Very degenerate elliptic equations: applications and regularity

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Partial differential equations
Benasque,
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Which equations and why?

- General form of these nonlinearities
- Variational interpretation
- Examples of degeneracy

Links with optimal transport

- Beckmann’s problem and duality
- Non-uniform metrics
- Simple congestion models

More refined models for congestion and the need for regularity

- Vector and scalar traffic intensity
- Heuristics
- Regular flows

Precise regularity results

- Sobolev
- $L^\infty$
- $C^0$
- Perspectives
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In all the talk we will be interested in the solutions of

$$\nabla \cdot F(\nabla u) = f$$

with possible boundary conditions in $\Omega \subset \mathbb{R}^d$, where $F : \mathbb{R}^d \to \mathbb{R}^d$ is
given by $F = \nabla \mathcal{H}^*$, with $\mathcal{H}^* : \mathbb{R}^d \to \mathbb{R}$ a given convex function.

This equation is the Euler-Lagrange equation of

$$\min \int_{\Omega} \mathcal{H}^*(\nabla u) + fu$$

and is linear whenever $\mathcal{H}^*$ is quadratic. For other power functions, one
gets the $p-$Laplacian operator.

Boundary conditions: Dirichlet, Neumann (i.e. $\nabla \mathcal{H}^*(\nabla u) \cdot n = 0$)...

Extensions: explicit dependence on $x$ (i.e. $\mathcal{H}^*(x, \nabla u)$)...

Simplest cases: radial functions $\mathcal{H}^*$, depending on the modulus only.
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and is linear whenever \( H^* \) is quadratic. For other power functions, one gets the \( p \)–Laplacian operator.

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**Simplest cases**: radial functions \( H^* \), depending on the modulus only.
Why $\mathcal{H}^*$? (convex duality)

Suppose that $\mathcal{H}^*$ is the Legendre transform of another function $\mathcal{H}$, i.e.,

$$\mathcal{H}^*(x) = \sup y \cdot x - \mathcal{H}(y) : y \in \mathbb{R}^d$$

Then our equation also appears when solving

$$\min \int \mathcal{H}(\nu) : \nabla \cdot \nu = f.$$ 

Actually, the optimality condition here reads

for all $w$ such that $\nabla \cdot w = 0$ we have $\int \nabla \mathcal{H}(\nu) \cdot w = 0$.

Orthogonality to all divergence-free vector fields means being a gradient:

$$\nabla \mathcal{H}(\nu) = \nabla u \Rightarrow \nu = \nabla \mathcal{H}^*(\nabla u),$$

which allows to compute the optimal $\nu$ if one solves $\nabla \cdot \nabla \mathcal{H}^*(\nabla u) = f$.

Notice that if $\mathcal{H}$ and $\mathcal{H}^*$ are strictly convex and differentiable then one has $\nabla \mathcal{H}^* = (\nabla \mathcal{H})^{-1}$ and in general

$$y \in \partial \mathcal{H}(x) \Leftrightarrow x \in \partial \mathcal{H}^*(y).$$
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$$y \in \partial \mathcal{H}(x) \Leftrightarrow x \in \partial \mathcal{H}^*(y).$$
The usual elliptic theory is based on the assumption $D^2\mathcal{H}^* \geq c > 0$. This assumption is not verified by the $p-$Laplace operator, where $D^2\mathcal{H}^*(z) = c|z|^{p-2}$ (for $p > 2$, this tends to 0 as $z \to 0$).

Yet, we are here interested in a much worse situation: suppose that $D^2\mathcal{H}^*$ identically vanish on a set.

This is the case for instance when one starts from a non-convex problem, with $\int W(\nabla u)$ and takes $\mathcal{H}^*$ as the convex envelop of $W$. This convexified case is the motivation of Carstensen and Müller in a paper studying similar questions to ours under some assumptions on $\mathcal{H}^*$ (in particular, quadratic growth).

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Another interesting case is obtained when one first chooses $\mathcal{H}$, and takes a function which is not differentiable. Suppose $B(0, r) \subset \partial \mathcal{H}(0)$. Then $\nabla \mathcal{H}^* = 0$ on $B(0, r)$. Then $F = \nabla \mathcal{H}^*$ vanishes on a whole ball!!

Examples:

- $\mathcal{H}(z) = |z|$ (but $\mathcal{H}^*$ is not real-valued, $\mathcal{H}^* = I_{B_1}$)
- $\mathcal{H}(z) \approx |z|$ for $z \approx 0$ but $\mathcal{H}$ is strictly convex and superlinear. For instance

$$\mathcal{H}_p(z) = |z| + \frac{1}{p} |z|^p, \quad \mathcal{H}^*_p(v) = \frac{1}{p'} (|v| - 1)^{p'}.$$
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Consider the Monge-Kantorovitch problem

\[(P) \quad \min \int |x - y| d\gamma : \gamma \in P(\Omega \times \Omega), (\pi_x)_\#\gamma = \mu, (\pi_y)_\#\gamma = \nu,\]

its dual

\[(D) \quad \max \int u d(\mu - \nu) : \ u \in \text{Lip}_1(\Omega),\]

as well as the minimal flow problem by Beckmann

\[(B) \quad \min \int |v| : \nabla \cdot v = \mu - \nu.\]

Thanks to inf-sup interchanging and to the equivalence

\[u \in \text{Lip}_1 \iff \forall x, y \ u(x) - u(y) \leq |x - y| \iff \forall x \ |\nabla u(x)| \leq 1\]

one can prove

\[(P) = (D) = (B).\]

Duality-based equivalences

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Non-uniform metrics

If, instead, one considers

$$\min \int k(x)|v(x)| : \nabla \cdot v = \mu - \nu,$$

then there is equivalence with the Monge problem for the distance

$$d_k(x, y) = \inf \left\{ L_k(\sigma) := \int_0^1 k(\sigma(t))|\sigma'(t)|dt \mid \sigma(0) = x, \sigma(1) = y \right\}$$

($L_k$ being the weighted length, with weight $k$, and $d_k$ the associated geodesic distance, a Riemannian distance with a conformal metric $k \cdot l_d$).

This works fine when $k$ is a geographical datum, given a priori; in traffic congestion, instead, $k$ is supposed to depend on the traffic “intensity”, i.e. on $|v|$ itself! One should consider $\int k(|v|)|v|...$
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The congested Beckmann’s problem

Let us call $H$ the function $t \mapsto k(t)t$. Take $\mathcal{H}(z) = H(|z|)$. We are again brought to consider

$$\min \int \mathcal{H}(v) : \nabla \cdot v = f := \mu - \nu.$$

- It is reasonable to suppose $H$ convex and superlinear.
- The easiest example is $H(t) = \frac{1}{p} t^p$, for $p > 1$.
- Yet, $\lim_{t \to 0^+} H(t)/t := k(0)$ should represent the metric when no traffic is present, and should not vanish.
- Hence, a more reasonable model is $H(t) = t + \frac{1}{p} t^p$.

Notice that superlinear minimization is more well-posed than $\min \int |v|$, which could fall out to the set of measures.
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Traffic intensity minimization

The main problem of the previous model is that using $|v|$ to represent traffic intensity can account for cancellations, which is not realistic. Hence, let us change the model.

Describe a traffic configuration through a measure $Q \in \mathcal{P}(C)$, where $C = \{\sigma : [0, 1] \to \Omega \text{ Lipschitz}\}$ is a set of path.

To such a $Q$, associate a traffic intensity measure $i_Q \in \mathcal{M}_+(\Omega)$ through

$$
\int C \varphi di_Q := \int C dQ(\sigma) \int_0^1 \varphi(\sigma(t))|\sigma'(t)|dt = \int L\varphi(\sigma)dQ(\sigma).
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Suppose for a while that $i_Q$ is a function. Then define the congested cost of a path $\sigma$ as $L_{k(i_Q)}(\sigma)$ and minimize the total cost

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\int C L_{k(i_Q)}(\sigma)dQ(\sigma) = \int k(i_Q(x))i_Q(x)dx = \int H(i_Q(x))dx.
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The constraints are $(e_0)_#Q = \mu$, $(e_1)_#Q = \nu$, (where $e_t : C \to \Omega$ is given by $e_t(\sigma) := \sigma(t)$). If $H$ is superlinear the minimization is well-posed.
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Network games: a typical question is to find an equilibrium when agents commute on the network and produce a measure $Q$ on the set of paths; find $Q$ such that no agent will change its mind once observed the payoffs $L_k(i_Q)$. This is called Wardrop equilibrium. Here the optimality conditions give: if $\overline{Q}$ minimizes $\int H(i_Q)$, and we set $\overline{k} := H'(i_Q)$, then $\overline{Q}$—a.e. path $\sigma$ satisfies

$$L_{\overline{k}}(\sigma) = d_{\overline{k}}(\sigma(0), \sigma(1)),$$

i.e. it is a geodesic for $\overline{k}$. It is a Wardrop equilibrium in a continuous (non-network) setting, for the congestion function $H'$ instead of $k$!

Warning: defining $d_{\overline{k}}$ is not evident for $\overline{k} \not\in C^0$ or $\overline{k}$ defined only a.e.

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Warning: defining $d_{\bar{k}}$ is not evident for $\bar{k} \notin C^0$ or $\bar{k}$ defined only a.e.

Our goal: proving regularity results and write PDEs for the equilibrium.


Wardrop equilibria

**Network games**: a typical question is to find an equilibrium when agents commute on the network and produce a measure $Q$ on the set of paths; find $Q$ such that no agent will change its mind once observed the payoffs $L_k(i_Q)$. This is called Wardrop equilibrium. Here the optimality conditions give: if $\overline{Q}$ minimizes $\int H(i_Q)$, and we set $\overline{k} := H'(i_Q)$, then $\overline{Q}$—a.e. path $\sigma$ satisfies

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Links between Beckmann and Wardrop, vector and scalar problems

Is the minimization of \( \int H(i_Q) \) equivalent or linked to that of \( \int H(|v|) \) under \( \nabla \cdot v = \mu - \nu \)?

To every \( Q \), associate a vector traffic intensity measure \( v_Q \in \mathcal{M}^d(\Omega) \) through

\[
\int \vec{\varphi} \cdot dv_Q := \int_C dQ(\sigma) \int_0^1 \vec{\varphi}(\sigma(t)) \cdot \sigma'(t) dt.
\]

It is easy to check \( \nabla \cdot v_Q = \mu - \nu \) and \( |v_Q| \leq i_Q \).

Hence

\[
\min \int H(i_Q) \geq \min \int H(|v_Q|) \geq \left( \min \int H(|v|) : \nabla \cdot v = \mu - \nu \right).
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To get the complete equivalence we need the opposite inequality, i.e. we need to take an optimal \( v \) and build a \( Q \) from it, guaranteeing \( i_Q \leq |v| \).

Idea: following the integral curves of \( v \).
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Filippo Santambrogio

Very degenerate elliptic equations: applications and regularity
The flow we need

An idea from Dacorogna-Moser & Evans-Gangbo: set $\mu_t := (1 - t)\mu + t\nu$, take the optimal vector field $\nu$ and suppose that everything is regular. Consider the vector field $w(t, x) = \nu(x)/\mu_t(x)$ and for every $x \in \Omega$

$$
\begin{align*}
  y'_x(t) &= w(t, y_x(t)), \\
  y_x(0) &= x
\end{align*}
$$

Call $Y(x)$ the curve $(y_x(t))_{t \in [0,1]}$ and consider then the measure $Q := Y_#\mu \in \mathcal{P}(C)$. Since both $(e_t)_#Q$ and $\mu_t$ solve the equation $\partial_t \rho + \nabla \cdot (\rho w) = 0$ with initial datum $\rho_0 = \mu$. By uniqueness, we get $(e_t)_#Q = \mu_t$ and hence $Q$ is admissible. Moreover, it is possible to check $i_Q = |\nu|$. This would solve the equivalence problem...

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Regularity needs

Everything would be fine if \( w = \nabla \bar{v}/\mu_t \) was Lipschitz continuous. We can add assumptions on \( \mu, \nu \) : let us suppose them to be a.c. with Lipschitz densities bounded away from 0. But what about \( \bar{v} \)? We have \( \bar{v} = \nabla H^*(\nabla u) \) with

\[ \nabla \cdot \nabla H^*(\nabla u) = \mu - \nu. \]

If \( H(t) = t^2 \): standard elliptic regularity!
If \( H^*(t) = t^p \): \( p \)–Laplacian!
How about the degenerate case \( H(t) = t + \frac{1}{p} t^p \)?

- can we expect \( u \in W^{2,\infty} \)? \( \text{NOT} \)
- can we expect something on \( \nabla H^*(\nabla u) \)? \( \text{YES} \)
- Less than Lipschitz could be enough? \( \text{YES} \)

We can use \textbf{DiPerna–Lions theory}. We need \( w \in W^{1,1} \) and \( \nabla \cdot w \in L^\infty \), i.e. we need to prove that \( \bar{v} \) is Sobolev and bounded.
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Here it is: Sobolevness and boundedness

Take $p \geq 2$ and consider the very degenerate elliptic equation

$$\nabla \cdot F_{p-1}(\nabla u) = f$$

where $F_r(z) = (|z| - 1)^r + \frac{z}{|z|}$. Suppose $f \in W^{1,p'}$:

1. $F_{p/2}(\nabla u) \in W^{1,2}$.
2. $|\nabla u| \in L^\infty$ (here $f \in L^{d+\varepsilon}$ is enough).
3. $F_{p-1}(\nabla u) \in W^{1,2}$.

Tools: for 1) adapt the incremental ratio method for the $p-$Laplacian, for 2) use suitable test functions based on $(|\nabla u| - 2)_+$.

Strange assumptions on the datum $f$. Usually to get $\nabla u \in W^{1,p}$ one needs $f \in L^p$, not $f \in W^{1,p}$. Actually we can arrive up to $f \in BV \cap L^{d+\varepsilon}$ but not better, we need at least some differentiability! This is due to the degeneracy, which always asks for more regularity (also on $\partial \Omega$).

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More on regularity

**Continuity in dimension two**

Solve

\[ \nabla \cdot F(\nabla u) = f \]

with \( F = \nabla \mathcal{H}^*, \ D^2 \mathcal{H}^*(z) \geq c_\delta I_d, \ c_\delta > 0, \) for all \( z \notin B_{1+\delta}, \ f \in L^{2+\varepsilon}, \ d = 2: \) suppose also \( F(\nabla u) \in W^{1,2} \cap L^\infty. \) Then \( g(\nabla u) \in C^0 \) for every \( g \in C^0(\mathbb{R}^2) \) with \( g = 0 \) sur \( B_1. \)

**Strategy** : consider first \( \nu_{e,\delta} = (\nabla u \cdot e - (1 + \delta)\,)_+ \) which solves a better equation, and prove continuity for it (actually the true assumption should be \( \nu_{e,\delta} \in W^{1,2} \) rather than \( F(\nabla u) \in W^{1,2}. \) \( C^0 \) estimates are uniform in \( e, \) but degenerate as \( \delta \to 0. \) Yet, as \( \delta \to 0 \) we have uniform convergence, and continuity is preserved. The modulus of continuity is very poor (for \( \delta > 0 \) it is logarithmic).

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Regularity : some perspectives

Anisotropy
The continuous congestion model leading to these PDEs is not the homogenization limit of network on grids. Instead, non-isotropic functions $H$ appear. Apart from modeling (traffic intensities depending on directions, not only on $i_Q$) and homogenization (random networks) questions one could study more general functions $\mathcal{H}$. Even for the easiest case

$$\mathcal{H}(v) = |v_1|^p + |v_2|^p + \cdots + |v_d|^p \quad (p > 2)$$

regularity results are not obvious, nor all known. (L. Brasco, G. Carlier)

The singular case
We only considered the degenerate case $p > 2$. What about continuity of $F_{p-1}(\nabla u)$ for $p < 2$? the idea is that $\nu_{e,\delta}$ could be easy to deal with, and then use the uniform limit as $\delta \to 0$. (L. Brasco, V. Julin)

Better continuity results
It seems that $C^{1,\alpha}$ techniques for the $p-$Laplacian could be used to prove $C^{0,\alpha}$ for $F_{p-1}(\nabla u)$. If it worked, it would improve the continuity result we have, and be valid in any dimension! (L. Caffarelli, A. Figalli)
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Transport density (but it does not really work)

Go back to \( \min \int |\nabla \cdot v| : \nabla \cdot v = \mu - \nu \). The measure \( m = |\nabla v| = i_Q \) (for the optimal field \( \nabla v \)) is usually called transport density. It solves with the Kantorovitch potential \( u \) the Monge-Kantorovitch system of PDEs

\[
\nabla \cdot (m \nabla u) = \mu - \nu; \quad |\nabla u| \leq 1; \quad |\nabla u| = 1 \text{ a.e. on } m > 0.
\]

Several regularity questions have been analyzed on the transport density \( m \), such as \( \mu, \nu \in L^p \Rightarrow m \in L^p \). But \( C^0 \) and differentiability are open.

**A strategy** : approximate through

\[
\min \int |v| + \frac{\varepsilon}{2} |v|^2 : \nabla \cdot v = \mu - \nu
\]

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**Problem** : the non linearity of the operator does not allow easy estimates.


Transport density (but it does not really work)

Go back to \( \min \int |\mathbf{v}| : \nabla \cdot \mathbf{v} = \mu - \nu \). The measure \( m = |\mathbf{v}| = i_Q \) (for the optimal field \( \mathbf{v} \)) is usually called transport density. It solves with the Kantorovitch potential \( u \) the Monge-Kantorovitch system of PDEs

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\nabla \cdot (m \nabla u) = \mu - \nu; \quad |\nabla u| \leq 1; \quad |\nabla u| = 1 \; \text{a.e. on} \; m > 0.
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Several regularity questions have been analyzed on the transport density \( m \), such as \( \mu, \nu \in L^p \Rightarrow m \in L^p \). But \( C^0 \) and differentiability are open.

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Filippo Santambrogio

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...thanks for your attention