Very degenerate elliptic equations: applications and regularity

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Partial differential equations
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1. Which equations and why ?
   - General form of these nonlinearities
   - Variational interpretation
   - Examples of degeneracy
2. Links with optimal transport
   - Beckmann's problem and duality
   - Non-uniform metrics
   - Simple congestion models
3. More refined models for congestion and the need for regularity
   - Vector and scalar traffic intensity
   - Heuristics
   - Regular flows
4. Precise regularity results
   - Sobolev
   - L^∞
   - C^0
   - Perspectives

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General form

In all the talk we will be interested in the solutions of

$$\nabla \cdot F(\nabla u) = f$$

with possible boundary conditions in $\Omega \subset \mathbb{R}^d$, where $F : \mathbb{R}^d \to \mathbb{R}^d$ is given by $F = \nabla \mathcal{H}^*$, with $\mathcal{H}^* : \mathbb{R}^d \to \mathbb{R}$ a given convex function.
This equation is the Euler-Lagrange equation of

$$\min \int_\Omega \mathcal{H}^*(\nabla u) + fu$$

and is linear whenever $\mathcal{H}^*$ is quadratic. For other power functions, one gets the $p-$Laplacian operator.
Boundary conditions : Dirichlet, Neumann (i.e. $\nabla \mathcal{H}^*(\nabla u) \cdot n = 0$) ...
Extensions : explicit dependence on x (i.e. $\mathcal{H}^*(x, \nabla u)$)...
Simplest cases : radial functions $\mathcal{H}^*$, depending on the modulus only.

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Why $\mathcal{H}^*$? (convex duality)

Suppose that $\mathcal{H}^*$ is the Legendre transform of another function $\mathcal{H}$, i.e.

$$\mathcal{H}^*(x) = \sup x \cdot y - \mathcal{H}(y) \ : \ y \in \mathbb{R}^d$$

Then our equation also appears when solving

$$\min \int \mathcal{H}(v) \quad : \quad \nabla \cdot v = f.$$

Actually, the optimality condition here reads

$$\text{for all } w \text{ such that } \nabla \cdot w = 0 \quad \text{we have} \quad \int \nabla \mathcal{H}(v) \cdot w = 0.$$

Orthogonality to all divergence-free vector fields means being a gradient :

$$\nabla \mathcal{H}(v) = \nabla u \Rightarrow v = \nabla \mathcal{H}^*(\nabla u),$$

which allows to compute the optimal v if one solves $\nabla \cdot \nabla \mathcal{H}^*(\nabla u) = f$.
Notice that if $\mathcal{H}$ and $\mathcal{H}^*$ are strictly convex and differentiable then one has $\nabla \mathcal{H}^* = (\nabla \mathcal{H})^{-1}$ and in general

$$y \in \partial \mathcal{H}(x) \Leftrightarrow x \in \partial \mathcal{H}^*(y).$$

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Degeneracy - 1

The usual elliptic theory is based on the assumption $D^2\mathcal{H}^* \geq c > 0$. This is assumption is not verified by the $p-$Laplace operator, where $D^2\mathcal{H}^*(z) = c|z|^{p-2}$ (for $p > 2$, this tends to 0 as $z \to 0$).

Yet, we are here interested in a much worse situation : suppose that $D^2\mathcal{H}^*$ identically vanish on a set.

This is the case for instance when one starts from a non-convex problem, with $\int W(\nabla u)$ and takes $\mathcal{H}^*$ as the convex envelop of W. This convexified case is the motivation of Carstensen and Müller in a paper studying similar questions to ours under some assumptions on $\mathcal{H}^*$ (in particular, quadratic growth).

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Degeneracy - 2

Another interesting case is obtained when one first chooses $\mathcal{H}$, and takes a function which is not differentiable. Suppose $B(0,r) \subset \partial\mathcal{H}(0)$. Then $\nabla\mathcal{H}^* = 0$ on $B(0,r)$. Then $F = \nabla\mathcal{H}^*$ vanishes on a whole ball ! !

Examples :

- $\mathcal{H}(z) = |z|$ (but $\mathcal{H}^*$ is not real-valued, $\mathcal{H}^* = I_{B_1}$)
- $\mathcal{H}(z) \approx |z|$ for $z \approx 0$ but $\mathcal{H}$ is strictly convex and superlinear. For instance

$$\mathcal{H}_p(z) = |z| + \frac{1}{p}|z|^p, \quad \mathcal{H}_p^*(v) = \frac{1}{p'}(|v|-1)_+^{p'}.$$

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Duality-based equivalences

Consider the Monge-Kantorovitch problem

$$(P) \quad \min \int |x-y| d\gamma \ : \ \gamma \in \mathcal{P}(\Omega \times \Omega), \ (\pi_x)_\# \gamma = \mu, \ (\pi_y)_\# \gamma = \nu,$$

its dual

$$(D) \quad \max \int u \, d(\mu - \nu) \ : \ u \in \mathrm{Lip}_1(\Omega),$$

as well as the minimal flow problem by Beckmann

$$(B) \quad \min \int |v| \ : \ \nabla \cdot v = \mu - \nu.$$

Thanks to inf-sup interchanging and to the equivalence

$$u \in \mathrm{Lip}_1 \Leftrightarrow \forall x, y \ u(x) - u(y) \leq |x-y| \Leftrightarrow \forall x \ |\nabla u(x)| \leq 1$$

one can prove

$$(P) = (D) = (B).$$

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Non-uniform metrics

If, instead, one considers

$$\min \int k(x)|v(x)| \; : \; \nabla \cdot v = \mu - \nu,$$

then there is equivalence with the Monge problem for the distance

$$d_k(x,y) = \inf \left\{ L_k(\sigma) := \int_0^1 k(\sigma(t))|\sigma'(t)|dt \quad \sigma(0) = x, \sigma(1) = y \right\}$$

(L_k being the weighted length, with weight k, and d_k the associated geodesic distance, a Riemannian distance with a conformal metric $k \cdot I_d$).

This works fine when k is a geographical datum, given a priori ; in traffic congestion, instead, k is supposed to depend on the traffic "intensity", i.e. on $|v|$ itself ! One should consider $\int k(|v|)|v|\dots$

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The congested Beckmann's problem

Let us call H the function $t \mapsto k(t)t$. Take $\mathcal{H}(z) = H(|z|)$. We are again brought to consider

$$\min \int \mathcal{H}(v) \ : \ \nabla \cdot v = f := \mu - \nu.$$

- It is reasonable to suppose H convex and superlinear.
- The easiest example is $H(t) = \frac{1}{p}t^p$, for $p > 1$.
- Yet, $\lim_{t \to 0^+} H(t)/t := k(0)$ should represent the metric when no traffic is present, and should not vanish.
- Hence, a more reasonable model is $H(t) = t + \frac{1}{p}t^p$.

Notice that superlinear minimization is more well-posed than $\min \int |v|$, which could fall out to the set of measures.

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Traffic intensity minimization

The main problem of the previous model is that using $|v|$ to represent traffic intensity can account for cancellations, which is not realistic. Hence, let us change the model.

Describe a traffic configuration through a measure $Q \in \mathcal{P}(C)$, where $C = \{\sigma : [0,1] \to \Omega \text{ Lipschitz}\}$ is a set of path.

To such a Q, associate a *traffic intensity measure* $i_Q \in \mathcal{M}_+(\Omega)$ through

$$\int \varphi di_Q := \int_C dQ(\sigma) \int_0^1 \varphi(\sigma(t))|\sigma'(t)|dt = \int L_\varphi(\sigma) dQ(\sigma).$$

Suppose for a while that i_Q is a function. Then define the congested cost of a path σ as $L_{k(i_Q)}(\sigma)$ and minimize the total cost

$$\int_C L_{k(i_Q)}(\sigma) dQ(\sigma) = \int k(i_Q(x)) i_Q(x) dx = \int H(i_Q(x)) dx.$$

The constraints are $(e_0)_\# Q = \mu$, $(e_1)_\# Q = \nu$, (where $e_t : C \to \Omega$ is given by $e_t(\sigma) := \sigma(t)$). If H is superlinear the minimization is well-posed.

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Wardrop equilibria

Network games : a typical question is to find an equilibrium when agents commute on the network and produce a measure Q on the set of paths ; find Q such that no agent will change its mind once observed the payoffs $L_{k(i_Q)}$. This is called Wardrop equilibrium.
Here the optimality conditions give : if $\overline{Q}$ minimizes $\int H(i_Q)$, and we set $\overline{k} := H'(i_Q)$, then $\overline{Q}-$a.e. path σ satisfies

$$L_{\overline{k}}(\sigma) = d_{\overline{k}}(\sigma(0), \sigma(1)),$$

i.e. it is a geodesic for $\overline{k}$. It is a Wardrop equilibrium in a continuous (non-network) setting, for the congestion function H' instead of k !

Warning : defining $d_{\overline{k}}$ is not evident for $\overline{k} \notin C^0$ or $\overline{k}$ defined only a.e.
Our goal : proving regularity results and write PDEs for the equilibrium.

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Here the optimality conditions give : if $\overline{Q}$ minimizes $\int H(i_Q)$, and we set $\overline{k} := H'(i_Q)$, then $\overline{Q}-$a.e. path σ satisfies

$$L_{\overline{k}}(\sigma) = d_{\overline{k}}(\sigma(0), \sigma(1)),$$

i.e. it is a geodesic for $\overline{k}$. It is a Wardrop equilibrium in a continuous (non-network) setting, for the congestion function H' instead of k !

Warning : defining $d_{\overline{k}}$ is not evident for $\overline{k} \notin C^0$ or $\overline{k}$ defined only a.e.

Our goal : proving regularity results and write PDEs for the equilibrium.

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Links between Beckmann and Wardrop, vector and scalar problems

Is the minimization of $\int H(i_Q)$ equivalent or linked to that of $\int H(|v|)$ under $\nabla \cdot v = \mu - \nu$?

To every Q, associate a vector traffic intensity measure $v_Q \in \mathcal{M}^d(\Omega)$ through

$$\int \vec{\varphi} \cdot dv_Q := \int_C dQ(\sigma) \int_0^1 \vec{\varphi}(\sigma(t)) \cdot \sigma'(t) dt.$$

It is easy to check $\nabla \cdot v_Q = \mu - \nu$ and $|v_Q| \leq i_Q$.
Hence

$$\min \int H(i_Q) \geq \min \int H(|v_Q|) \geq \left(\min \int H(|v|) \ : \ \nabla \cdot v = \mu - \nu \right).$$

To get the complete equivalence we need the opposite inequality, i.e. we need to take an optimal v and build a Q from it, guaranteeing $i_Q \leq |v|$.
Idea : following the integral curves of v.

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The flow we need

An idea from Dacorogna-Moser & Evans-Gangbo : set $\mu_t := (1-t)\mu + t\nu$, take the optimal vector field $\overline{v}$ and suppose that everything is regular. Consider the vector field $w(t,x) = \overline{v}(x)/\mu_t(x)$ and for every $x \in \Omega$

$$\begin{cases} y_x'(t) = w(t, y_x(t)), \\ y_x(0) = x \end{cases}$$

Call $Y(x)$ the curve $(y_x(t))_{t\in[0,1]}$ and consider then the measure $Q := Y_\# \mu \in \mathcal{P}(C)$.

Since both $(e_t)_\# Q$ and μ_t solve the equation $\partial_t \rho + \nabla \cdot (\rho w) = 0$ with initial datum $\rho_0 = \mu$. By uniqueness, we get $(e_t)_\# Q = \mu_t$ and hence Q is admissible. Moreover, it is possible to check $i_Q = |\overline{v}|$.

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Regularity needs

Everything would be fine if $w = \overline{v}/\mu_t$ was Lipschitz continuous.

We can add assumptions on μ, ν : let us suppose them to be a.c. with Lipschitz densities bounded away from 0.

But what about $\overline{v}$? We have $\overline{v} = \nabla\mathcal{H}^*(\nabla u)$ with

$$\nabla \cdot \nabla\mathcal{H}^*(\nabla u) = \mu - \nu.$$

If $H(t) = t^2$: **standard elliptic regularity !**
If $H^*(t) = t^p$: $p-$**Laplacian !**
How about the degenerate case $H(t) = t + \frac{1}{p}t^p$?

can we expect $u \in W^{2,\infty}$?	NOT
can we expect something on $\nabla\mathcal{H}^*(\nabla u)$?	YES
Less than Lipschitz could be enough ?	YES

We can use **DiPerna-Lions theory**. We need $w \in W^{1,1}$ and $\nabla \cdot w \in L^\infty$, i.e. we need to prove that $\overline{v}$ is Sobolev and bounded.

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Here it is : Sobolevness and boundedness

Take $p \geq 2$ and consider the very degenerate elliptic equation

$$\nabla \cdot F_{p-1}(\nabla u) = f$$

where $F_r(z) = (|z| - 1)_+^r \frac{z}{|z|}$. Suppose $f \in W^{1,p'}$: then

1. $F_{p/2}(\nabla u) \in W^{1,2}$.
2. $|\nabla u| \in L^\infty$ (here $f \in L^{d+\varepsilon}$ is enough).
3. $F_{p-1}(\nabla u) \in W^{1,2}$.

Tools : for 1) adapt the incremental ratio method for the $p-$Laplacian, for 2) use suitable test functions based on $(|\nabla u| - 2)_+$.

Strange assumptions on the datum f. Usually to get $\nabla u \in W^{1,p}$ one needs $f \in L^p$, not $f \in W^{1,p}$. Actually we can arrive up to $f \in BV \cap L^{d+\varepsilon}$ but not better, we need at least some differentiability ! This is due to the degeneracy, which always asks for more regularity (also on $\partial\Omega$).

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More on regularity

Continuity in dimension two
Solve

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with $F = \nabla \mathcal{H}^*$, $D^2\mathcal{H}^*(z) \geq c_\delta I_d$, $c_\delta > 0$, for all $z \notin B_{1+\delta}$, $f \in L^{2+\varepsilon}$, $d = 2$: suppose also $F(\nabla u) \in W^{1,2} \cap L^\infty$. Then $g(\nabla u) \in C^0$ for every $g \in C^0(\mathbb{R}^2)$ with $g = 0$ sur B_1.

Strategy : consider first $v_{e,\delta} = (\nabla u \cdot e - (1+\delta))_+$ which solves a better equation, and prove continuity for it (actually the true assumption should be $v_{e,\delta} \in W^{1,2}$ rather than $F(\nabla u) \in W^{1,2}$). C^0 estimates are uniform in e, but degenerate as $\delta \to 0$. Yet, as $\delta \to 0$ we have uniform convergence, and continuity is preserved. The modulus of continuity is very poor (for $\delta > 0$ it is logarithmic).

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Regularity : some perspectives

Anisotropy

The continuous congestion model leading to these PDEs is not the homogenization limit of network on grids. Instead, non-isotropic functions H appear. Apart from modeling (traffic intensities depending on directions, not only on i_Q) and homogenization (random networks) questions one could study more general functions $\mathcal{H}$. Even for the easiest case

$$\mathcal{H}(v) = |v_1|^p + |v_2|^p + \cdots + |v_d|^p \quad (p > 2)$$

regularity results are not obvious, nor all known. (*L. Brasco, G. Carlier*)

The singular case

We only considered the degenerate case $p > 2$. What about continuity of $F_{p-1}(\nabla u)$ for $p < 2$? the idea is that $v_{e,\delta}$ could be easy to deal with, and then use the uniform limit as $\delta \to 0$. (*L. Brasco, V. Julin*)

Better continuity results

It seems that $C^{1,\alpha}$ techniques for the $p-$Laplacian could be used to prove $C^{0,\alpha}$ for $F_{p-1}(\nabla u)$. If it worked, it would improve the continuity result we have, and be valid in any dimension ! (*L. Caffarelli, A. Figalli*)

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regularity results are not obvious, nor all known. (*L. Brasco, G. Carlier*)

The singular case

We only considered the degenerate case $p > 2$. What about continuity of $F_{p-1}(\nabla u)$ for $p < 2$? the idea is that $v_{e,\delta}$ could be easy to deal with, and then use the uniform limit as $\delta \to 0$. (*L. Brasco, V. Julin*)

Better continuity results

It seems that $C^{1,\alpha}$ techniques for the $p-$Laplacian could be used to prove $C^{0,\alpha}$ for $F_{p-1}(\nabla u)$. If it worked, it would improve the continuity result we have, and be valid in any dimension ! (*L. Caffarelli, A. Figalli*).

Transport density (but it does not really work)

Go back to min $\int |v| \,:\, \nabla\cdot v = \mu-\nu$. The measure $m = |\overline{v}| = i_Q$ (for the optimal field $\overline{v}$) is usually called *transport density*. It solves with the Kantorovitch potential $u$the *Monge-Kantorovitch system* of PDEs

$$\nabla\cdot(m\nabla u) = \mu-\nu; \quad |\nabla u|\le 1; \quad |\nabla u| = 1 \text{ a.e. on } m>0.$$

Several regularity questions have been analyzed on the transport density m, such as $\mu,\nu\in L^p \Rightarrow m\in L^p$. But C^0 and differentiability are open.

A strategy : approximate through

$$\min \int |v| + \frac{\varepsilon}{2}|v|^2 \,:\, \nabla\cdot v = \mu-\nu$$

We have $H(t) = t+\frac{\varepsilon}{2}t^2$, $H^*(t) = \frac{1}{2\varepsilon}(t-1)_+^2$. Study, as $\varepsilon\to 0$, the PDE

$$\nabla F_1(\nabla u) = \varepsilon f$$

Problem : the non linearity of the operator does not allow easy estimates.

G. Bouchitté and G. Buttazzo, Characterization of optimal shapes and masses through Monge-. Kantorovich equation. *J. Eur. Math. Soc.* 2001,
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THE END . . .

. . .thanks for your attention