Stabilization for solving linear systems in context of adjoint techniques

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DLR

Evaluation of Gradients

Gradient based shape optimization:

Cost function I and the optimization problem be stated as

$$I = f(\alpha), \quad \min_{w.r.t.\,\alpha} I(\alpha)$$

α … design parameters

Problem: Find $\dfrac{\mathrm{d}I}{\mathrm{d}\alpha}$

subject to **flow equation R(W, D) = 0** with $A = \dfrac{\partial R}{\partial W}$

$$\frac{\mathrm{d}I}{\mathrm{d}\alpha} = \frac{\partial I}{\partial \alpha} + \Lambda^T B$$

$$A^T \Lambda = -b^T$$

Evaluation of Gradients
Solution methods

General solution method: $$M(W^{n+1} - W^n) = -R(W^n)$$

LU-SGS: $$M = (D+L)D^{-1}(D+U)$$

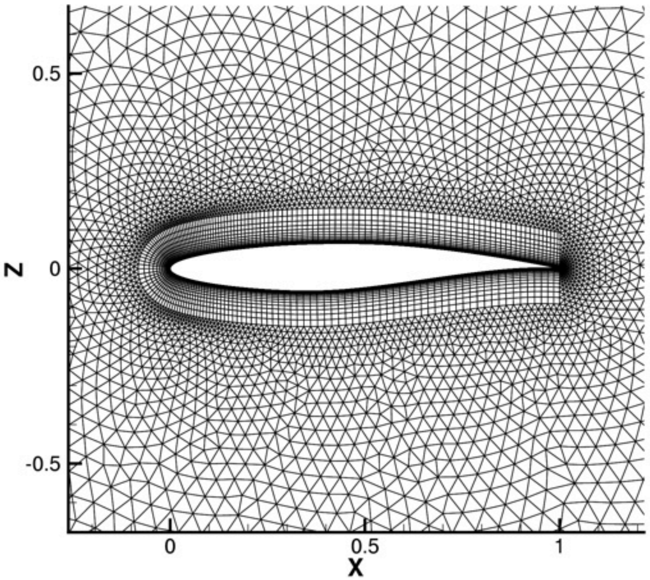
Adjoint

$$M^T(\Lambda^{n+1} - \Lambda^n) = b^T - A^T\Lambda^n$$

$$M^T = (D+L)^T D^{-T}(D+U)^T$$

Linear Iteration on RAE2822

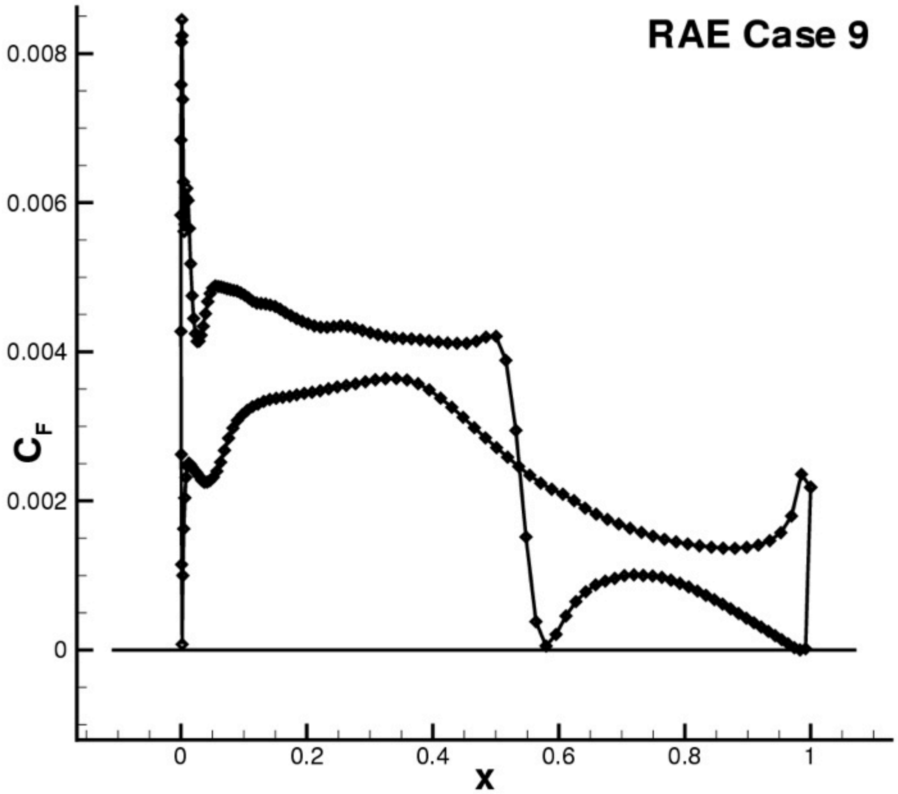
Effect of Separation

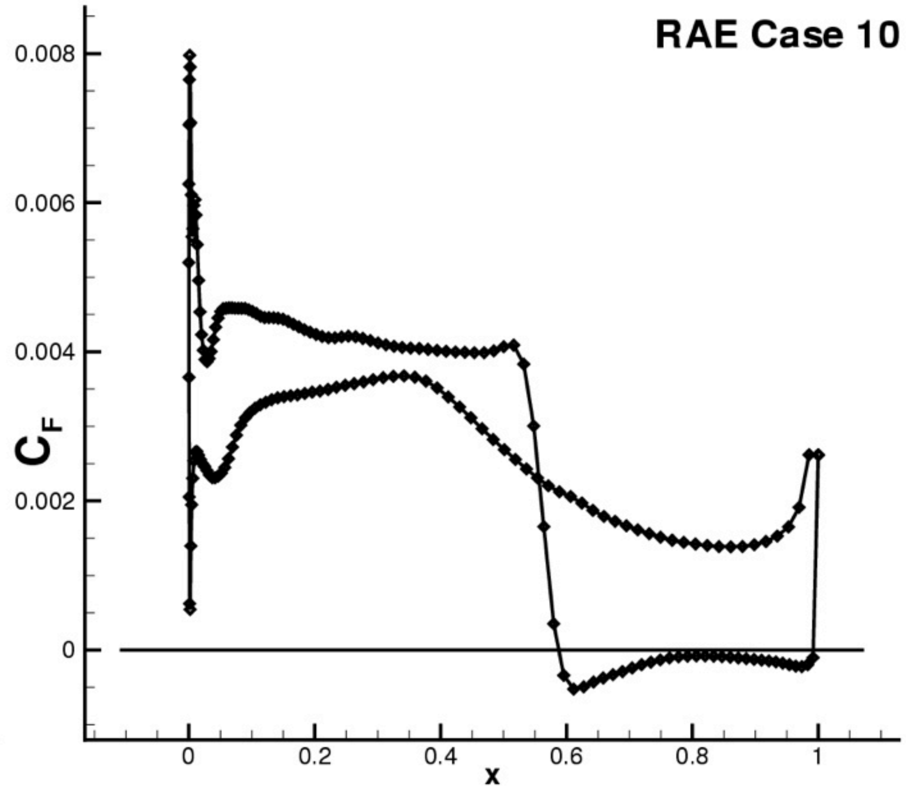


RAE Case 9	RAE Case 10
Alpha = 2.80	Alpha = 2.80
Mach = 0.73	Mach = 0.75
Re = 6.5×10^6	Re = 6.2×10^6

Linear Iteration on RAE2822

Case 9 and 10 Differences

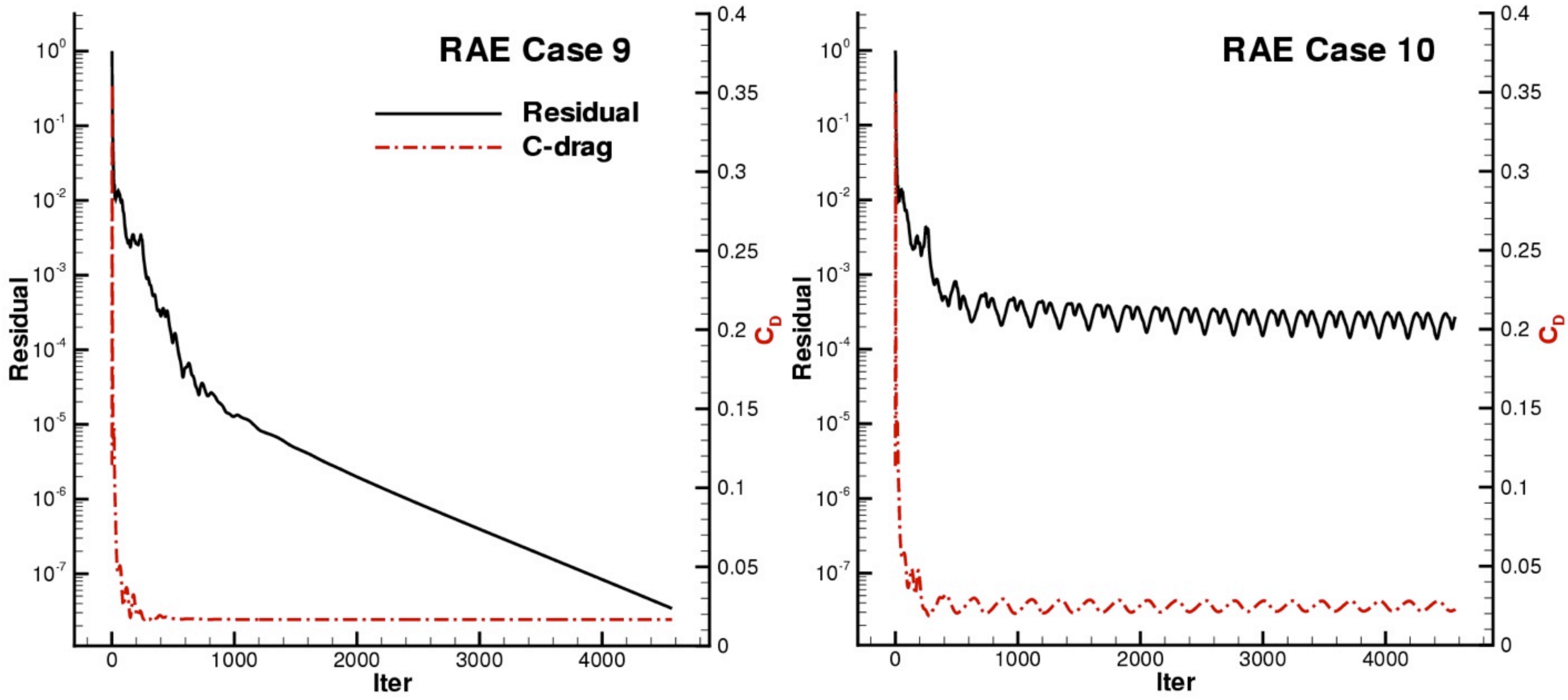




$$C_f = \frac{2}{\rho_\infty V_\infty^2}\mu_{eff}\sqrt{\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2}$$

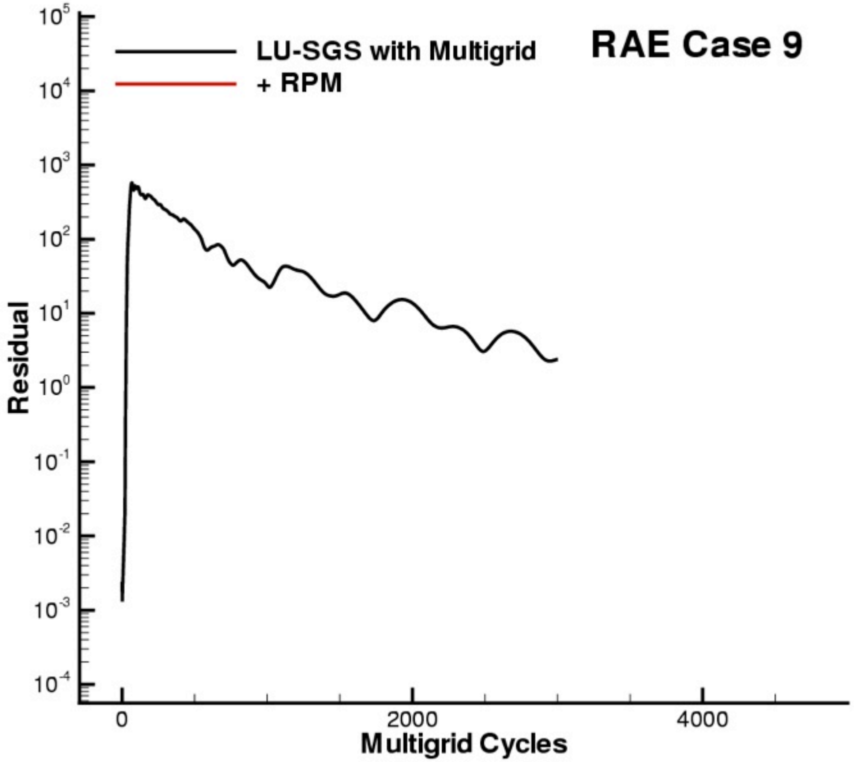
Linear Iteration on RAE2822

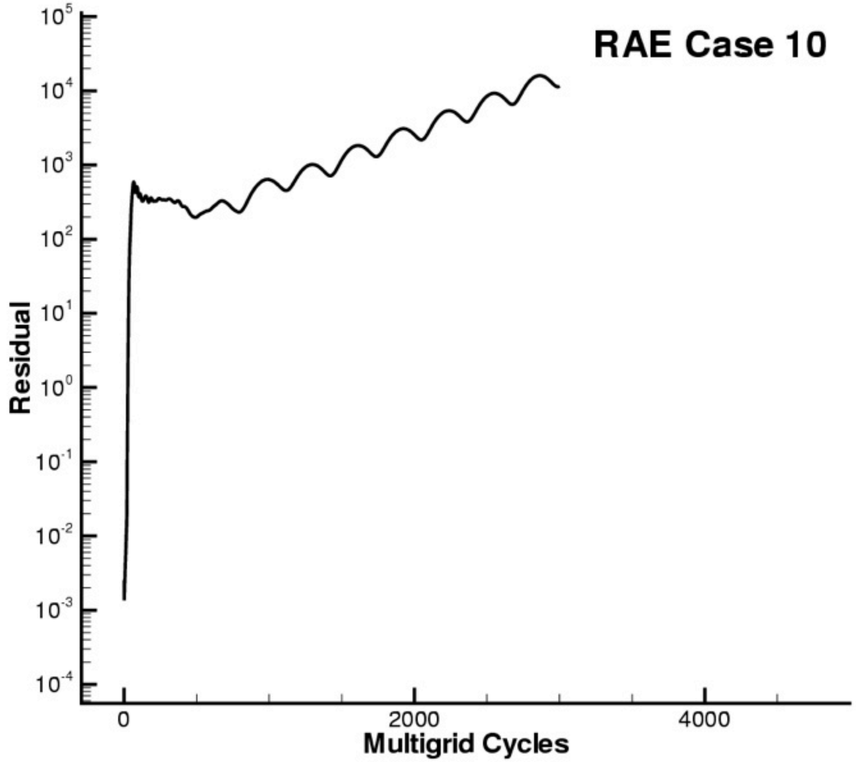
Non-linear Convergence



Linear Iteration on RAE2822

Linear Convergence





Recursive Projection Method

Idea

Problem: Iterative method N(●) is unstable.

Observation: Probably not *all* components of the iteration are unstable.

Idea:

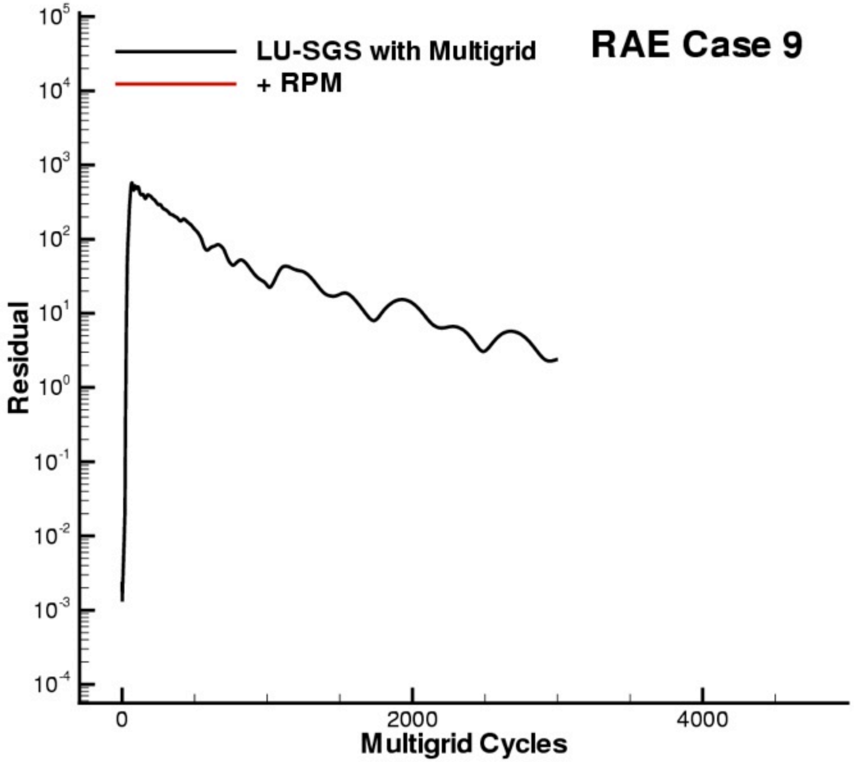
- Since it's a linear problem, decompose the solution space into a sum of eigenvectors of the iteration operator, $M^{-1}A$.

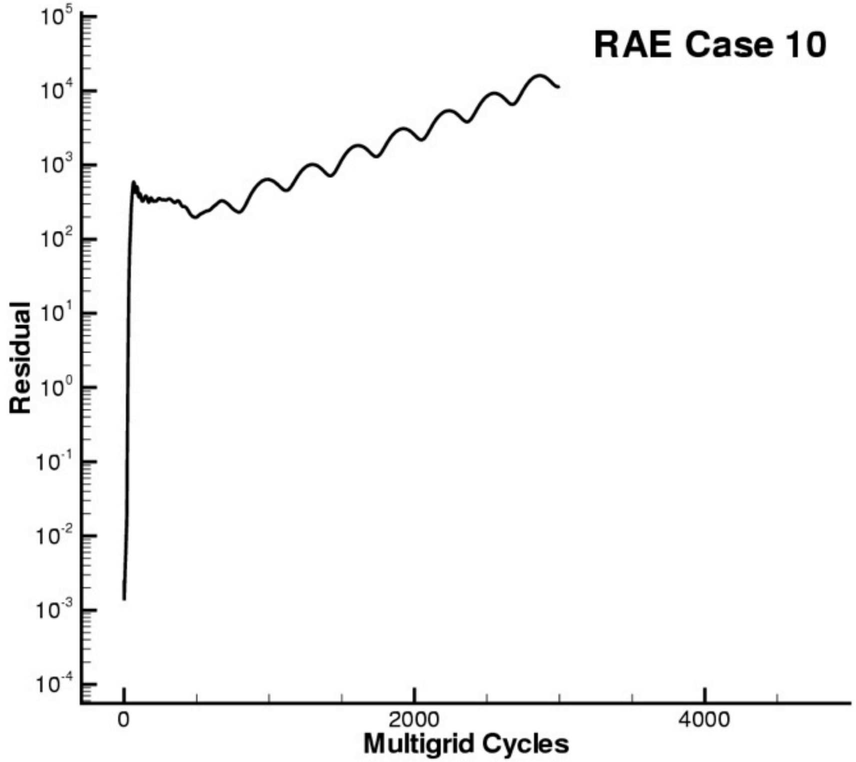
-The unstable eigenvectors give a subspace ***P***, apply a Newton method.

-Apply $M^{-1}A$ to the remaining subspace ***Q***, where it is stable.

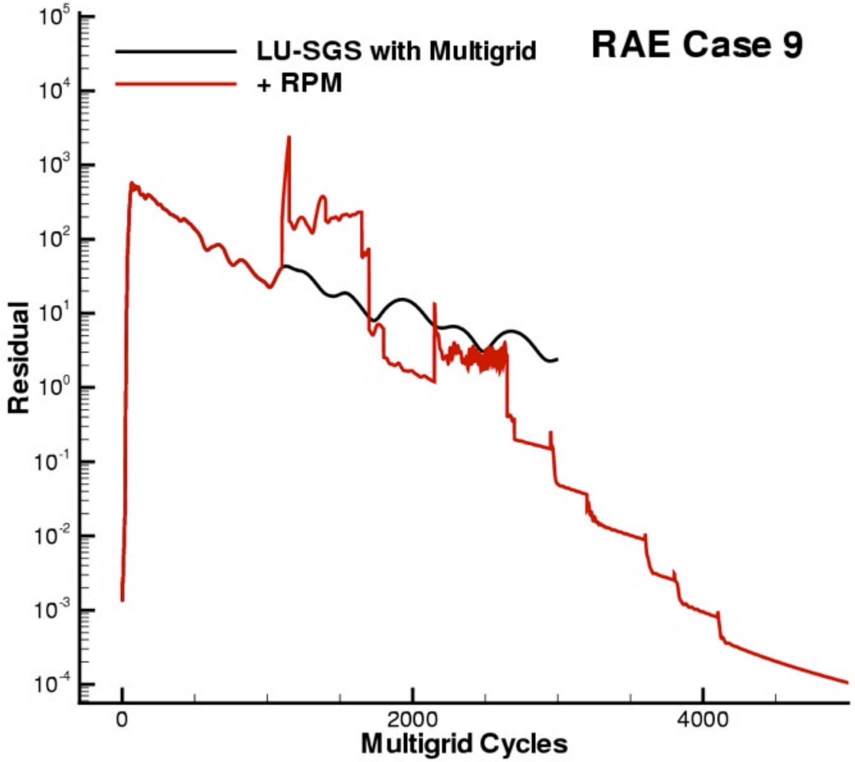
-Obtain eigenvectors with a *power iteration*.

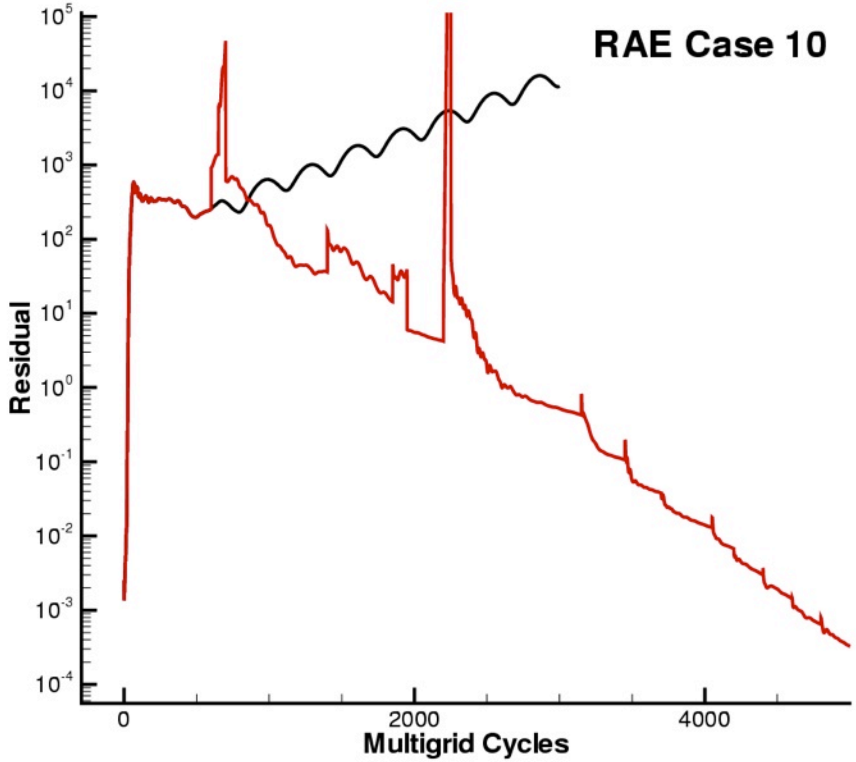
Linear Iteration on RAE2822
Linear Convergence **with RPM**





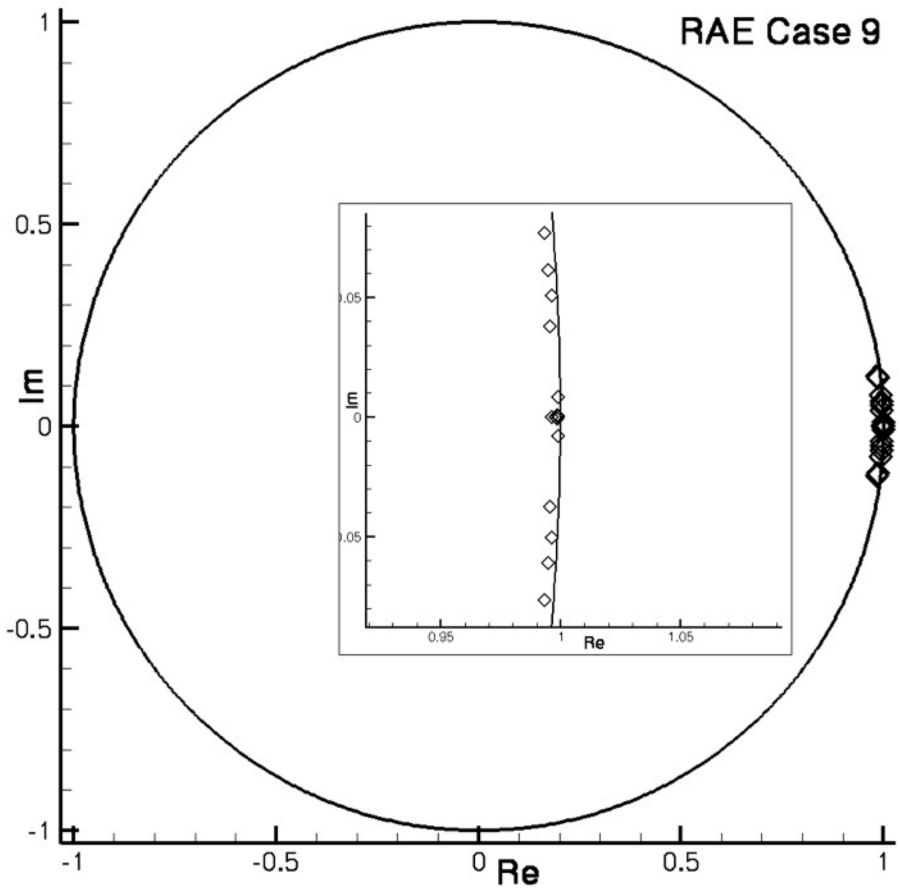
Linear Iteration on RAE2822
Linear Convergence **with RPM**

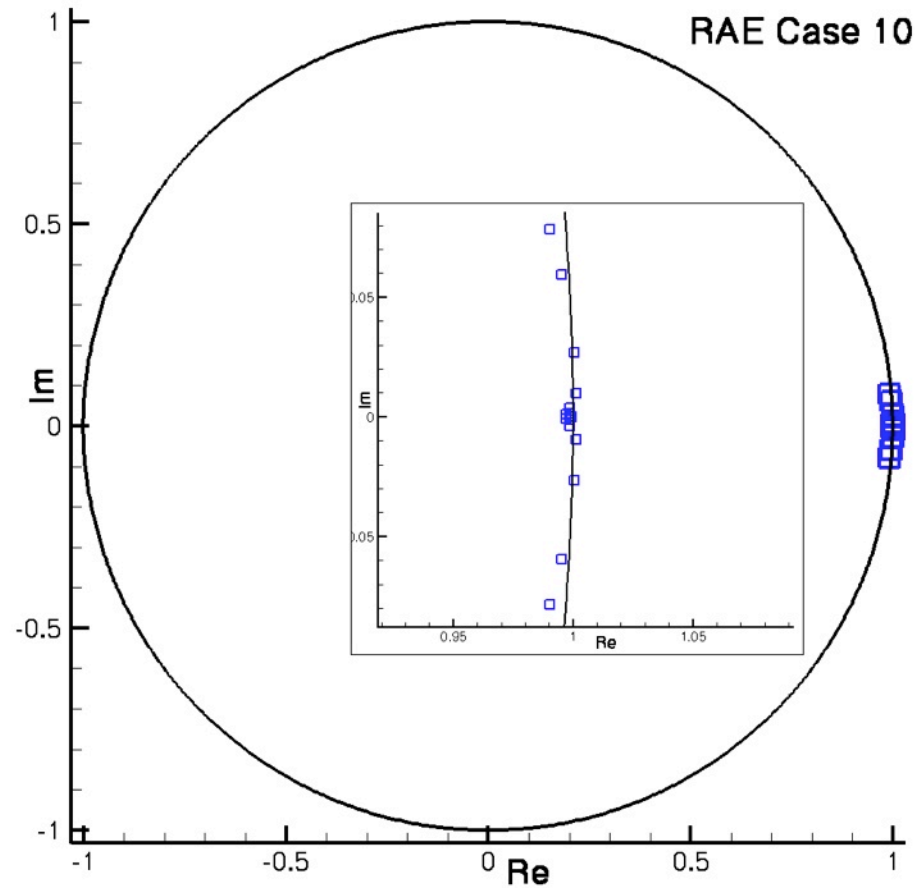




Eigenvalues of the Iteration Operator M-1A

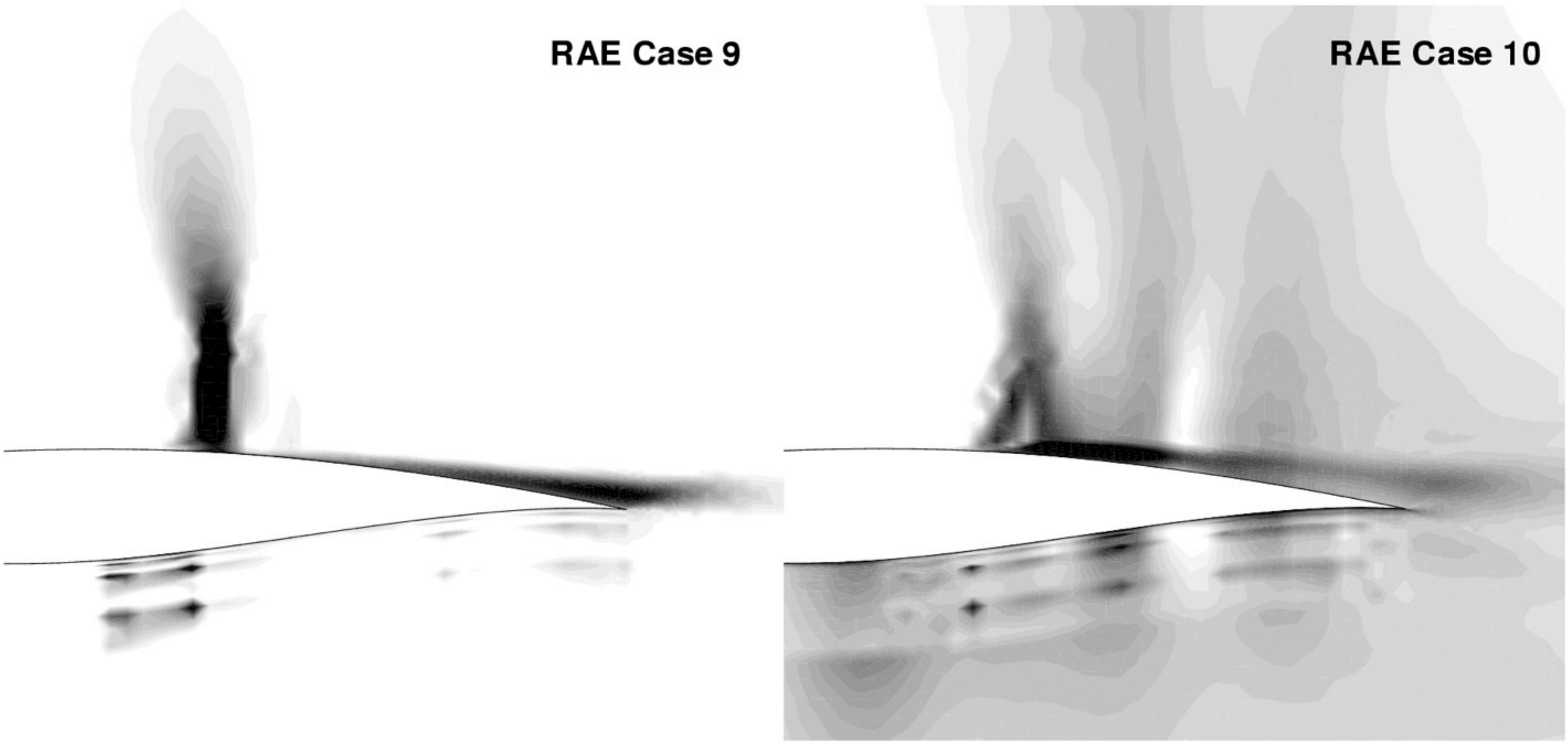
Stable and Unstable





Eigenvectors of the Iteration Operator M-1A

“Map of the Instability of the Iteration”



Conclusions & Question Marks

- We have experience in solving such equations from the non-linear world.
- In practice the solution of the linear equations is **more** difficult than the non-linear due to unstable modes.
- Need of stabilization techniques, here RPM
  - but expensive in terms of computational costs
  - very difficult to parallelize - therefor industrial applicability is restricted

- Questions?
  - Are their more efficient stabilization techniques for solving linear equations?
  - Has someone experience for parallelization of RPM?
  - Applicable stabilization techniques for linear systems?
  - May their be active stabilization instead of passive methods?

Test Case
DLR-F6

Re=$3x10^6$

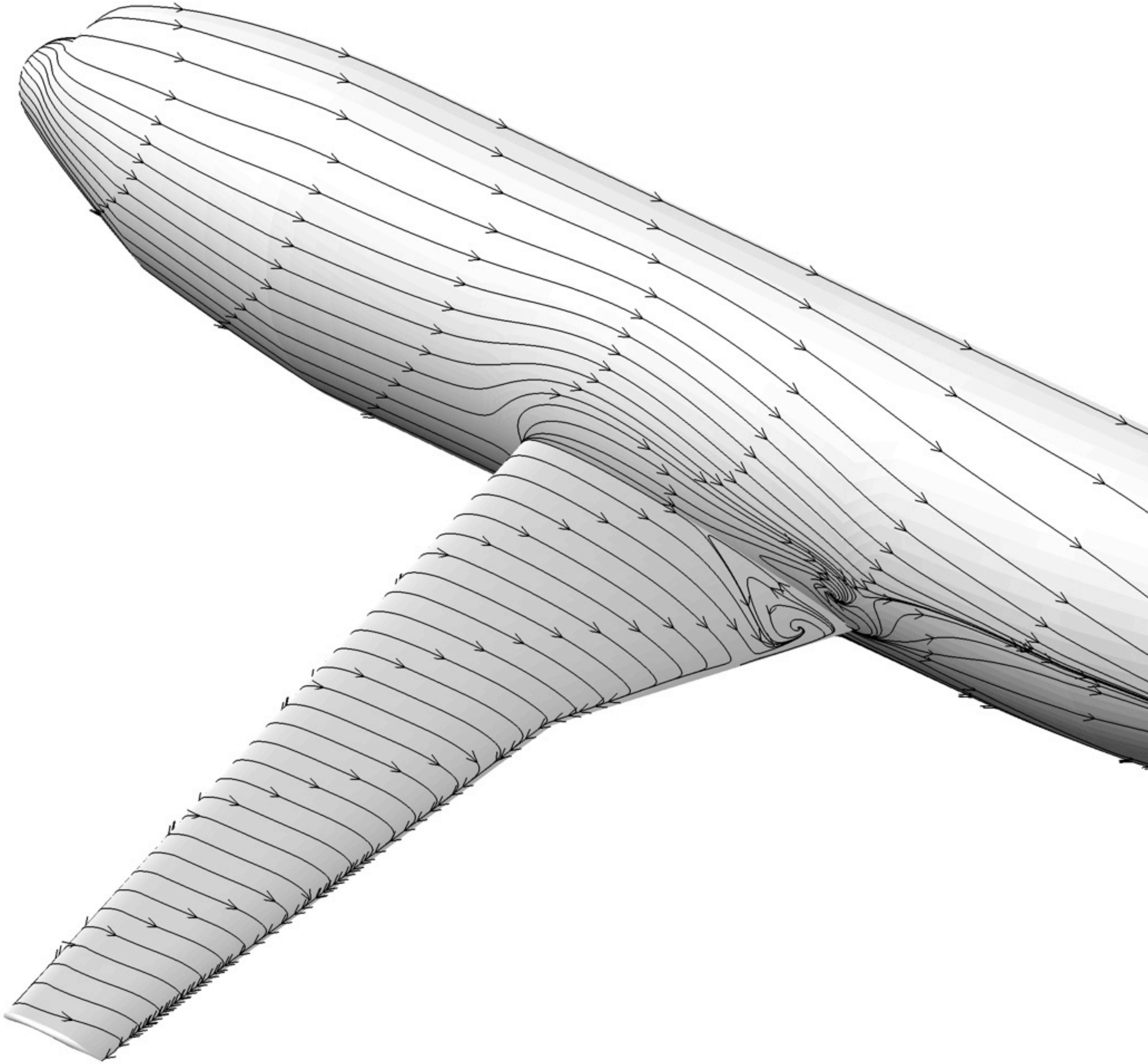
Mach=0.75

C_L=0.5

TAU calculation

JST convective flux

SAE 1-eq turbulence



Test Case
Convergence

Convergence of:

Newton iteration (on P).

Iteration on Q.

Total residual.

Test Case
Convergence

Convergence of:

Newton iteration (on P).

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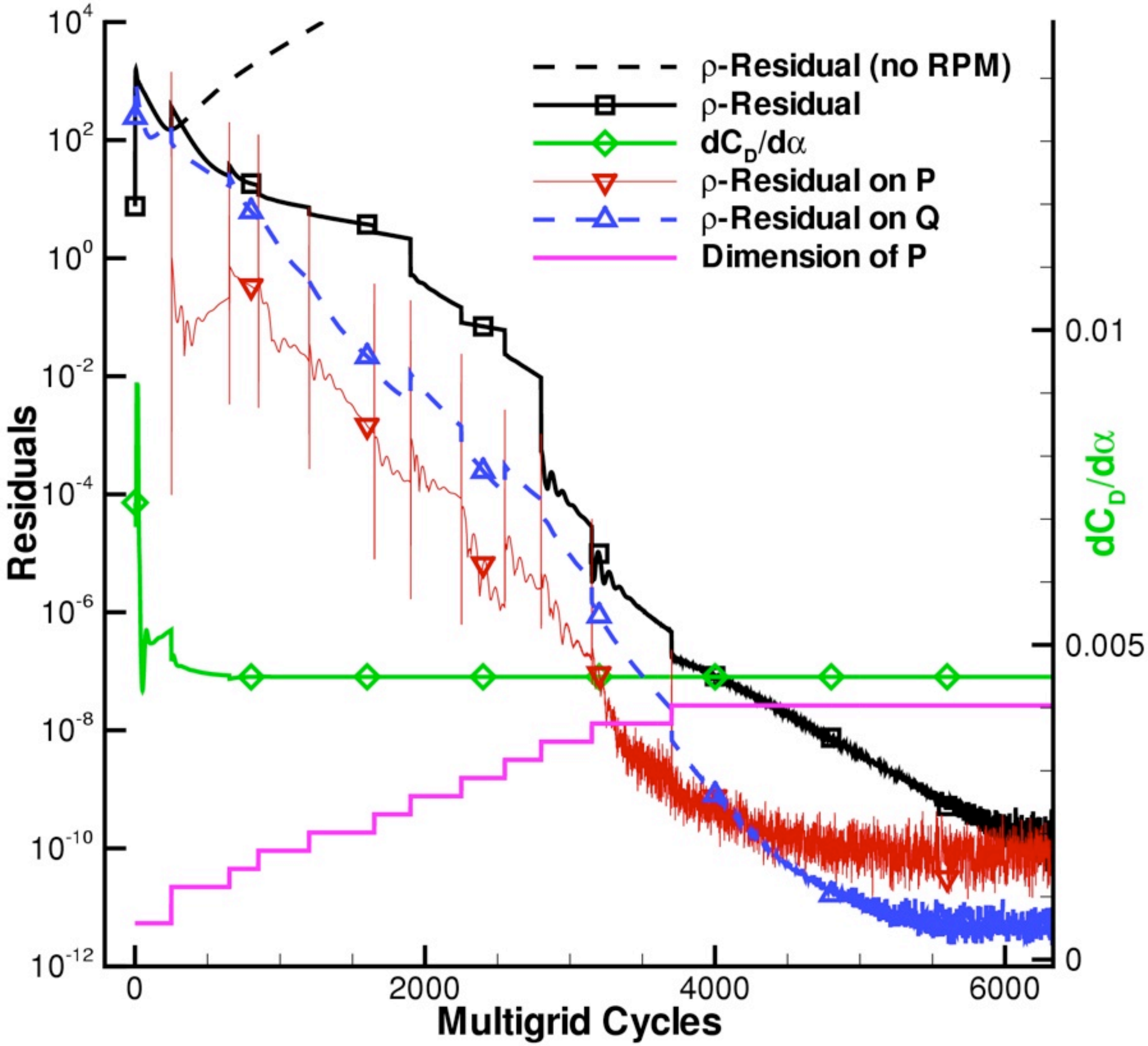
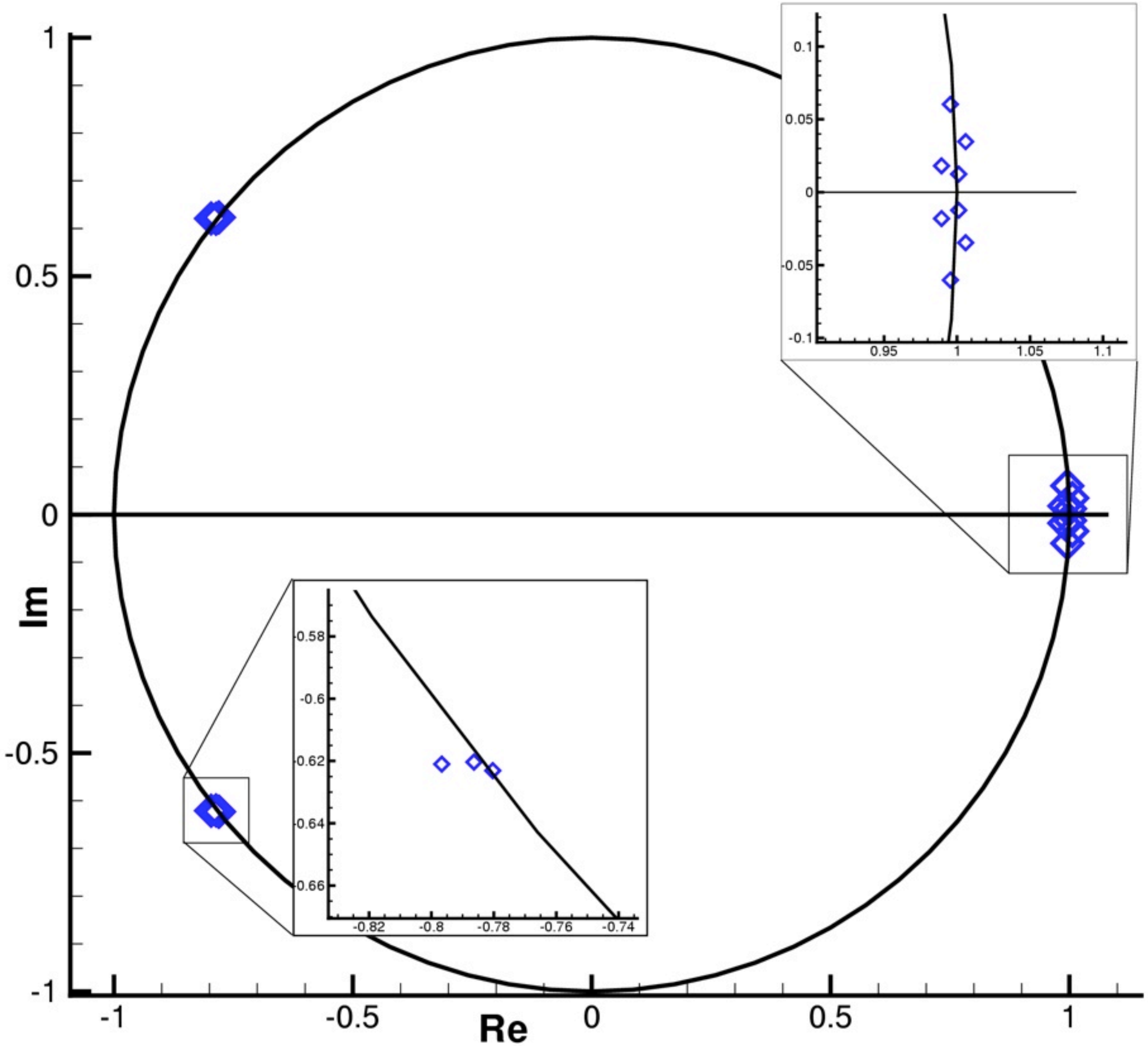


Dominant Eigenvalues of DLR-F6
With LU-SGS and Multigrid

RPM found subspace of dimension 14:

8 unstable modes
6 slowly converging modes



Recursive Projection Method
Idea

Concern solution of linear system:

$$Ax = b \qquad A \in \mathbb{R}^{N \times N},\, b \in \mathbb{R}^N \tag{1}$$

fixed-point iterative family:

$$x^{l+1} = F(x^l) = \Phi x^l + M^{-1} b \tag{2}$$

where

$$\Phi = I - M^{-1} A, \qquad M \text{...preconditioner} \tag{3}$$

is the iterative matrix of numerical scheme and if A and M are nonsingular then (2) converges.

But if eigenvalues of (3) become

$$|\lambda_1|, ..., |\lambda_n| \geq 1$$

it diverges.

Recursive Projection Method

Idea

define divergent subspace with the eigenvalues from (3):

$$\mathbb{P} = \mathrm{span}\{e_1, ..., e_m\}, \qquad \mathbb{Q} = \mathbb{P}^{\perp}$$

therefor every vector can be decomposed in a unique way as the sum of

$$\forall x \in \mathbb{R}^N; \quad \exists (x_p, x_q) \in \mathbb{P} \times \mathbb{Q}: \quad x = x_p + x_q$$

orthogonal projectors onto subspace called P and Q, then define from an orthonormal basis:

$$V \in \mathbb{R}^{N \times m}; \quad P = VV^T \text{ and } Q = I - VV^T$$

with

$$VV^T = 1; \qquad Q\Phi P = 0$$

Recursive Projection Method
Idea

RPM becomes:

$$
\begin{aligned}
x_q^{l+1} &= QF(x^l) \\
x_p^{l+1} &= x_p + (I - PQP)^{-1}(PF(x^l) - x_p^l) \\
x^{l+1} &= x_q^{l+1} + x_p^{l+1} \\
\text{with } \; x_p^0 = Px^0; \;\; x_q^0 &= Qx_q^0
\end{aligned}
$$

and it is proved by Schroff and Keller that in the limit:

$$\lim_{l \to \infty} x_p^l + x_q^l = A^{-1}b$$

Recursive Projection Method

Idea

Construction of the basis V:

$$k \in N^*$$

$$v_1 = \Delta x_q^{l-k+1}; \quad \Delta x_q^j = x_q^{j+1} - x_q$$

$$\hat{A} = Q\Phi Q$$

Remember that (2) converges when Φ contains only the stable eigenspectrum!

So define Krylov subspace of dimension k generated by v1 and hat A

$$\mathbb{K} = \mathrm{span}\{v_1, \hat{A}v_1, ..., \hat{A}^{k-1}v_1\}$$

$$\text{and } \; K_k = (v_1, \hat{A}v_1, ..., \hat{A}^{k-1}v_1)$$

be a matrix whose columns span the subspace $\mathbb{K}$

With equation:

$$x_q^{l+1} = QF(x^l) \;\text{ we have }\; \hat{A}^j \Delta x_q^0 = \Delta x_q^j$$

K is easy computed from successive solutions xq and perform a QR factorization on K

Recursive Projection Method

Idea

If solution procedure fails to converge - chose V

$$\left|\frac{R_{j,j}}{R_{j+1,j+1}}\right| > \kappa, \quad \kappa > 1$$

kappa is the krylov acceptance ratio.
Size of V is crucial - preventing of adding stable eigenvalues to V consider orthogonal projection for approximating eigenvalues of PΦP
Let (lambda, y) be an eigenpair of

$$V^T \Phi V \in \mathbb{R}^{m \times m}$$

$$V^T(P\Phi P x - \lambda x) = 0$$

$$x = Vy$$

eigenvalues verify:

$$|\lambda| \leq 1 - \delta \ \text{ with } \ 0 \leq \delta \ll 1$$

compute new basis without stable modes.
delta allows selection of stable modes clustered close to the unit circle!

Recursive Projection Method
Idea

Implementation of RPM:

$$K_k \leftarrow (\Delta x_q^l, \Delta x_q^{l+1}, ..., \Delta x_q^{l-k+1}); j \leftarrow 0$$

$$(Q_k R_k) \leftarrow QR\ \ factorization(K_k)$$

$$r \leftarrow \{r_i = |R_{i,i}| : |Ri, i| \geq |Ri+1, i+1|\}; i = 1, ..., k-1$$

$$\text{loop}\ \ (i = 1; k-1)$$

$$\quad \text{if}\ \ (r_i > \kappa r_{i+1})$$

$$\qquad j \leftarrow i$$

$$V \leftarrow (V, Q_j)$$