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Wave propagation in discrete heterogeneous media

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Finite difference approximation of the homogeneous wave equation on a non-uniform grid $y = (y_j)_{0\leq j\leq N+1}$ of $(0,1)$:

$$\begin{cases} u_j''(t) - \dfrac{\frac{u_{j+1}(t)-u_j(t)}{y_{j+1}-y_j} - \frac{u_j(t)-u_{j-1}(t)}{y_j-y_{j-1}}}{\frac{y_{j+1}-y_{j-1}}{2}} = 0, \\ u_0(t) = u_{N+1}(t) = 0, \ u_j(0) = u_j^0, u_j'(0) = u_j^1, \ 1 \leq j \leq N. \end{cases} \tag{1}$$

Aim: analyze the observability inequality for (1):

$$E_h(\mathbf{u}^{h,0}, \mathbf{u}^{h,1}) \leq C_h(T) \int_0^T |\partial_n^h \mathbf{u}^h(t)|^2 \, dt,$$

where $\partial_n^h \mathbf{u}^h(t) = -u_N(t)/(1-y_N)$ and

$$E_h(\mathbf{u}^{h,0}, \mathbf{u}^{h,1}) = \frac{1}{2}\sum_{j=1}^{N} \frac{y_{j+1}-y_{j-1}}{2} |u_j'(t)|^2 + \frac{1}{2}\sum_{j=0}^{N} (y_{j+1}-y_j) \left| \frac{u_{j+1}(t)-u_j(t)}{y_{j+1}-y_j} \right|^2.$$

Pathologies and remedies for numerical approximations of waves on **uniform** media $y = x = [0:h:1]$, $h = 1/(N+1)$:

Ervedoza, Zuazua, *The wave equation: control and numerics*, CIME Subseries, Springer.

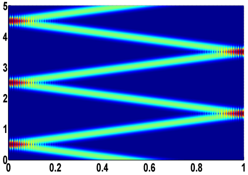
Rays of Geometric Optics:

Bardos, Lebeau, Rauch, *Sharp sufficient conditions for observation, control and stabilization of waves from the boundary*, SIAM J. Cont. Optim., 1992.

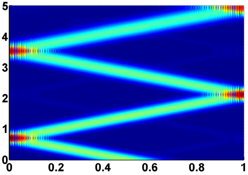
Rays of GO - continuous and discrete on uniform meshes cases

Continuous rays: $x(t) = x^* \pm t$

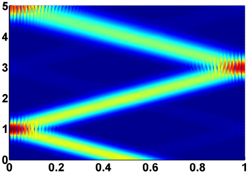
Discrete rays on uniform meshes: $x(t) = x^* \pm t\omega_h'(\xi_0)$, where $\omega_h(\xi) := 2\sin(\xi h/2)/h$.



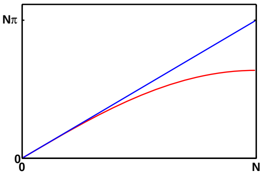
(a) continuous



(b) discrete, $\xi_0 = \pi/2h$



(c) discrete, $\xi_0 = 2\pi/3h$



(d) Continuous (blue) and discrete (red) dispersion relations

RESOLVENT ESTIMATES

Ervedoza, *Spectral conditions for admissibility and observability of wave systems*, Numer. Mathematik, 2009.

The observability inequality for general finite element semi-discretizations of the wave equation corresponding to a convergent approximation of order h^θ of the Laplacian

holds uniformly as $h \to 0$

in a class of truncated solutions generated by eigenvalues of order $(\epsilon/h^\theta)^2$, for some $\epsilon > 0$, which does not include the critical scale $1/h^2$ appearing in the numerical approximations on uniform meshes.

MIXED FINITE ELEMENTS METHODS

Ervedoza, *On the mixed finite element method for the $1-d$ wave equation on non-uniform meshes*, ESAIM:COCV, 2010.

The numerical scheme:

$$\frac{y_{j+1}-y_j}{4}(u''_{j+1}(t)+u''_j(t))+\frac{y_j-y_{j-1}}{4}(u''_j(t)+u''_{j-1}(t))-\frac{u_{j+1}(t)-u_j(t)}{y_{j+1}-y_j}-\frac{u_j(t)-u_{j-1}(t)}{y_j-y_{j-1}}=0.$$

Eigenvalues λ:

$$\frac{k\pi}{2}=\sum_{j=0}^{N}\arctan\frac{\lambda(y_{j+1}-y_j)}{2},\ 1\le k\le N.$$

SPECTRAL DISTRIBUTION OF LOCALLY TOEPLITZ SEQUENCE MATRICES

Beckerman, Serra-Capizzano, *On the asymptotic spectrum of finite element matrix sequences*, SINUM, 2007.

Tilli, *Locally Toeplitz sequences: spectral properties and applications*, Lin. Alg. Appl., 1998.

Szegö Theorem. Uniform mesh + constant coefficients. $T_N(\omega)$ = Toeplitz matrix whose diagonals are Fourier coefficients ω_j $(0 \leq j \leq N)$ of $\omega : (-\pi, \pi) \to \mathbb{C}$ and $(\lambda_j)_{1 \leq j \leq N}$ are the eigenvalues of $T_N(\omega)$. Then $\forall F \in C_c^b(\mathbb{R})$:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} F(\lambda_j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega(\xi))\, d\xi.$$

Uniform mesh + variable coefficients. $T_N(\omega, a)$ = locally Toeplitz matrix, $a : (0,1) \to \mathbb{R}$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} F(\lambda_j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{0}^{1} F(\omega(\xi) a(x))\, dx\, d\xi.$$

Non-uniform mesh + variable coefficients. For $-(a(x)u_x)_x = b(x), x \in (0,1)$ discretized using a scheme generating a Toeplitz matrix whose diagonals are generated by ω and on a non-uniform mesh $y : (0,1) \to (0,1)$, we obtain the locally Toeplitz matrix $T_N(\omega, a, y)$ s.t.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} F(\lambda_j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{0}^{1} F\left(\frac{a(y(x))}{|y'(x)|^2} \omega(\xi) \right)\, dx\, d\xi.$$

The $1-d$ transport equation:

$$\rho(x)u_t^\epsilon(x,t) - u_x^\epsilon(x,t) = 0,\ x\in\mathbb{R}, t>0, \quad u^\epsilon(x,0) = \varphi^\epsilon(x),\ x\in\mathbb{R}. \tag{2}$$

$||u^\epsilon(\cdot,t)||L^2(\mathbb{R},\rho)$ is conserved in time.

If $\rho\in L^\infty(\mathbb{R})$, $\rho\geq 0$ and φ^ϵ is uniformly bounded in $L^2(\mathbb{R})$, then

$\rho(x)|u^\epsilon(x,t)|^2$ is uniformly bounded in $L^1(\mathbb{R})$, so that

$$\rho(\cdot)|u^\epsilon(\cdot,t)|^2 \rightharpoonup \mu, \text{ weakly } \star \text{ in } \mathcal{M}(\mathbb{R}).$$

The ρ-weighted Wigner transform of u^ϵ:

$$w^\epsilon(x,t,\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} \rho(x)u^\epsilon(x+\epsilon z/2,t)\overline{u}^\epsilon(x-\epsilon z/2,t)\exp(-i\xi z)\,dz. \tag{3}$$

$f^\epsilon(x,t,z) := (\mathcal{F}_{\xi\to z}w^\epsilon)(x,t,z)$ - the Fourier transform of w^ϵ in ξ.

f^ϵ satisfies the following equation:

$$\begin{aligned} f_t^\epsilon(x,t,z) &= \frac{1}{2}\Big(\frac{1}{\rho(x+\epsilon z/2)} + \frac{1}{\rho(x-\epsilon z/2)}\Big)f_x^\epsilon(x,t,z) \\ &- \frac{\rho'(x)}{\rho(x)}\frac{1}{2}\Big(\frac{1}{\rho(x+\epsilon z/2)} + \frac{1}{\rho(x-\epsilon z/2)}\Big)f^\epsilon(x,t,z) \\ &+ \frac{1}{\epsilon}\frac{1}{2}\Big(\frac{1}{\rho(x+\epsilon z/2)} - \frac{1}{\rho(x-\epsilon z/2)}\Big)f_z^\epsilon(x,t,z). \end{aligned} \tag{4}$$

$$K_1^\epsilon(x,\xi) := \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{2}\Big(\frac{1}{\rho(x+\epsilon z/2)} + \frac{1}{\rho(x-\epsilon z/2)}\Big)\exp(-i\xi z)\,dz \tag{5}$$

$$K_2^\epsilon(x,\xi) := \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{\epsilon z}\Big(\frac{1}{\rho(x+\epsilon z/2)} - \frac{1}{\rho(x-\epsilon z/2)}\Big)\exp(-i\xi z)\,dz. \tag{6}$$

The Wigner transform $w^\epsilon(x,t,\xi)$ satisfies the following equation:

$$\begin{aligned} w_t^\epsilon(x,t,\xi) = K_1^\epsilon(x,\cdot) * w_x^\epsilon(x,t,\xi) - \frac{\rho'(x)}{\rho(x)} K_1^\epsilon(x,\cdot) * w^\epsilon(x,t,\xi) \\ - K_2^\epsilon(x,\cdot) * w^\epsilon(x,t,\xi) - K_2^\epsilon(x,\cdot) * \xi w_\xi^\epsilon(x,t,\xi). \end{aligned} \tag{7}$$

Formally, $K_1^\epsilon(x,\xi) \to 1/\rho(x)\delta_0(\xi)$ and $K_2^\epsilon(x,\xi) \to (1/\rho)'(x)\delta_0(\xi)$, so that $w^\epsilon(x,t,\xi)$ converges to a function $w(x,t,\xi)$ which verifies the equation:

$$w_t(x,t,\xi) = \frac{1}{\rho(x)} w_x(x,t,\xi) + \frac{\rho'(x)}{\rho^2(x)}\xi w_\xi(x,t,\xi). \tag{8}$$

Characteristics verifying the Hamiltonian system:

$$x'(t) = -\frac{1}{\rho(x(t))} \text{ and } \xi'(t) = -\frac{\rho'(x(t))}{\rho^2(x(t))}\xi(t).$$

Uniqueness for the Hamiltonian system iff $\rho \in C^{1,1}(\mathbb{R})$.

Gérard, Markowich, Mauser, Poupaud, *Homogenization limits and Wigner transforms*, Comm. Pure Appl. Math., 1997.

Lions P.-L., Paul, *Sur les mesures de Wigner*, Rev. Matemática Iberoamericana, 1993.

Schrödinger equation with potential $ih\psi_t^h = -h^2\psi_{xx}^h + V(x)\psi^h,\ (x,t) \in \mathbb{R} \times \mathbb{R}$,

is transformed into Liouville equation $w_t + \xi w_x - V_x w_\xi = 0$, which needs $V \in C^{1,1}(\mathbb{R})$ for uniqueness.

FINITE DIFFERENCE APPROXIMATION of the transport equation

$$\rho(x)u_t^\epsilon(x,t) - \frac{u^\epsilon(x+\epsilon,t) - u^\epsilon(x-\epsilon,t)}{2\epsilon} = 0,\ \ x \in \mathbb{R}, t > 0. \tag{9}$$

$f^\epsilon(x,t,z)$ verifies the equation:

$$\begin{aligned} f_t^\epsilon(x,t,z) = {} & \rho(x)\frac{f^\epsilon(x+\epsilon/2,t,z+1)}{2\epsilon\rho(x+\epsilon z/2)\rho(x+\epsilon/2)} - \rho(x)\frac{f^\epsilon(x-\epsilon/2,t,z-1)}{2\epsilon\rho(x+\epsilon z/2)\rho(x-\epsilon/2)} \\ & + \rho(x)\frac{f^\epsilon(x+\epsilon/2,t,z-1)}{2\epsilon\rho(x-\epsilon z/2)\rho(x+\epsilon/2)} - \rho(x)\frac{f^\epsilon(x-\epsilon/2,t,z+1)}{2\epsilon\rho(x-\epsilon z/2)\rho(x-\epsilon/2)}. \end{aligned} \tag{10}$$

$$
\begin{aligned}
&w_t^\epsilon(x,t,\xi) = \\
&= \frac{\rho(x)}{4}\Big(\frac{1}{\rho(x+\epsilon/2)} + \frac{1}{\rho(x-\epsilon/2)}\Big) K_1^\epsilon(x,\cdot) * 2\cos(\xi)\frac{w^\epsilon(x+\epsilon/2,t,\xi) - w^\epsilon(x-\epsilon/2,t,\xi)}{\epsilon} \\
&+ \frac{\rho(x)}{8}\Big(\frac{1}{\rho(x+\epsilon/2)} + \frac{1}{\rho(x-\epsilon/2)}\Big) K_2^\epsilon(x,\cdot) * \Big(-2\sin(\xi)\big(w_\xi^\epsilon(x+\epsilon/2,t,\xi) + w_\xi^\epsilon(x-\epsilon/2,t,\xi)\big) \\
&- 2\cos(\xi)\big(w^\epsilon(x+\epsilon/2,t,\xi) + w^\epsilon(x-\epsilon/2,t,\xi)\big)\Big) \\
&+ \frac{\rho(x)}{4\epsilon}\Big(\frac{1}{\rho(x+\epsilon/2)} - \frac{1}{\rho(x-\epsilon/2)}\Big) K_1^\epsilon(x,\cdot) * 2\cos(\xi)(w^\epsilon(x+\epsilon/2,t,\xi) + w^\epsilon(x-\epsilon/2,t,\xi)) \\
&+ \epsilon\frac{\rho(x)}{8\epsilon}\Big(\frac{1}{\rho(x+\epsilon/2)} - \frac{1}{\rho(x-\epsilon/2)}\Big) K_2^\epsilon(x,\cdot) * \Big(-2\sin(\xi)\big(w_\xi^\epsilon(x+\epsilon/2,t,\xi) - w_\xi^\epsilon(x-\epsilon/2,t,\xi)\big) \\
&- 2\cos(\xi)\big(w^\epsilon(x+\epsilon/2,t,\xi) - w^\epsilon(x-\epsilon/2,t,\xi)\big)\Big).
\end{aligned}
$$

The limit equation is

$$
w_t(x,t,\xi) = \frac{1}{\rho(x)}\cos(\xi) w_x(x,t,\xi) + \frac{\rho'(x)}{\rho^2(x)}\sin(\xi) w_\xi(x,t,\xi).
$$

For the transport equation on a non-uniform grid $y = g(x)$, we have

$$
w_t(x,t,\xi) = \frac{1}{\rho(g(x))g'(x)}\cos(\xi) w_x(x,t,\xi) + \Big(\frac{1}{\rho(g(x))g'(x)}\Big)'\sin(\xi) w_\xi(x,t,\xi).
$$

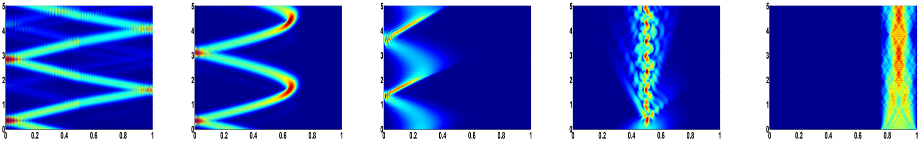
$$\boxed{u^0(y) = \varphi^{\gamma}_{\xi_0}(y - y_0)\exp(i\xi_0 y_0)}$$

Legend:
(e) $h = 1/200$, uniform grid of size h for $y \in (0, 1/2)$ and $h/2$ for $y \in (1/2, 1)$ and $y_0 = 1/4$
(f-g) $h = 1/200$, $y = \tan(\pi x/4)$, $y_0 = 1/4$; **(h)** $h = 1/200$, $y = \sin(\pi x/3)$ for $x \in (0, 1/2)$, $y = 1 - \sin(\pi(1 - x)/3)$ for $x \in (1/2, 1)$ and $y_0 = 1/2$; **(i)** $h = 1/100$, uniform grid of size $h/8$ and $h/4$ for $y \in (0, 1/4)$ and $y \in (3/4, 1)$, $y = 1/4 + \tan(x/4)/2$ for $y \in (1/4, 3/4)$, and $y_0 = 7/8$

We illustrate some phenomena, most of them being pathological and requiring further analysis:

☐ reflection-transmission problem at the interface between two piecewise uniform discrete media (see Fig. 4(e))

☐ torsion of the rays of Geometric Optics, reflecting before touching the boundary of the domain (see Fig. 4(f-i))



(e) $\xi_0 = \pi/2h$ (f) $\xi_0 = \pi/2h_{min}$ (g) $\xi_0 = \pi/h_{min}$ (h) $\xi_0 = \pi/2h_{min}$ (i) $\xi_0 = \pi/2h_{min}$

In this talk:

- We give a meaning to the notion of rays of Geometric Optics by constructing appropriate transport equations.

Open problems:

- To solve the Hamiltonian systems, we need $\rho(g(x))g'(x) \in C^{1,1}(\mathbb{R})$. Study the case when the non-homogeneity of the grid is less regular.
- Adapt the multiplier techniques to prove the observability inequality for the non-uniform mesh case (a posteriori error estimates techniques).
- Adapt the filtering techniques (the bi-grid ones) to remedy the pathological effects of the high-frequency spurious solutions.
- Study more sophisticated methods for the wave equation (DG ones, higher order ones) on non-uniform meshes.
- The multi-d case.
- Dispersive estimates for the Schrödinger equation on non-uniform meshes.
- Meshes which are given randomly.

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Thank you very much for your attention!