Adaptive mesh refinement techniques for well-balanced schemes for shallow water flows

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Outline

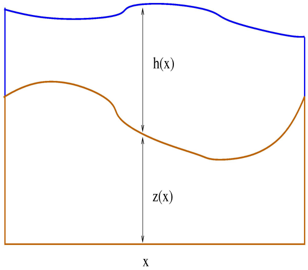
1. Shock capturing schemes for Shallow water flows
2. Adaptive Mesh Refinement
   - Adaptive schemes
   - Grid hierarchy
3. Well-balanced AMR
   - Well-balanced schemes
   - Homogeneous discretization for SWE
   - Well-balanced interpolation
4. Numerical results
   - Numerical results
5. Conclusions

Shallow water flow

- Shallow water equations (SWE) are obtained from incompressible Navier-Stokes equations by depth-averaging and neglecting some terms:

$$h_t + \text{div}(hv) = 0$$

$$(hv)_t + \text{div}(hv \otimes v + \frac{gh^2}{2} I_2) = -gh\nabla z$$



$h \equiv$ water height,
$v = (\bar{v}^x, \bar{v}^y) \equiv$ depth-averaged velocity,
$g \equiv$ gravity acceleration,
$z \equiv$ bottom topography.

- To simplify, we do the exposition in 1D:

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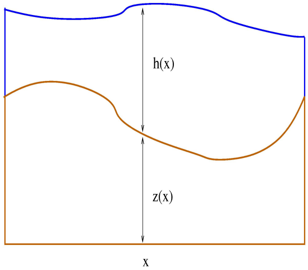

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Shock capturing schemes

- Use notation:

$$u = \begin{bmatrix} h \\ hv \end{bmatrix}, f(u) = \begin{bmatrix} hv \\ hv^2 + \frac{gh^2}{2} \end{bmatrix}, s(x,u) = \begin{bmatrix} 0 \\ -ghz_x \end{bmatrix}$$

  so that SWE system can be written as: $\boxed{u_t + f(u)_x = s(x,u)}$.

- Nonlinear hyperbolic system $\Rightarrow$ solutions can develop discontinuities. $\Rightarrow$ use **shock capturing** schemes:

$$u_i^{n+1} = u_i^n - \Delta t \Big(\frac{\hat{f}^n_{i+1/2} - \hat{f}^n_{i-1/2}}{\Delta x} - s_i^n \Big),$$

  where $s_i^n(u(x,t)) \approx s(x_i, u(x_i, t_n))$ and the numerical fluxes $\hat{f}_{i+1/2} = \hat{f}(u_{i-s}, \dots, u_{i+s+1})$ verify

$$\left[\frac{\hat{f}^n_{i+1/2} - \hat{f}^n_{i-1/2}}{\Delta x} \right] (u(x,t)) \approx f(u)_x(x_i, t_n), \quad x_i = i\Delta x, t_n = n\Delta t$$

  and appropriate stability conditions (through **upwinding** and adding numerical viscosity to comply with entropy conditions).

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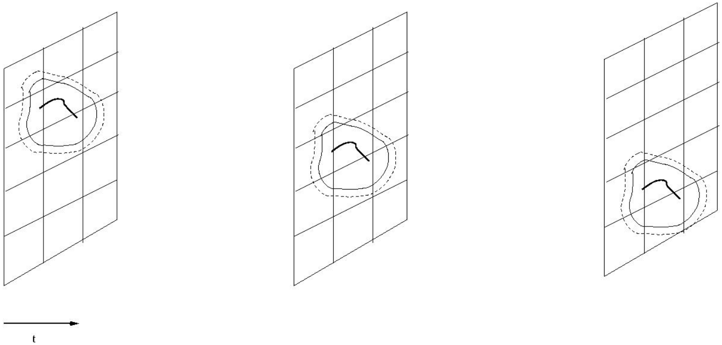
Outline

Adaptive schemes

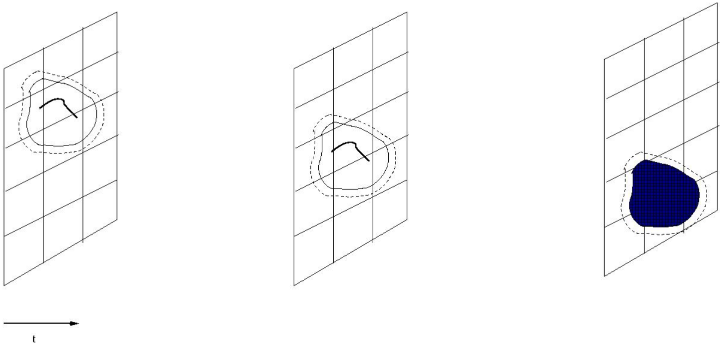
- For $N = 1/\Delta$ and d dimensions, computational cost is $\mathcal{O}(N^{d+1})$, storage is $\mathcal{O}(N^d)$, huge to get small errors.
- Numerical errors are not uniformly distributed:
  - larger errors at discontinuities
  - smaller errors at smooth regions
- An **Adaptive Scheme**, with a smaller Δ where higher errors, would be necessary for $d \geq 2$ and high precision needs.
- Many approaches, we briefly review the (**S**tructured) **A**daptive **M**esh **R**efinement algorithm, proposed by [Berger and Oliger, 1984] and extended by many authors (Colella, Quirk, $\cdots$) to FV schemes.

AMR algorithm



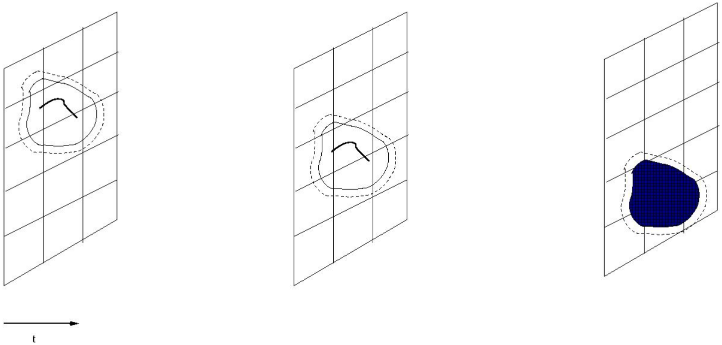
- Time evolution for some grid size $\Delta \equiv \Delta x$ and Δt.

AMR algorithm



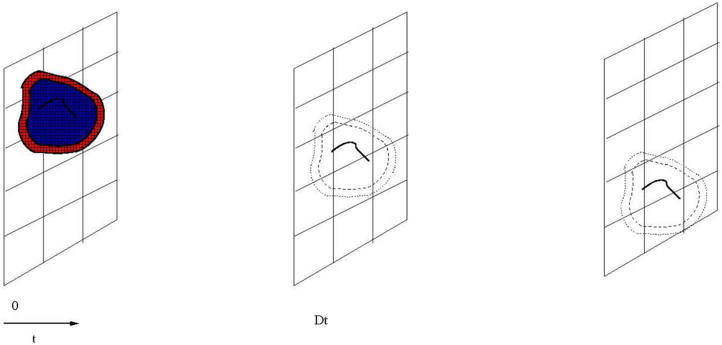
- Want to zoom at **R**egion **O**f **I**nterest, say by using $\Delta/2$.

AMR algorithm



- A: use **interpolation** (zoom), but this causes large errors near shocks.
- B: discard results with Δ, start over with $\Delta/2$.
- C: track region of interest through time evolution.

AMR algorithm



- Before going to B plan, notice that solution on $\Omega \times [0, \Delta t]$ (hopefully) depends on solution at **D**omain of **D**ependence $\widetilde{\Omega} \times \{0\}$ (**by hyperbolicity**).
- Can compute solution at $\Omega \times \{\frac{\Delta t}{2}\}$ (assuming $\Delta/2$ at ROI, same CFL)

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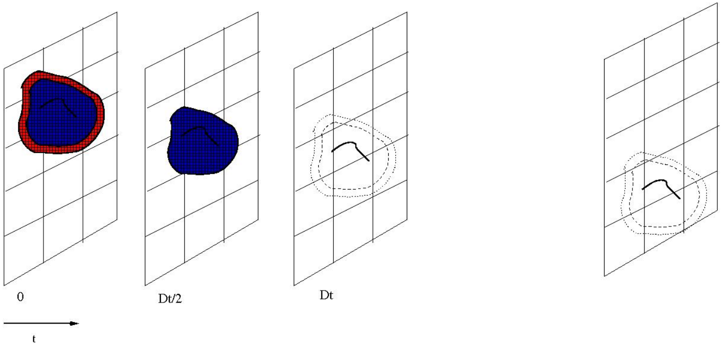

- How can new DD of region of interest be computed?
- **Zooming by** (x, t)**-interpolation**, OK at (supposedly smooth) surrounding band (**coarse** $\rightarrow$ **fine interpolation**)

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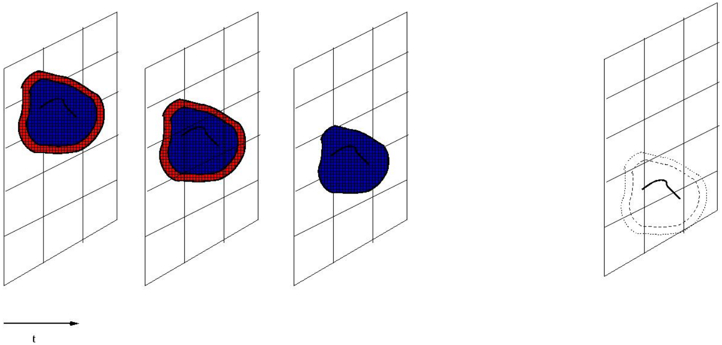

- Recursion $\Rightarrow$ need **nested Grid Hierarchy** (for interpolation).
- Must synchronize data through GH at same (x, t) (**fine $\rightarrow$ coarse project.**)
- More (shorter) time steps at finer resolutions (**local time stepping**).

Outline

Grid hierarchy

- Grid hierarchy indexed by level l from $l = 0$ (**coarsest**) to $l = L$ (**finest**).
- **Point value approach:** Points in the grid hierarchy: $x_i^l = i\Delta_0/2^l$, $i = 0, \ldots, N_0 2^l$. Since $x_{2i}^{l+1} = x_i^l$ (even indexed points in level $l+1$ are aligned with points in level l), project solution by just copying

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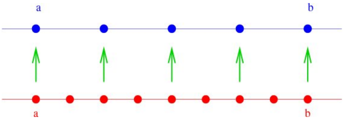
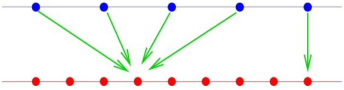


fine $\to$ coarse



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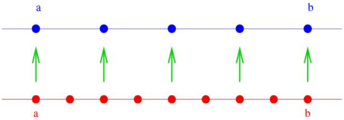

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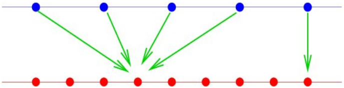

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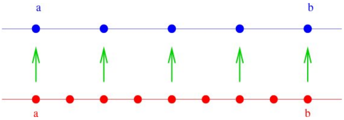

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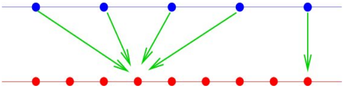

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- **Cell-based approach:** Points in the grid hierarchy: $x_i^l = (i + \frac{1}{2})\Delta_0/2^l$, $i = 0, \dots, N_0 2^l - 1$ (**cell centers**).
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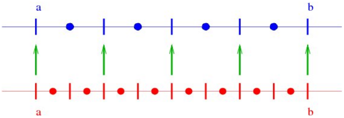

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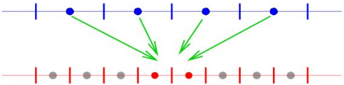

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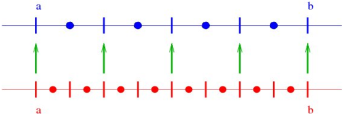

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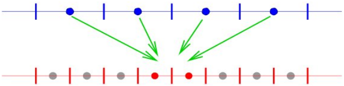

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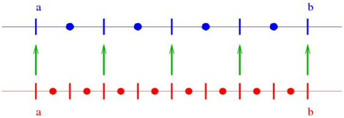

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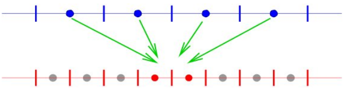

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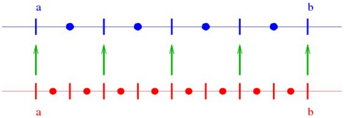

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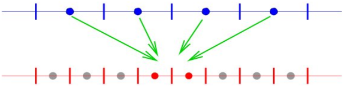

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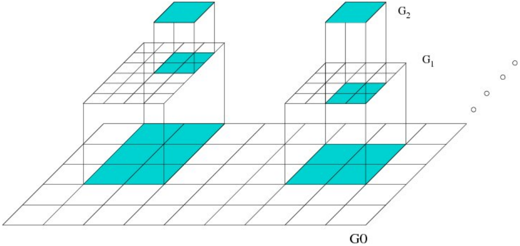
AMR algorithm

- Nested grids as in 2D example with 2 levels. In a time snapshot we have data where marked. At level 0 all the data is available.



- AMR algorithm ≡ **"time evolution"** of **grid functions** $(u_0^{t_0}, G_0^{t_0}), \dots, (u_L^{t_L}, G_L^{t_L})$ with data $u_l^{t_l}$ attached to grid points indexed by subsets $G_l^{t_l}$ and associated to times $t_0 \geq t_1 \geq \cdots \geq t_L$ (coarser levels evolve "faster" to provide interpolation data to finer levels).
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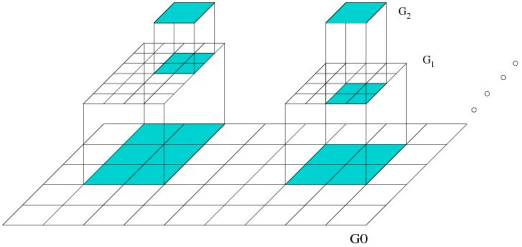

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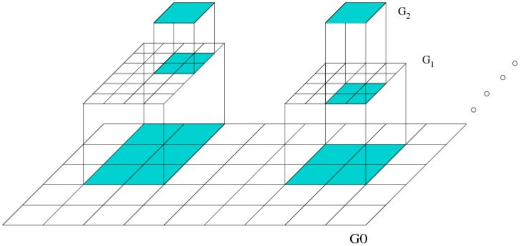

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Well-balanced schemes

- The convergence of the scheme is usually proved (when possible) through its consistence and stability (this being the harder part).
- But, when seeking convergence to a steady state, it is plausible to require the scheme to **preserve steady states**.
- When the scheme

$$u_i^{n+1} = u_i^n - \Delta t \left(\frac{\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n}{\Delta x} - s_i^n \right)$$

does so, that is:

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Well-balanced schemes

- Special steady state for SWE, **water at rest** ($h + z =$constant, $v = 0$).
- If a scheme preserves this steady state solution, then the scheme is said to verify the C-property [Bermudez and Vazquez, 1994].
- It is not easy to obtain well-balanced schemes: for example, the **centered** choice $s_i^n = s(x_i, u_i^n)$ seldom yields a well-balanced scheme, for this would imply that the finite differencing of the fluxes would be exact (what is not to be expected):

$$s(x_i, u(x_i, t_n)) = f(u(x))_x(x_i, t_n) \qquad \equiv u \textbf{ solution of PDE}$$

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$$s(x_i, u(x_i, t_n)) = \left[\frac{\hat{f}^n_{i+1/2} - \hat{f}^n_{i-1/2}}{\Delta x}\right](u(x)) \quad \equiv \textbf{well-balancing}$$

- Need to synchronize the discretization of $f(u)_x$ and $s(x, u)$, i.e., need to do an **upwind** discretization of $s(x, u)$.

Well-balanced schemes

- Special steady state for SWE, **water at rest** ($h + z =$constant, $v = 0$).
- If a scheme preserves this steady state solution, then the scheme is said to verify the C-property [Bermudez and Vazquez, 1994].
- It is not easy to obtain well-balanced schemes: for example, the **centered** choice $s_i^n = s(x_i, u_i^n)$ seldom yields a well-balanced scheme, for this would imply that the finite differencing of the fluxes would be exact (what is not to be expected):

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Outline

Homogeneous discretization

- We build on [Gascón and Corberán, 2001, Caselles et al., 2009, Donat and Martínez-Gavara, 2011]: we can re-write PDE in "homogeneous" form:

$$u_t + f(u)_x = s(x, u) \Leftrightarrow u_t + g[u]_x = 0$$

  where the **functional** g (dependent on f and s) acts on $u = u(x, t)$ as:

$$g[u](x, t) = f(u(x, t)) + b[u](x, t), \quad b[u](x, t) = -\int_0^x s(r, u(r, t))\, dr$$

- Notice that $u(x)$ is a (stationary) solution of PDE $\Leftrightarrow g[u]_x = 0$.
- We can derive upwind numerical methods for **non-homogeneous** conservation law from well established techniques for **homogeneous** conservation laws.

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Homogeneous discretization

- [Donat and Martínez-Gavara, 2011] propose a **Lax-Wendroff**-type discretization for $u_t + g[u]_x = 0$, which is hybridized with a first order monotone scheme through **flux-limiting** techniques that can be written in terms of $\Delta\hat{g}^n_{i-\frac{1}{2}}, \Delta\hat{g}^n_{i+\frac{1}{2}}$ as follows:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(\overbrace{A_i^n \Delta\hat{g}^n_{i-\frac{1}{2}} + B_i^n \Delta\hat{g}^n_{i+\frac{1}{2}}}^{G^n_{i+\frac{1}{2}} - G^n_{i-\frac{1}{2}}})$$

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Outline

Well-balanced AMR

- Ingredients of AMR algorithm:
  - Basic numerical scheme.
  - Coarse to fine communication (interpolation).
  - Fine to coarse communication (projection).
- If AMR algorithm is to preserve stationary solutions $\Rightarrow$ each ingredient should preserve them $\Rightarrow$ need well-balanced interpolation and projection.

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Well-balanced interpolation: cell approach

- In cell-based grid hierarchy, projection is given by $h_{i+\frac{1}{2}} = \frac{1}{2}(h_i + h_{i+1})$, where indexes indicate the point the data is attached to.
- If $h_i = h(x_i)$ correspond to a water at rest solution, does $h_{i+\frac{1}{2}} = \frac{1}{2}(h_i + h_{i+1})$ correspond to point values (at $x_{i+\frac{1}{2}}$) of the solution?
- If it were so, from $h(x) = \eta - z(x)$ we get

$$h_{i+\frac{1}{2}} = \frac{1}{2}\Big(h(x_i) + h(x_{i+1})\Big) = \eta - \frac{1}{2}\Big(z(x_i) + z(x_{i+1})\Big)$$
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so z should verify

$$\frac{z(x_i) + z(x_{i+1})}{2} = z\Big(\frac{x_i + x_{i+1}}{2}\Big), \forall i,$$

which does not hold for general z.

- Projection OK if h_i are cell-averages of stationary solution, but then underlying scheme should preserve them (which is not clear, because the well-balancing of an scheme tipically depends on point values).

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Well-balanced interpolation: point-value approach

- For point value grid hierarchy, the projection from level $l+1$ to level l is given by copying values with even indexes, corresponding to the same point-values, so this projection is automatically well-balanced.
- How can we get a **well-balanced** interpolation?
- If we can re-write $f(u)_x = s(x,u)$ as $V(x,u)_x = 0$, then

$$u(x) \text{ is solution of PDE} \Leftrightarrow V(x, u(x)) \text{ is constant.}$$

- $V(x,u) \equiv$ **equilibrium variables**
- In case of SWE, the equilibrium variables are:

$$V(x, \begin{bmatrix} h \\ hv \end{bmatrix}) = \begin{bmatrix} \frac{v^2}{2} + g(h + z(x)) \\ hv \end{bmatrix}$$

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- Given interpolator $I((w_i); x)$ (i.e., $I((w_i), x_j) = w_j$), if $V(x, \cdot)$ is bijective onto some relevant range then can define a **well-balanced** interpolator by

$$\tilde{I}((u_i); x) = V(x, \cdot)^{-1}(I((V_i); x)), \quad V_i = V(x_i, u_i)$$

(i.e., $\tilde{I}$ interpolates V variables obtained from u and gets back to u variables)

- Since I preserves constants, then $\tilde{I}$ preserves stationary states:

$V(x, u(x)) = K, \forall x \Rightarrow V(x, \tilde{I}((u(x_i)); x)) = K, \forall x \Rightarrow \tilde{I}((u(x_i)); x)$ is a stationary solution

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(i.e., interpolate total heights, then subtract bottom height) and get an interpolation that preserves water at rest solutions.

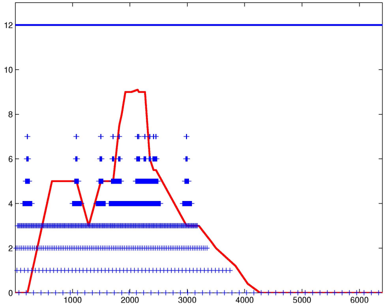
Outline

Tests setup

- We use point-value-based grid hierarchy, with well-balanced interpolation based on linear interpolation.
- Refinement criterion: mark cells to refine when interpolation error exceeds some relative error **rtol** on the maximum interpolation error at each level.
- Δt is adjusted according to the maximum of characteristic speeds to get a Courant number of 0.4.

Test for stationary solutions

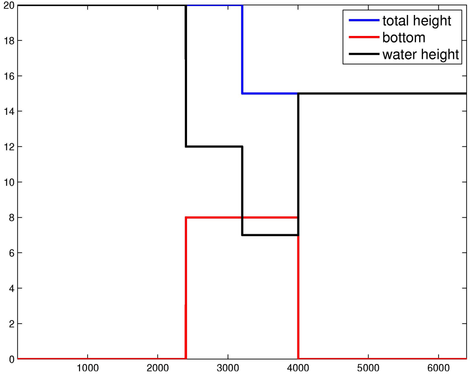
- Water at rest solution of total height=12, bottom topography below. Solution at $T = 200$.
- Have used rtol=10^{-1} $N_0 = 50$, and eight levels ($L = 7$, $N_7 = 6400$) to obtain:



- 4.59% of total integrations (with respect to equivalent finest fixed grid computation), CPU speedup ≈ 11.5.
- Scheme gives approximated solution with water height h such that $||h + z - 12||_\infty = 1.06 \cdot 10^{-14}$ and $||v||_\infty = 3.36 \cdot 10^{-14} \Rightarrow$ **C-property OK to double precision**.

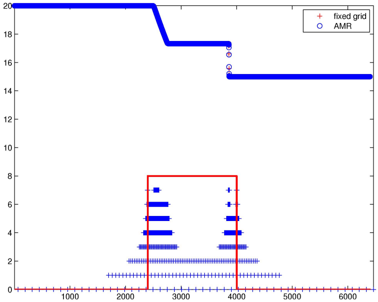
Test for non stationary 1D solutions

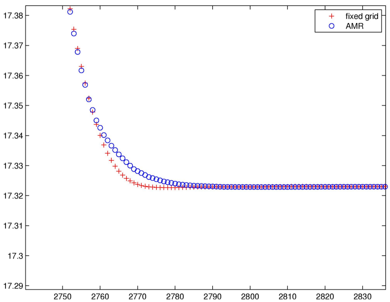
- Dam break problem with bottom topography. Solution at $T = 15$:



Test for non stationary 1D solutions

- Have used rtol=10^{-3}, $N_0 = 50$, and eight levels ($L = 7$, $N_7 = 6400$) to obtain:

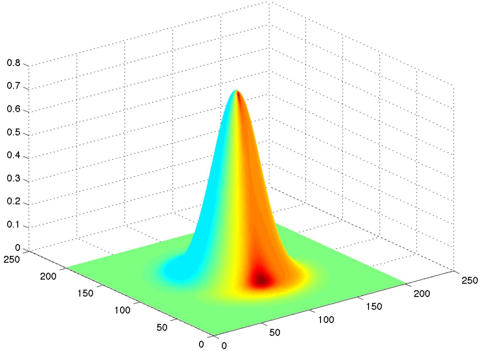




- 4.74% of total integrations (with respect to equivalent finest fixed grid computation), CPU speedup ≈ 14.04.
- Scheme gives approximated solution with water height h such that $||h_{AMR} - h_{fixed}|| = 1.44 \cdot 10^{-4}$, $||v_{AMR} - v_{fixed}|| = 1.47 \cdot 10^{-4}$

Test for stationary 2D solutions

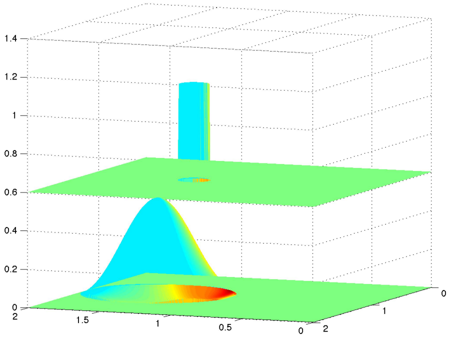
- Water at rest, total height$= 1$ and bottom:



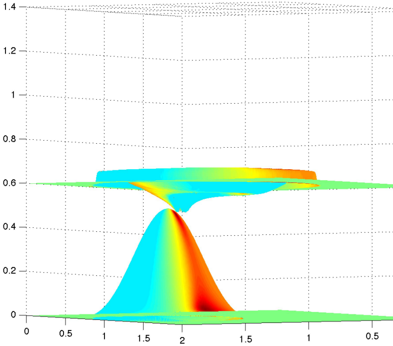
- Have used rtol=10^{-1}, $N_0 = 25$, and 4 levels ($L = 3$, $N_3 = 200$), $T = 0.1$ to obtain: $||h + z - 1||_\infty = 1.1102e - 15$, $||v^x||_\infty = 3.5162e - 15$, $||v^y||_\infty = 3.8820e - 15 \Rightarrow$ **C-property OK to double precision.**
- 22.77% of total integrations, cpu speedup=3.96

Test for non stationary 2D solutions

- Circular dam break problem. Have used rtol=10^{-1}, $N_0 = 25$, and 5 levels ($L = 4$, $N_4 = 400$), $T = 0.25$



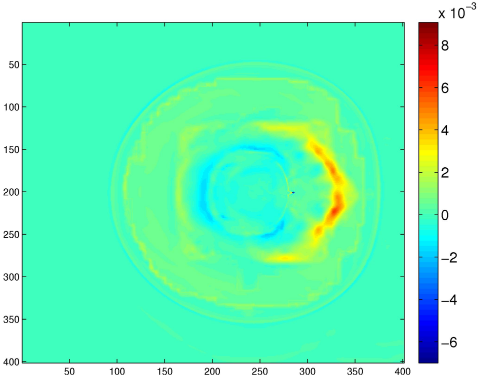
$T = 0$



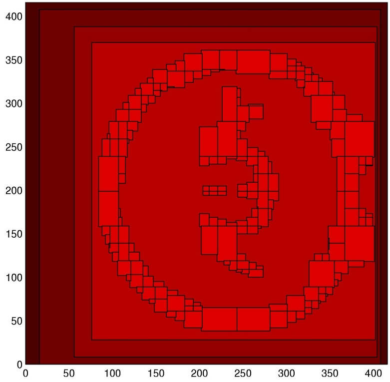
$T = 0.25$

- 29.22% of total integrations, cpu speedup=3.72
- $\|h_{AMR} - h_{fixed}\|_\infty = 0.009$, difference of mass$\approx$ 0.02%.

Test for non stationary 2D solutions



$h_{AMR} - h_{fixed}$



Grid hierarchy (the lighter, the finer)

Conclusions

- We have presented a technique for obtaining well-balanced point-value-based adaptive mesh refinement schemes for shallow water equations.
- We have seen some of the difficulties for getting a well-balanced cell-based adaptive mesh refinement schemes for SWE.
- We have tested the scheme with Donat&Martinez-Gavara homogenized SWE solver and we have obtained an adaptive scheme with the exact C-property.
- We are working on extending it to deal with dry zones.
- Possibility of getting an adaptive scheme that preserves more stationary solutions if underlying scheme does so.

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