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A stabilized Finite Element Method for a solid immersed in a viscous incompressible fluid

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Presentation

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u + \nabla p &= f \quad \text{in } \mathcal{F}, \\
\operatorname{div} u &= 0 \quad \text{in } \mathcal{F}, \\
u &= 0 \quad \text{on } \partial\mathcal{O}, \\
u &= g \quad \text{on } \partial\mathcal{S}, \\
u(x,0) &= u_0(x) \quad \text{in } \mathcal{F}.
\end{aligned}
$$

We define on $\partial\mathcal{S}$

$$
\begin{aligned}
\lambda(u,p) &= \sigma(u,p)n \\
&= 2\nu D(u)n - pn \\
&= \nu\left(\nabla u + \nabla u^T\right) n - pn.
\end{aligned}
$$

The classical variational formulation

The weak formulation of this Stokes problem is

$$2\nu \int_{\mathcal{F}} D(u) : D(v) - \int_{\mathcal{F}} p \mathrm{div}\ v - \int_{\partial\mathcal{S}} v \cdot \lambda \mathrm{d}\Gamma = \int_{\mathcal{F}} f \cdot v$$

$$\int_{\mathcal{F}} q \mathrm{div}\ u = 0$$

$$\int_{\partial\mathcal{S}} \mu \cdot u \mathrm{d}\Gamma = \int_{\partial\mathcal{S}} \mu \cdot g \mathrm{d}\Gamma$$

for any $(v, q, \mu) \in \mathbf{V} \times \mathrm{L}_0^2(\mathcal{F}) \times \mathbf{H}^{-1/2}(\partial\mathcal{S})$, where $V = \mathbf{H}_0^1(\mathcal{F})$. The solution can be viewed as the stationary point of the Lagrangian

$$\begin{aligned} L(u, p, \lambda) &= \nu \int_{\mathcal{F}} |D(u)|^2 - \int_{\mathcal{F}} p \mathrm{div}\ u - \int_{\mathcal{F}} f \cdot u \\ &\quad - \int_{\partial\mathcal{S}} \lambda \cdot (u - g) \mathrm{d}\Gamma. \end{aligned}$$

The extended Lagrangian

$$
\begin{aligned}
L(u,p,\lambda) \quad = \quad & \nu \int_{\mathcal{F}} |D(u)|^2 - \int_{\mathcal{F}} p \mathrm{div}\ u - \int_{\mathcal{F}} f \cdot u \\
& - \int_{\partial\mathcal{S}} \lambda \cdot (u - g) \mathrm{d}\Gamma \\
& - \frac{\gamma}{2} \int_{\partial\mathcal{S}} |\lambda - \sigma(u,p)n|^2
\end{aligned}
$$

Some references

J. Haslinger, Y. Renard, *A new fictitious domain approach inspired by the extended finite element method*, SIAM J. Numer. Anal., 2009.

H. J. C. Barbosa, T. J. R. Hughes, *The finite element method with Lagrange multipliers on the boundary : circumventing the Babuška-Brezzi condition*, Comput. Meth. Appl. Mech. Engrg., 1991.

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The stabilized formulation

$$2\nu \int_{\mathcal{F}} D(u) : D(v) - \int_{\mathcal{F}} p \,\mathrm{div}\ v - \int_{\mathcal{F}} f \cdot v - \int_{\partial\mathcal{S}} \lambda \cdot v \mathrm{d}\Gamma$$
$$+2\nu\gamma \int_{\partial\mathcal{S}} (\lambda - \sigma(u,p)n) \cdot (D(v)n) \,\mathrm{d}\Gamma = \int_{\mathcal{F}} f \cdot v$$

$$-\int_{\mathcal{F}} q \,\mathrm{div}\ u - \gamma \int_{\partial\mathcal{S}} q\,(\lambda - \sigma(u,p)n) \cdot n \mathrm{d}\Gamma = 0$$

$$-\int_{\partial\mathcal{S}} \mu \cdot (u - g) - \gamma \int_{\partial\mathcal{S}} (\lambda - \sigma(u,p)n) \cdot \mu \mathrm{d}\Gamma$$
$$= -\int_{\partial\mathcal{S}} \mu \cdot g \mathrm{d}\Gamma$$

The discretized matrices

$$\begin{pmatrix} A_{uu} & A_{up} & A_{u\lambda} \\ A_{up}^T & A_{pp} & A_{p\lambda} \\ A_{u\lambda}^T & A_{p\lambda}^T & A_{\lambda\lambda} \end{pmatrix} \begin{pmatrix} u \\ p \\ \lambda \end{pmatrix} = \begin{pmatrix} F \\ 0 \\ G \end{pmatrix}$$

Open problems

- The choice of coefficient γ
- Proving the "inf-sup" condition for this penalized formulation : For γ_0 small enough, there exists $C > 0$ independent of h such that

$$C \left|\left|\left|(u^h, p^h, \lambda^h)\right|\right|\right| \leq \sup_{0 \neq (v^h, q^h, \mu^h) \in E^h} \frac{\mathcal{B}(u^h, p^h, \lambda^h; u^h, q^h, \mu^h)}{\left|\left|\left|(v^h, q^h, \mu^h)\right|\right|\right|}.$$

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Making the solid deformable

The Dirichlet condition on $\partial\mathcal{S}$ for the rigid solid

$$u(x,t) = h'(t) + \theta'(x - h(t))^{\perp}$$

becomes, in the deformable case

$$u(x,t) = h'(t) + \theta'(x - h(t))^{\perp} + w(x,t)$$

where

$$\begin{aligned} w(x,t) &= w^*\left(\mathbf{R}_{\theta(t)}(x - h(t)), t\right), \\ \frac{\partial X^*}{\partial t} &= w^*(X^*, t). \end{aligned}$$