From the control of Galerkin approximations to the approximate controllability of the Schrödinger equation

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Controllability of finite-dimensional control systems: compatible vector fields

M compact manifold

$$\dot{q} = f(q,u), \quad u \in U \subset \mathbf{R}^m, \quad q \in M$$

A vector field g is compatible with $\mathcal{F} = \{f(\cdot,u) \mid u \in U\}$ if

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If $\overline{\mathcal{A}_{\mathcal{F}}(q)} = M$ and $\mathrm{Lie}_q\mathcal{F} = T_qM$ for all $q \in M$ then the system is controllable.

Convexification and closure: convergence of flows

Z_τ non-autonomous vector field. We write $\overrightarrow{\exp} \int_{t_0}^{t_1} Z_\tau \mathrm{d}\tau : M \to M$ for the flow at time t_1 of the Cauchy problem

$$\dot{q}(\tau) = Z_\tau(q(\tau)), \quad q(t_0) = \mathrm{I.\,C.}$$

Lemma

Let $u_j(\cdot)$, $j \in \mathbb{N}$, be a bounded sequence in $L^\infty([0, t_1], U)$ and Z_τ be a non-autonomous vector field on M. If

$$\int_0^t f(\cdot, u_j(\tau))\mathrm{d}\tau \to \int_0^t Z_\tau \mathrm{d}\tau, \quad j \to \infty,$$

then

$$\overrightarrow{\exp} \int_0^t f(\cdot, u_j(\tau))\mathrm{d}\tau \to \overrightarrow{\exp} \int_0^t Z_\tau \mathrm{d}\tau, \quad j \to \infty,$$

both convergences being uniform with respect to $(t, q) \in [0, t_1] \times M$ with all derivatives in q.

Bilinear Schrödinger equation

Consider the approximate controllability of

$$\dot{\psi} = A\psi + uB\psi, \quad u \in (0, \delta), \quad \|\psi\| = 1,$$

where A and B are skew-Hermitian and A has discrete spectrum.
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Letting $w(t) = \int_0^t v(\tau)d\tau$ and $y(t) = e^{-w(t)A}\psi(t)$ we get

$$\dot{y} = e^{-wA}Be^{wA}y$$

In the basis of eigenvectors of A the (j,k)-th element of $e^{-wA}Be^{wA}$ is $b_{j,k}e^{i(\lambda_k - \lambda_j)w}$, where $(i\lambda_k)_{k\in\mathbf{N}}$ is the spectrum of A.

Convexification

Let n be such that $\Pi_n y(0)$ is close to $y(0)$.
Let $N > n$ and $M_N(w) = (b_{j,k} e^{i(\lambda_k - \lambda_j)w})_{j,k=1}^N$.
If $|\lambda_k - \lambda_j| \neq |\lambda_m - \lambda_l|$ for $\{k,j\} \neq \{m,l\}$ and $1 \leq k,j \leq n$, then

$$E_{j,k} = \nu \begin{pmatrix} e_{j,k} b_{j,k} + e_{k,j} b_{k,j} & 0_{n\times(N-n)} \\ 0_{(N-n)\times n} & \star \end{pmatrix} \in \overline{\mathrm{conv}(\{M_N(w) \mid w\})}$$

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The curves $y_{(n)}(t) = (y_1(t), \ldots, y_n(t))^T$ and $y_{(n,N)}(t) = (y_{n+1}(t), \ldots, y_N(t))^T$ satisfy

$$\begin{pmatrix} \dot{y}_{(n)}(t) \\ \dot{y}_{(n,N)}(t) \end{pmatrix} = M_N(t) \begin{pmatrix} y_{(n)}(t) \\ y_{(n,N)}(t) \end{pmatrix} + \begin{pmatrix} H(t) \\ I(t) \end{pmatrix}$$

with $\|H\|_\infty$ arbitrarily small (for N large) and $\|I\|_\infty \leq C(N)$.
Let $M_N(t)$ converge in the integral sense to $E_{j,k}$. If $b_{j,k} \neq 0$, we can approximately perform an arbitrary probability transfer between the j-th and k-th components letting the other first n components unchanged.

Approximate controllability result (with U. Boscain, M. Caponigro, T. Chambrion)

If there are enough pairs (j, k) such that

- $|\lambda_k - \lambda_j| \neq |\lambda_m - \lambda_l|$ for $\{k, j\} \neq \{m, l\}$
- $b_{j,k} \neq 0$

then the system is approximately controllable.

To compute how many such (j, k) are enough we consider finite dimensional systems, whose controllability is proved using compatible vector fields obtained by Poisson stability and Lie brackets.