gamma > 0 \qquad \begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = \chi_\omega(x,y) f(x,y,t) \\ u(\pm 1, y, t) = 0\,,\ u(x,0,t) = 0 = u(x,1,t) \\ u(x,y,0) = u_0(x,y) \end{cases} \qquad (G)$$

remarks

- Existence and uniqueness

$$T > 0\,,\ u_0 \in L^2(\Omega)\,,\ f \in L^2(\Omega \times (0,T))$$

$$\Rightarrow \exists! u \in C([0,T];L^2(\Omega)) \text{ such that } \forall t \in (0,T)\,,\ \phi \in C^2(\Omega \times [0,T])$$

$$\int_\Omega [u(t)\phi(t) - u(0)\phi(0)] = \int_0^t \int_\Omega u(\partial_t \phi + \partial_x^2 \phi + |x|^{2\gamma} \partial_y^2 \phi) + \chi_\omega f \phi$$

- Unique continuation Garofalo – Vassilev (2005)
  ($\Rightarrow$ approximate controllability $\forall \gamma > 0$)
- Hypoellipticity
  $A = \partial_x^2 + |x|^{2\gamma} \partial_y^2$ satisfies Hörmander's condition $\forall \gamma \in \mathbb{N}$:

$$X_1(x,y) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2(x,y) := \begin{pmatrix} 0 \\ x^\gamma \end{pmatrix}.$$

then

$$[X_1, X_2](x,y) = \begin{pmatrix} 0 \\ \gamma x^{\gamma-1} \end{pmatrix}, \ [X_1,[X_1,X_2]](x,y) = \begin{pmatrix} 0 \\ \gamma(\gamma-1)x^{\gamma-2} \end{pmatrix}, \dots$$

remarks

- Existence and uniqueness

$$T > 0\,,\ u_0 \in L^2(\Omega)\,,\ f \in L^2(\Omega \times (0,T))$$

$\Rightarrow\ \exists! u \in C([0,T]; L^2(\Omega))$ such that $\forall t \in (0,T)\,,\ \phi \in C^2(\Omega \times [0,T])$

$$\int_\Omega [u(t)\phi(t) - u(0)\phi(0)] = \int_0^t \int_\Omega u(\partial_t \phi + \partial_x^2 \phi + |x|^{2\gamma} \partial_y^2 \phi) + \chi_\omega f \phi$$

- Unique continuation   Garofalo – Vassilev (2005)
  ($\Rightarrow$ approximate controllability $\forall \gamma > 0$)
- Hypoellipticity
  $A = \partial_x^2 + |x|^{2\gamma}\partial_y^2$ satisfies Hörmander's condition $\forall \gamma \in \mathbb{N}$:

$$X_1(x,y) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \quad X_2(x,y) := \begin{pmatrix} 0 \\ x^\gamma \end{pmatrix}\,.$$

then

$$[X_1, X_2](x,y) = \begin{pmatrix} 0 \\ \gamma x^{\gamma-1} \end{pmatrix}\,, [X_1,[X_1,X_2]](x,y) = \begin{pmatrix} 0 \\ \gamma(\gamma-1)x^{\gamma-2} \end{pmatrix}\,, \dots$$

remarks

- Existence and uniqueness

$$T > 0\,,\;\; u_0 \in L^2(\Omega)\,,\;\; f \in L^2(\Omega \times (0,T))$$

$$\Rightarrow \exists! u \in C([0,T];L^2(\Omega)) \text{ such that } \forall t \in (0,T)\,,\;\; \phi \in C^2(\Omega \times [0,T])$$

$$\int_\Omega [u(t)\phi(t) - u(0)\phi(0)] = \int_0^t \int_\Omega u(\partial_t \phi + \partial_x^2 \phi + |x|^{2\gamma} \partial_y^2 \phi) + \chi_\omega f \phi$$

- Unique continuation   Garofalo – Vassilev (2005)
  ($\Rightarrow$ approximate controllability $\forall \gamma > 0$)

- Hypoellipticity
  $A = \partial_x^2 + |x|^{2\gamma}\partial_y^2$ satisfies Hörmander's condition $\forall \gamma \in \mathbb{N}$:

$$X_1(x,y) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2(x,y) := \begin{pmatrix} 0 \\ x^\gamma \end{pmatrix}.$$

then

$$[X_1, X_2](x,y) = \begin{pmatrix} 0 \\ \gamma x^{\gamma-1} \end{pmatrix}, \; [X_1,[X_1,X_2]](x,y) = \begin{pmatrix} 0 \\ \gamma(\gamma-1)x^{\gamma-2} \end{pmatrix},$$

remarks

- Existence and uniqueness

$$T > 0\,,\ u_0 \in L^2(\Omega)\,,\ f \in L^2(\Omega \times (0,T))$$
$$\Rightarrow \exists! u \in C([0,T]; L^2(\Omega)) \text{ such that } \forall t \in (0,T)\,,\ \phi \in C^2(\Omega \times [0,T])$$
$$\int_\Omega [u(t)\phi(t) - u(0)\phi(0)] = \int_0^t \int_\Omega u(\partial_t \phi + \partial_x^2 \phi + |x|^{2\gamma} \partial_y^2 \phi) + \chi_\omega f \phi$$

- Unique continuation   Garofalo – Vassilev (2005)
  ($\Rightarrow$ approximate controllability $\forall \gamma > 0$)
- Hypoellipticity
  $A = \partial_x^2 + |x|^{2\gamma}\partial_y^2$ satisfies Hörmander's condition $\forall \gamma \in \mathbb{N}$:

$$X_1(x,y) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2(x,y) := \begin{pmatrix} 0 \\ x^\gamma \end{pmatrix}.$$

then

$$[X_1, X_2](x,y) = \begin{pmatrix} 0 \\ \gamma x^{\gamma-1} \end{pmatrix}, [X_1,[X_1,X_2]](x,y) = \begin{pmatrix} 0 \\ \gamma(\gamma-1)x^{\gamma-2} \end{pmatrix}, \ldots$$

null controllability

Theorem (Beauchard – C – Guglielmi, 2011)

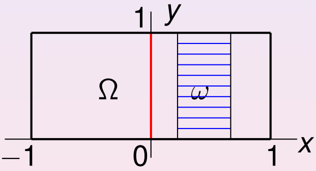
$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = \chi_\omega(x,y) f(x,y,t) \\ u(\pm 1, y, t) = 0\,,\ u(x,0,t) = 0 = u(x,1,t) \\ u(x,y,0) = u_0(x,y) \end{cases} \qquad (G)$$

- $0 < \gamma < 1 \quad \Rightarrow \quad (G)$ *null controllable* $\forall T > 0$
- $\gamma > 1 \quad \Rightarrow \quad (G)$ *not null controllable*
- $\gamma = 1 \quad \Rightarrow \quad \exists T^* > 0$ *such that* (G) *is*
  - *null controllable* $\forall T > T^*$
  - *not null controllable* $\forall T < T^*$

null controllability

Theorem (Beauchard – C – Guglielmi, 2011)

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = \chi_\omega(x,y) f(x,y,t) \\ u(\pm 1, y, t) = 0\,,\ u(x,0,t) = 0 = u(x,1,t) \\ u(x,y,0) = u_0(x,y) \end{cases} \tag{G}$$

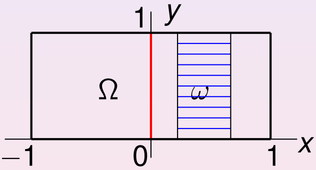


- $0 < \gamma < 1 \quad \Rightarrow \quad (G)$ *null controllable* $\forall T > 0$
- $\gamma > 1 \quad \Rightarrow \quad (G)$ *not null controllable*
- $\gamma = 1 \quad \Rightarrow \quad \exists T^* > 0$ *such that* (G) *is*
  - *null controllable* $\forall T > T^*$
  - *not null controllable* $\forall T < T^*$

null controllability

Theorem (Beauchard – C – Guglielmi, 2011)

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma} \partial_y^2 u = \chi_\omega(x,y) f(x,y,t) \\ u(\pm 1, y, t) = 0\,, \ u(x,0,t) = 0 = u(x,1,t) \\ u(x,y,0) = u_0(x,y) \end{cases} \qquad (G)$$

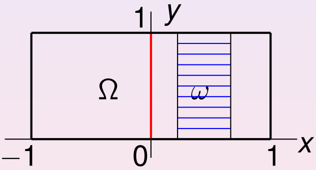


- $0 < \gamma < 1 \quad \Rightarrow \quad (G)$ *null controllable* $\quad \forall T > 0$
- $\gamma > 1 \quad \Rightarrow \quad (G)$ *not null controllable*
- $\gamma = 1 \quad \Rightarrow \quad \exists T^* > 0$ *such that* (G) *is*
  - *null controllable* $\quad \forall T > T^*$
  - *not null controllable* $\quad \forall T < T^*$

null controllability

Theorem (Beauchard – C – Guglielmi, 2011)

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = \chi_\omega(x,y) f(x,y,t) \\ u(\pm 1, y, t) = 0\,, \; u(x,0,t) = 0 = u(x,1,t) \\ u(x,y,0) = u_0(x,y) \end{cases} \qquad (G)$$

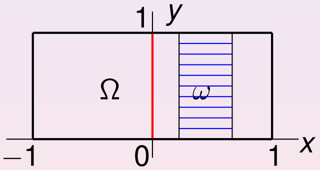


- $0 < \gamma < 1 \quad \Rightarrow \quad (G)$ *null controllable* $\forall T > 0$
- $\gamma > 1 \quad \Rightarrow \quad (G)$ *not null controllable*
- $\gamma = 1 \quad \Rightarrow \quad \exists T^* > 0$ *such that* (G) *is*
  - *null controllable* $\forall T > T^*$
  - *not null controllable* $\forall T < T^*$

null controllability

Theorem (Beauchard – C – Guglielmi, 2011)

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = \chi_\omega(x,y) f(x,y,t) \\ u(\pm 1, y, t) = 0\,,\ u(x,0,t) = 0 = u(x,1,t) \\ u(x,y,0) = u_0(x,y) \end{cases} \qquad (G)$$



- $0 < \gamma < 1 \quad \Rightarrow \quad (G)$ *null controllable* $\forall T > 0$
- $\gamma > 1 \quad \Rightarrow \quad (G)$ *not null controllable*
- $\gamma = 1 \quad \Rightarrow \quad \exists T^* > 0$ *such that* (G) *is*
  - *null controllable* $\forall\, T > T^*$
  - *not null controllable* $\forall\, T < T^*$

observability

$$\begin{cases} \partial_t v - \partial_x^2 v - |x|^{2\gamma}\partial_y^2 v = 0 \\ v(\pm 1, y, t) = 0\,,\ v(x,0,t) = 0 = v(x,1,t) \\ v(x,y,0) = v_0(x,y) \end{cases} \qquad (G^*)$$

observable in ω in time T $\quad \exists C > 0$ such that $\forall v_0 \in L^2(\Omega)$

$$\int_\Omega |v(x,y,T)|^2 dxdy \le C \int_0^T \int_\omega |v(x,y,t)|^2 dxdy \qquad (O)$$

Theorem (Beauchard – C – Guglielmi, 2011)

- $0 < \gamma < 1 \quad \Rightarrow \quad (G^*) \quad$ *observable* $\quad \forall T > 0$
- $\gamma > 1 \quad \Rightarrow \quad (G^*) \quad$ *not observable*
- $\gamma = 1 \quad \Rightarrow \quad \exists T^* > 0 \quad$ *such that* $\quad (G^*) \quad$ *is*
  - *observable* $\quad \forall\, T > T^*$
  - *not observable* $\quad \forall\, T < T^*$

observability

$$\begin{cases} \partial_t v - \partial_x^2 v - |x|^{2\gamma} \partial_y^2 v = 0 \\ v(\pm 1, y, t) = 0\,, \;\; v(x,0,t) = 0 = v(x,1,t) \\ v(x,y,0) = v_0(x,y) \end{cases} \qquad (G^*)$$

observable in ω in time T $\quad \exists C > 0$ such that $\forall v_0 \in L^2(\Omega)$

$$\int_\Omega |v(x,y,T)|^2 dxdy \le C \int_0^T \int_\omega |v(x,y,t)|^2 dxdy \qquad (O)$$

Theorem (Beauchard – C – Guglielmi, 2011)

- $0 < \gamma < 1 \quad \Rightarrow \quad (G^*) \quad$ *observable* $\quad \forall T > 0$
- $\gamma > 1 \quad \Rightarrow \quad (G^*) \quad$ *not observable*
- $\gamma = 1 \quad \Rightarrow \quad \exists T^* > 0 \quad$ *such that* $\quad (G^*) \quad$ *is*
  - *observable* $\quad \forall T > T^*$
  - *not observable* $\quad \forall T < T^*$

observability

$$\begin{cases} \partial_t v - \partial_x^2 v - |x|^{2\gamma}\partial_y^2 v = 0 \\ v(\pm 1, y, t) = 0\,,\ \ v(x,0,t) = 0 = v(x,1,t) \\ v(x,y,0) = v_0(x,y) \end{cases} \qquad (G^*)$$

observable in ω in time T $\quad \exists C > 0$ such that $\forall v_0 \in L^2(\Omega)$

$$\int_\Omega |v(x,y,T)|^2 dxdy \le C \int_0^T \int_\omega |v(x,y,t)|^2 dxdy \qquad (O)$$

Theorem (Beauchard – C – Guglielmi, 2011)

- $0 < \gamma < 1 \quad \Rightarrow \quad (G^*)$ *observable* $\forall T > 0$
- $\gamma > 1 \quad \Rightarrow \quad (G^*)$ *not observable*
- $\gamma = 1 \quad \Rightarrow \quad \exists T^* > 0$ *such that* (G^*) *is*
  - *observable* $\forall\, T > T^*$
  - *not observable* $\forall\, T < T^*$

Fourier decomposition

$$\begin{cases} \partial_t v - \partial_x^2 v - |x|^{2\gamma}\partial_y^2 v = 0 \\ v(\pm 1, y, t) = 0\,,\ \ v(x,0,t) = 0 = v(x,1,t) \\ v(x,y,0) = v_0(x,y) \end{cases} \qquad (G^*)$$

- $v(x,y,t) = \sum_{n=1}^{\infty} v_n(x,t)\varphi_n(y)$ with $\varphi_n(y) := \sqrt{2}\sin(n\pi y)$

  where $v_n(x,t) := \int_0^1 v(x,y,t)\varphi_n(y)dy$ satisfies

$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2|x|^{2\gamma} v_n = 0 & (x,t) \in (-1,1)\times(0,T) \\ v_n(\pm 1, t) = 0 & t \in (0,T) \\ v_n(x,0) = v_{0,n}(x) & x \in (-1,1) \end{cases} \qquad (G_n^*)$$

- $\int_\Omega |v(x,y,T)|^2 dxdy = \sum_{n=1}^{\infty} \int_{-1}^{1} |v_n(x,T)|^2 dx$

  $\int_{\omega=(a,b)\times(0,1)} |v(x,y,t)|^2 dxdy = \sum_{n=1}^{\infty} \int_a^b |v_n(x,t)|^2 dx$

Fourier decomposition

$$\begin{cases} \partial_t v - \partial_x^2 v - |x|^{2\gamma}\partial_y^2 v = 0 \\ v(\pm 1, y, t) = 0\,,\ v(x,0,t) = 0 = v(x,1,t) \\ v(x,y,0) = v_0(x,y) \end{cases} \qquad (G^*)$$

- $v(x,y,t) = \sum_{n=1}^{\infty} v_n(x,t)\varphi_n(y)$ with $\varphi_n(y) := \sqrt{2}\sin(n\pi y)$

  where $v_n(x,t) := \int_0^1 v(x,y,t)\varphi_n(y)dy$ satisfies

$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2|x|^{2\gamma} v_n = 0 & (x,t)\in(-1,1)\times(0,T) \\ v_n(\pm 1, t) = 0 & t \in (0,T) \\ v_n(x,0) = v_{0,n}(x) & x \in (-1,1) \end{cases} \qquad (G_n^*)$$

- $\int_\Omega |v(x,y,T)|^2 dxdy = \sum_{n=1}^{\infty}\int_{-1}^{1}|v_n(x,T)|^2 dx$

  $\int_{\omega=(a,b)\times(0,1)} |v(x,y,t)|^2 dxdy = \sum_{n=1}^{\infty}\int_a^b |v_n(x,t)|^2 dx$

Fourier decomposition

$$\begin{cases} \partial_t v - \partial_x^2 v - |x|^{2\gamma}\partial_y^2 v = 0 \\ v(\pm 1, y, t) = 0\,, \ v(x,0,t) = 0 = v(x,1,t) \\ v(x,y,0) = v_0(x,y) \end{cases} \qquad (G^*)$$

- $v(x,y,t) = \sum_{n=1}^{\infty} v_n(x,t)\varphi_n(y)$ with $\varphi_n(y) := \sqrt{2}\sin(n\pi y)$

  where $v_n(x,t) := \int_0^1 v(x,y,t)\varphi_n(y)dy$ satisfies

$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2|x|^{2\gamma} v_n = 0 & (x,t) \in (-1,1)\times(0,T) \\ v_n(\pm 1, t) = 0 & t \in (0,T) \\ v_n(x,0) = v_{0,n}(x) & x \in (-1,1) \end{cases} \qquad (G_n^*)$$

- $\int_\Omega |v(x,y,T)|^2 dxdy = \sum_{n=1}^{\infty} \int_{-1}^{1} |v_n(x,T)|^2 dx$

  $\int_{\omega=(a,b)\times(0,1)} |v(x,y,t)|^2 dxdy = \sum_{n=1}^{\infty} \int_a^b |v_n(x,t)|^2 dx$

Fourier decomposition

$$\begin{cases} \partial_t v - \partial_x^2 v - |x|^{2\gamma}\partial_y^2 v = 0 \\ v(\pm 1, y, t) = 0\,, \;\; v(x,0,t) = 0 = v(x,1,t) \\ v(x,y,0) = v_0(x,y) \end{cases} \qquad (G^*)$$

- $v(x,y,t) = \sum_{n=1}^{\infty} v_n(x,t)\varphi_n(y)$ with $\varphi_n(y) := \sqrt{2}\sin(n\pi y)$

  where $v_n(x,t) := \int_0^1 v(x,y,t)\varphi_n(y)dy$ satisfies

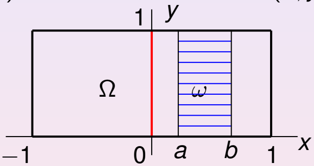
$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2|x|^{2\gamma} v_n = 0 & (x,t) \in (-1,1)\times(0,T) \\ v_n(\pm 1, t) = 0 & t \in (0,T) \\ v_n(x,0) = v_{0,n}(x) & x \in (-1,1) \end{cases} \qquad (G_n^*)$$

- $$\int_\Omega |v(x,y,T)|^2 dxdy = \sum_{n=1}^{\infty}\int_{-1}^{1} |v_n(x,T)|^2 dx$$

  $$\int_{\omega=(a,b)\times(0,1)} |v(x,y,t)|^2 dxdy = \sum_{n=1}^{\infty}\int_a^b |v_n(x,t)|^2 dx$$

uniform observability

$$\begin{cases} \partial_t v - \partial_x^2 v - |x|^{2\gamma}\partial_y^2 v = 0 & \Omega \times (0,T) \\ v(\pm 1, y, t) = 0\,,\ v(x,0,t) = 0 = v(x,1,t) & t \in (0,T) \\ v(x,y,0) = v_0(x,y) & (x,y) \in \Omega \end{cases} \qquad (G^*)$$



$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2 |x|^{2\gamma} v_n = 0 & (x,t) \in (-1,1) \times (0,T) \\ v_n(\pm 1, t) = 0 & t \in (0,T) \\ v_n(x,0) = v_{0,n}(x) & x \in (-1,1) \end{cases} \qquad (G_n^*)$$

observability for (G^*) in ω $\iff$ uniform observability for (G_n^*) in (a,b)

$$\int_{-1}^{1} |v_n(x,T)|^2 dx \le C \int_0^T \int_a^b |v_n(x,t)|^2 dxdt \qquad \forall n \ge 1 \qquad (UO)$$

for the proof of (UO): dissipation speed

- define $A_n : D(A_n) \subset L^2(-1,1) \to L^2(-1,1)$ by

$$D(A_n) := H^2 \cap H^1_0(-1,1)\,, \qquad A_n\varphi := -\varphi'' + (n\pi)^2|x|^{2\gamma}\varphi$$

- λ_n := the first eigenvalue of A_n so that

$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2|x|^{2\gamma} v_n = 0 & (x,t) \in (-1,1)\times(0,T) \\ v_n(\pm1,t) = 0 & t \in (0,T) \end{cases}$$

satisfies

$$\int_{-1}^{1} |v_n(x,T)|^2 dx \le e^{-\lambda_n(T-t)} \int_{-1}^{1} |v_n(x,t)|^2 dx \quad \forall t \in [0,T] \qquad (D_n)$$

Lemma (dissipation speed)

(ub) $\forall \gamma > 0 \quad \exists c^* > 0$ *such that* $\lambda_n \le c^* n^{\frac{2}{1+\gamma}}$

(lb) $\forall \gamma \in (0,1] \quad \exists c_* > 0$ *such that* $\lambda_n \ge c_* n^{\frac{2}{1+\gamma}}$

for the proof of (UO): dissipation speed

- define $A_n : D(A_n) \subset L^2(-1,1) \to L^2(-1,1)$ by

$$D(A_n) := H^2 \cap H_0^1(-1,1), \qquad A_n\varphi := -\varphi'' + (n\pi)^2|x|^{2\gamma}\varphi$$

- λ_n := the first eigenvalue of A_n so that

$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2|x|^{2\gamma} v_n = 0 & (x,t) \in (-1,1)\times(0,T) \\ v_n(\pm 1, t) = 0 & t \in (0,T) \end{cases}$$

satisfies

$$\int_{-1}^{1} |v_n(x,T)|^2 dx \le e^{-\lambda_n(T-t)} \int_{-1}^{1} |v_n(x,t)|^2 dx \quad \forall t \in [0,T] \qquad (D_n)$$

Lemma (dissipation speed)

(ub) $\forall \gamma > 0 \quad \exists c^* > 0$ *such that* $\lambda_n \le c^* n^{\frac{2}{1+\gamma}}$

(lb) $\forall \gamma \in (0,1] \quad \exists c_* > 0$ *such that* $\lambda_n \ge c_* n^{\frac{2}{1+\gamma}}$

for the proof of (UO): dissipation speed

- define $A_n : D(A_n) \subset L^2(-1,1) \to L^2(-1,1)$ by

$$D(A_n) := H^2 \cap H^1_0(-1,1)\,, \qquad A_n\varphi := -\varphi'' + (n\pi)^2|x|^{2\gamma}\varphi$$

- λ_n := the first eigenvalue of A_n so that

$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2|x|^{2\gamma} v_n = 0 & (x,t) \in (-1,1)\times(0,T) \\ v_n(\pm 1, t) = 0 & t \in (0,T) \end{cases}$$

satisfies

$$\int_{-1}^{1} |v_n(x,T)|^2 dx \le e^{-\lambda_n (T-t)} \int_{-1}^{1} |v_n(x,t)|^2 dx \quad \forall t \in [0,T] \qquad (D_n)$$

Lemma (dissipation speed)

(ub) $\forall \gamma > 0 \quad \exists c^* > 0$ *such that* $\lambda_n \le c^* n^{\frac{2}{1+\gamma}}$

(lb) $\forall \gamma \in (0,1] \quad \exists c_* > 0$ *such that* $\lambda_n \ge c_* n^{\frac{2}{1+\gamma}}$

for the proof of (UO): dissipation speed

- define $A_n : D(A_n) \subset L^2(-1,1) \to L^2(-1,1)$ by
$$D(A_n) := H^2 \cap H^1_0(-1,1)\,, \qquad A_n\varphi := -\varphi'' + (n\pi)^2|x|^{2\gamma}\varphi$$

- λ_n := the first eigenvalue of A_n so that
$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2|x|^{2\gamma} v_n = 0 & (x,t) \in (-1,1)\times(0,T) \\ v_n(\pm 1, t) = 0 & t \in (0,T) \end{cases}$$
satisfies
$$\int_{-1}^{1} |v_n(x,T)|^2 dx \le e^{-\lambda_n(T-t)} \int_{-1}^{1} |v_n(x,t)|^2 dx \quad \forall t \in [0,T] \qquad (D_n)$$

Lemma (dissipation speed)

(ub) $\forall \gamma > 0 \quad \exists c^* > 0$ *such that* $\lambda_n \le c^* n^{\frac{2}{1+\gamma}}$

(lb) $\forall \gamma \in (0,1] \quad \exists c_* > 0$ *such that* $\lambda_n \ge c_* n^{\frac{2}{1+\gamma}}$

proof of (UO): $0<\gamma<1$ and $\gamma=1$

- Carleman estimate: $\exists C>0$ such that $\forall T>0$ $\exists n_T\geq 1$ with

$$\int_{\frac{T}{3}}^{\frac{2T}{3}}\int_{-1}^{1}|v_n(x,t)|^2dxdt\leq e^{Cn}\int_{\frac{T}{3}}^{\frac{2T}{3}}\int_{a}^{b}|v_n(x,t)|^2dxdt \qquad \forall n\geq n_T \quad (C_n)$$

- combine (D_n), (C_n), and Dissipation Lemma to obtain

$$\begin{aligned}\frac{T}{3}\int_{-1}^{1}|v_n(x,T)|^2dx &\leq e^{-\lambda_n\frac{T}{3}}\int_{\frac{T}{3}}^{\frac{2T}{3}}\int_{-1}^{1}|v_n(x,t)|^2dxdt\\ &\leq e^{Cn-c_*\frac{T}{3}n^{\frac{2}{1+\gamma}}}\int_{\frac{T}{3}}^{\frac{2T}{3}}\int_{a}^{b}|v_n(x,t)|^2dxdt\end{aligned}$$

- conclude
  - $0<\gamma<1\Rightarrow Cn-c_*\frac{T}{3}n^{\frac{2}{1+\gamma}}\to-\infty\Rightarrow(UO)\ \forall T>0$
  - $\gamma=1\Rightarrow Cn-c_*\frac{T}{3}n^{\frac{2}{1+\gamma}}\to-\infty$ for $T>3C/c_*\Rightarrow(UO)$

proof of (UO): $0 < \gamma < 1$ and $\gamma = 1$

- Carleman estimate: $\exists C > 0$ such that $\forall T > 0$ $\exists n_T \geq 1$ with

$$\int_{\frac{T}{3}}^{\frac{2T}{3}} \int_{-1}^{1} |v_n(x,t)|^2 dxdt \leq e^{Cn} \int_{\frac{T}{3}}^{\frac{2T}{3}} \int_{a}^{b} |v_n(x,t)|^2 dxdt \qquad \forall n \geq n_T \quad (C_n)$$

- combine (D_n), (C_n), and Dissipation Lemma to obtain

$$\begin{aligned} \frac{T}{3} \int_{-1}^{1} |v_n(x,T)|^2 dx \quad &\leq \quad e^{-\lambda_n \frac{T}{3}} \int_{\frac{T}{3}}^{\frac{2T}{3}} \int_{-1}^{1} |v_n(x,t)|^2 dxdt \\ &\leq \quad e^{Cn - c_* \frac{T}{3} n^{\frac{2}{1+\gamma}}} \int_{\frac{T}{3}}^{\frac{2T}{3}} \int_{a}^{b} |v_n(x,t)|^2 dxdt \end{aligned}$$

- conclude
  - $0 < \gamma < 1 \Rightarrow Cn - c_* \frac{T}{3} n^{\frac{2}{1+\gamma}} \to -\infty \Rightarrow (UO)\ \forall T > 0$
  - $\gamma = 1 \Rightarrow Cn - c_* \frac{T}{3} n^{\frac{2}{1+\gamma}} \to -\infty$ for $T > 3C/c_* \Rightarrow (UO)$

proof of (UO): $0 < \gamma < 1$ and $\gamma = 1$

- Carleman estimate: $\exists C > 0$ such that $\forall T > 0$ $\exists n_T \geq 1$ with

$$\int_{\frac{T}{3}}^{\frac{2T}{3}} \int_{-1}^{1} |v_n(x,t)|^2 dxdt \leq e^{Cn} \int_{\frac{T}{3}}^{\frac{2T}{3}} \int_{a}^{b} |v_n(x,t)|^2 dxdt \qquad \forall n \geq n_T \quad (C_n)$$

- combine (D_n), (C_n), and Dissipation Lemma to obtain

$$\begin{aligned} \frac{T}{3} \int_{-1}^{1} |v_n(x,T)|^2 dx \quad &\leq \quad e^{-\lambda_n \frac{T}{3}} \int_{\frac{T}{3}}^{\frac{2T}{3}} \int_{-1}^{1} |v_n(x,t)|^2 dxdt \\ &\leq \quad e^{Cn - c_* \frac{T}{3} n^{\frac{2}{1+\gamma}}} \int_{\frac{T}{3}}^{\frac{2T}{3}} \int_{a}^{b} |v_n(x,t)|^2 dxdt \end{aligned}$$

- conclude
  - $0 < \gamma < 1 \Rightarrow Cn - c_* \frac{T}{3} n^{\frac{2}{1+\gamma}} \to -\infty \Rightarrow (UO)\ \forall T > 0$
  - $\gamma = 1 \Rightarrow Cn - c_* \frac{T}{3} n^{\frac{2}{1+\gamma}} \to -\infty$ for $T > 3C/c_* \Rightarrow (UO)$

proof of (UO): $0<\gamma<1$ and $\gamma=1$

- Carleman estimate: $\exists C>0$ such that $\forall T>0$ $\exists n_T\geq 1$ with

$$\int_{\frac{T}{3}}^{\frac{2T}{3}}\int_{-1}^{1}|v_n(x,t)|^2dxdt \leq e^{Cn}\int_{\frac{T}{3}}^{\frac{2T}{3}}\int_{a}^{b}|v_n(x,t)|^2dxdt \qquad \forall n\geq n_T \quad (C_n)$$

- combine (D_n), (C_n), and Dissipation Lemma to obtain

$$\begin{aligned}\frac{T}{3}\int_{-1}^{1}|v_n(x,T)|^2dx &\leq e^{-\lambda_n\frac{T}{3}}\int_{\frac{T}{3}}^{\frac{2T}{3}}\int_{-1}^{1}|v_n(x,t)|^2dxdt\\ &\leq e^{Cn-c_*\frac{T}{3}n^{\frac{2}{1+\gamma}}}\int_{\frac{T}{3}}^{\frac{2T}{3}}\int_{a}^{b}|v_n(x,t)|^2dxdt\end{aligned}$$

- conclude
  - $0<\gamma<1 \Rightarrow Cn-c_*\frac{T}{3}n^{\frac{2}{1+\gamma}}\to-\infty \Rightarrow (UO)\ \forall T>0$
  - $\gamma=1 \Rightarrow Cn-c_*\frac{T}{3}n^{\frac{2}{1+\gamma}}\to-\infty$ for $T>3C/c_* \Rightarrow (UO)$

proof of (UO): $0 < \gamma < 1$ and $\gamma = 1$

- Carleman estimate: $\exists C > 0$ such that $\forall T > 0$ $\exists n_T \geq 1$ with

$$\int_{\frac{T}{3}}^{\frac{2T}{3}} \int_{-1}^{1} |v_n(x,t)|^2 dxdt \leq e^{Cn} \int_{\frac{T}{3}}^{\frac{2T}{3}} \int_{a}^{b} |v_n(x,t)|^2 dxdt \qquad \forall n \geq n_T \quad (C_n)$$

- combine (D_n), (C_n), and Dissipation Lemma to obtain

$$\begin{aligned} \frac{T}{3} \int_{-1}^{1} |v_n(x,T)|^2 dx \quad &\leq \quad e^{-\lambda_n \frac{T}{3}} \int_{\frac{T}{3}}^{\frac{2T}{3}} \int_{-1}^{1} |v_n(x,t)|^2 dxdt \\ &\leq \quad e^{Cn - c_* \frac{T}{3} n^{\frac{2}{1+\gamma}}} \int_{\frac{T}{3}}^{\frac{2T}{3}} \int_{a}^{b} |v_n(x,t)|^2 dxdt \end{aligned}$$

- conclude
  - $0 < \gamma < 1 \Rightarrow Cn - c_* \frac{T}{3} n^{\frac{2}{1+\gamma}} \to -\infty \Rightarrow (UO)\ \forall T > 0$
  - $\gamma = 1 \Rightarrow Cn - c_* \frac{T}{3} n^{\frac{2}{1+\gamma}} \to -\infty$ for $T > 3C/c_* \Rightarrow (UO)$

failure of (UO): $\gamma > 1$ and $\gamma = 1$

- take eigenfunctions w_n of A_n associated with λ_n

$$\begin{cases} -w_n''(x) + \left[(n\pi)^2|x|^{2\gamma} - \lambda_n\right] w_n(x) = 0 & x \in (-1,1) \\ w_n(\pm 1) = 0\,, \quad w_n \geq 0\,, \quad \|w_n\|_{L^2(-1,1)} = 1 \end{cases}$$

- $v_n(x,t) := e^{-\lambda_n t} w_n(x)$ solution to

$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2|x|^{2\gamma} v_n = 0 & (x,t) \in (-1,1) \times (0,T) \\ v_n(\pm 1, t) = 0 & t \in (0,T) \end{cases}$$

- use a comparison argument and Dissipation Lemma to show

$$\frac{\int_0^T\int_a^b |v_n(x,t)|^2 dxdt}{\int_{-1}^1 |v_n(x,T)|^2 dx} = \frac{e^{2\lambda_n T} - 1}{2\lambda_n} \int_a^b |w_n(x)|^2 dx \leq e^{2n\left(\frac{\lambda_n}{n} T - C_\gamma\right)} R(n) \to 0$$

technical because $(n\pi)^2|x|^{2\gamma} - \lambda_n$ changes sign in $[-1,1]$

failure of (UO): $\gamma > 1$ and $\gamma = 1$

- take eigenfunctions w_n of A_n associated with λ_n

$$\begin{cases} -w_n''(x) + \big[(n\pi)^2|x|^{2\gamma} - \lambda_n\big] w_n(x) = 0 & x \in (-1,1) \\ w_n(\pm 1) = 0\,, \quad w_n \geq 0\,, \quad \|w_n\|_{L^2(-1,1)} = 1 \end{cases}$$

- $v_n(x,t) := e^{-\lambda_n t} w_n(x)$ solution to

$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2 |x|^{2\gamma} v_n = 0 & (x,t) \in (-1,1)\times(0,T) \\ v_n(\pm 1, t) = 0 & t \in (0,T) \end{cases}$$

- use a comparison argument and Dissipation Lemma to show

$$\frac{\int_0^T\int_a^b |v_n(x,t)|^2 dxdt}{\int_{-1}^1 |v_n(x,T)|^2 dx} = \frac{e^{2\lambda_n T} - 1}{2\lambda_n} \int_a^b |w_n(x)|^2 dx \leq e^{2n\left(\frac{\lambda_n}{n} T - C_\gamma\right)} R(n) \to 0$$

technical because $(n\pi)^2|x|^{2\gamma} - \lambda_n$ changes sign in $[-1,1]$

failure of (UO): $\gamma > 1$ and $\gamma = 1$

- take eigenfunctions w_n of A_n associated with λ_n

$$\begin{cases} -w_n''(x) + \left[(n\pi)^2|x|^{2\gamma} - \lambda_n\right] w_n(x) = 0 & x \in (-1,1) \\ w_n(\pm 1) = 0\,, \quad w_n \geq 0\,, \quad \|w_n\|_{L^2(-1,1)} = 1 \end{cases}$$

- $v_n(x,t) := e^{-\lambda_n t} w_n(x)$ solution to

$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2|x|^{2\gamma} v_n = 0 & (x,t) \in (-1,1) \times (0,T) \\ v_n(\pm 1, t) = 0 & t \in (0,T) \end{cases}$$

- use a comparison argument and Dissipation Lemma to show

$$\frac{\int_0^T\int_a^b |v_n(x,t)|^2 dxdt}{\int_{-1}^1 |v_n(x,T)|^2 dx} = \frac{e^{2\lambda_n T} - 1}{2\lambda_n} \int_a^b |w_n(x)|^2 dx \leq e^{2n\left(\frac{\lambda_n}{n}T - C_\gamma\right)} R(n) \to 0$$

technical because $(n\pi)^2|x|^{2\gamma} - \lambda_n$ changes sign in $[-1,1]$

failure of (UO): $\gamma > 1$ and $\gamma = 1$

- take eigenfunctions w_n of A_n associated with λ_n

$$\begin{cases} -w_n''(x) + \left[(n\pi)^2|x|^{2\gamma} - \lambda_n\right] w_n(x) = 0 & x \in (-1,1) \\ w_n(\pm 1) = 0\,, \quad w_n \geq 0\,, \quad \|w_n\|_{L^2(-1,1)} = 1 \end{cases}$$

- $v_n(x,t) := e^{-\lambda_n t} w_n(x)$ solution to

$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2|x|^{2\gamma} v_n = 0 & (x,t) \in (-1,1) \times (0,T) \\ v_n(\pm 1, t) = 0 & t \in (0,T) \end{cases}$$

- use a comparison argument and Dissipation Lemma to show

$$\frac{\int_0^T\int_a^b |v_n(x,t)|^2 dxdt}{\int_{-1}^1 |v_n(x,T)|^2 dx} = \frac{e^{2\lambda_n T} - 1}{2\lambda_n} \int_a^b |w_n(x)|^2 dx \leq e^{2n\left(\frac{\lambda_n}{n} T - C_\gamma\right)} R(n) \to 0$$

technical because $(n\pi)^2|x|^{2\gamma} - \lambda_n$ changes sign in $[-1,1]$

failure of (UO): $\gamma > 1$ and $\gamma = 1$

- take eigenfunctions w_n of A_n associated with λ_n

$$\begin{cases} -w_n''(x) + \big[(n\pi)^2|x|^{2\gamma} - \lambda_n\big] w_n(x) = 0 & x \in (-1,1) \\ w_n(\pm 1) = 0\,, \quad w_n \geq 0\,, \quad \|w_n\|_{L^2(-1,1)} = 1 \end{cases}$$

- $v_n(x,t) := e^{-\lambda_n t} w_n(x)$ solution to

$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2|x|^{2\gamma} v_n = 0 & (x,t) \in (-1,1) \times (0,T) \\ v_n(\pm 1, t) = 0 & t \in (0,T) \end{cases}$$

- use a comparison argument and Dissipation Lemma to show

$$\frac{\int_0^T\int_a^b |v_n(x,t)|^2 dxdt}{\int_{-1}^1 |v_n(x,T)|^2 dx} = \frac{e^{2\lambda_n T} - 1}{2\lambda_n} \int_a^b |w_n(x)|^2 dx \leq e^{2n\left(\frac{\lambda_n}{n}T - C_\gamma\right)} R(n) \to 0$$

technical because $(n\pi)^2|x|^{2\gamma} - \lambda_n$ changes sign in $[-1,1]$

open problems

- boundary degeneracy
  - null controllability with boundary control for weakly degenerate operators
  - degenerate problems in 3 or more space dimensions
  - systems of DPO (Maniar et al.)
- interior degeneracy
  - ok for strong degeneracy in 1D with control on both sides of nondegenerate set
  - what about weak degeneracy in 1D with control on one side?
  - study Grushin-type operators with lower order terms (Laplace-Beltrami operator on Grushin plane)
  - relate controllability properties to the length of Lie brackets needed to satisfy Hörmander's condition

thank you for your attention

open problems

- boundary degeneracy
  - null controllability with boundary control for weakly degenerate operators
  - degenerate problems in 3 or more space dimensions
  - systems of DPO (Maniar et al.)
- interior degeneracy
  - ok for strong degeneracy in 1D with control on both sides of nondegenerate set
  - what about weak degeneracy in 1D with control on one side?
  - study Grushin-type operators with lower order terms (Laplace-Beltrami operator on Grushin plane)
  - relate controllability properties to the length of Lie brackets needed to satisfy Hörmander's condition

thank you for your attention

open problems

- boundary degeneracy
  - null controllability with boundary control for weakly degenerate operators
  - degenerate problems in 3 or more space dimensions
  - systems of DPO (Maniar et al.)
- interior degeneracy
  - ok for strong degeneracy in 1D with control on both sides of nondegenerate set
  - what about weak degeneracy in 1D with control on one side?
  - study Grushin-type operators with lower order terms (Laplace-Beltrami operator on Grushin plane)
  - relate controllability properties to the length of Lie brackets needed to satisfy Hörmander's condition

thank you for your attention

open problems

- boundary degeneracy
  - null controllability with boundary control for weakly degenerate operators
  - degenerate problems in 3 or more space dimensions
  - systems of DPO (Maniar et al.)
- interior degeneracy
  - ok for strong degeneracy in 1D with control on both sides of nondegenerate set
  - what about weak degeneracy in 1D with control on one side?
  - study Grushin-type operators with lower order terms (Laplace-Beltrami operator on Grushin plane)
  - relate controllability properties to the length of Lie brackets needed to satisfy Hörmander's condition

thank you for your attention

open problems

- boundary degeneracy
  - null controllability with boundary control for weakly degenerate operators
  - degenerate problems in 3 or more space dimensions
  - systems of DPO (Maniar et al.)
- interior degeneracy
  - ok for strong degeneracy in 1D with control on both sides of nondegenerate set
  - what about weak degeneracy in 1D with control on one side?
  - study Grushin-type operators with lower order terms (Laplace-Beltrami operator on Grushin plane)
  - relate controllability properties to the length of Lie brackets needed to satisfy Hörmander's condition

thank you for your attention

open problems

- boundary degeneracy
  - null controllability with boundary control for weakly degenerate operators
  - degenerate problems in 3 or more space dimensions
  - systems of DPO (Maniar et al.)
- interior degeneracy
  - ok for strong degeneracy in 1D with control on both sides of nondegenerate set
  - what about weak degeneracy in 1D with control on one side?
  - study Grushin-type operators with lower order terms (Laplace-Beltrami operator on Grushin plane)
  - relate controllability properties to the length of Lie brackets needed to satisfy Hörmander's condition

thank you for your attention

open problems

- boundary degeneracy
  - null controllability with boundary control for weakly degenerate operators
  - degenerate problems in 3 or more space dimensions
  - systems of DPO (Maniar et al.)
- interior degeneracy
  - ok for strong degeneracy in 1D with control on both sides of nondegenerate set
  - what about weak degeneracy in 1D with control on one side?
  - study Grushin-type operators with lower order terms (Laplace-Beltrami operator on Grushin plane)
  - relate controllability properties to the length of Lie brackets needed to satisfy Hörmander's condition

thank you for your attention

open problems

- boundary degeneracy
  - null controllability with boundary control for weakly degenerate operators
  - degenerate problems in 3 or more space dimensions
  - systems of DPO (Maniar et al.)
- interior degeneracy
  - ok for strong degeneracy in 1D with control on both sides of nondegenerate set
  - what about weak degeneracy in 1D with control on one side?
  - study Grushin-type operators with lower order terms (Laplace-Beltrami operator on Grushin plane)
  - relate controllability properties to the length of Lie brackets needed to satisfy Hörmander's condition

thank you for your attention

open problems

- boundary degeneracy
  - null controllability with boundary control for weakly degenerate operators
  - degenerate problems in 3 or more space dimensions
  - systems of DPO (Maniar et al.)
- interior degeneracy
  - ok for strong degeneracy in 1D with control on both sides of nondegenerate set
  - what about weak degeneracy in 1D with control on one side?
  - study Grushin-type operators with lower order terms (Laplace-Beltrami operator on Grushin plane)
  - relate controllability properties to the length of Lie brackets needed to satisfy Hörmander's condition

thank you for your attention

open problems

- boundary degeneracy
  - null controllability with boundary control for weakly degenerate operators
  - degenerate problems in 3 or more space dimensions
  - systems of DPO (Maniar et al.)
- interior degeneracy
  - ok for strong degeneracy in 1D with control on both sides of nondegenerate set
  - what about weak degeneracy in 1D with control on one side?
  - study Grushin-type operators with lower order terms (Laplace-Beltrami operator on Grushin plane)
  - relate controllability properties to the length of Lie brackets needed to satisfy Hörmander's condition

thank you for your attention

open problems

- boundary degeneracy
  - null controllability with boundary control for weakly degenerate operators
  - degenerate problems in 3 or more space dimensions
  - systems of DPO (Maniar et al.)
- interior degeneracy
  - ok for strong degeneracy in 1D with control on both sides of nondegenerate set
  - what about weak degeneracy in 1D with control on one side?
  - study Grushin-type operators with lower order terms (Laplace-Beltrami operator on Grushin plane)
  - relate controllability properties to the length of Lie brackets needed to satisfy Hörmander's condition

thank you for your attention