Partial differential equations

Benasque

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Estimation of the velocity of a fluid flow from boundary pressure measurements

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Plan of the talk

1. Problem and models

Navier-Stokes equations with mixed boundary conditions

Feedback control with partial information

2. Classical tools from control theory

Coupling between feedback control law and estimation

3. Estimation problems for the Navier-Stokes equations

Detectability of the linearized Navier-Stokes equations for different measure operators

Equations for the estimator

4. An extended linearized system

Stabilizability and detectability of the extended system

(5. Local feedback stabilization of Navier-Stokes system with partial information)

1. Problem and models

- We consider a fluid flow governed by the N.S.E.

- The goal. Given an unstable stationary solution w_s, we want to stabilize the fluid about w_s in the case of partial information.

- The partial information is a noisy boundary measurement (pressure, stress tensor)

- We have is to find an estimator and to couple it with a feedback control law (corresponding to a boundary control in a Dirichlet B.C.).

The estimator and the control law will be determined for the linearized model. Next, we want to prove a local stabilization result for the Navier-Stokes system.

• What is new ?

Feedback control of the N.S.E. in the case of mixed B.C..

The well posedness of the estimator for boundary pressure measurements.

The coupling between control and estimation.

For regular domain with Dirichlet B.C., see Barbu, Lasiecka, Triggiani, Fursikov, Badra, Raymond, Rowley, Sipp....

Some results are avalaible in the engineering literature.

No estimation results for pressure measure at the boundary in the case of a boundary control.

• What is not yet solved ?

How to determine an estimator of finite dimension ?

The unstable stationary solution w_s of the N.S.E.

$$-\nu\Delta w_s + (w_s \cdot \nabla) w_s + \nabla p_s = 0, \quad \text{in } \Omega,$$

$$\text{div } w_s = 0 \quad \text{in } \Omega, \quad w_s = u_s \text{ on } \Gamma_e \quad + \text{ Other B.C. on } \Gamma \setminus \Gamma_e.$$

The stabilization problem with partial observation

Using the obervation $y_{obs}(t) = Hw(t) + \eta(t)$, find an estimation w_e of w

and a control u in the form $u(t) = K(w_e(t) - w_s)$,

s.t. $|w(t) - w_s|_{L^2} \longrightarrow 0$ as $t \longrightarrow \infty$,

$$\frac{\partial w}{\partial t} - \nu\Delta w + (w \cdot \nabla) w + \nabla q = 0, \qquad \text{div } w = 0 \quad \text{in } Q,$$

$$w = u_s \text{ on } \Sigma_e = \Gamma_e \times (0,\infty), \quad w = Mu \text{ on } \Sigma_c = \Gamma_c \times (0,\infty),$$

$$+ \text{ Other B.C. on } \Sigma \setminus (\Sigma_e \cup \Sigma_c), \qquad w(0) = w_0 = w_s + z_0 \text{ in } \Omega.$$

Set $z = w - w_s$, $p = q - p_s$. The linearized (resp. nonlinear) equation is

$$\frac{\partial z}{\partial t} - \nu\Delta z + (w_s \cdot \nabla)z + (z \cdot \nabla)w_s + (z \cdot \nabla)z + \nabla p = 0,$$

$$\operatorname{div} z = 0 \quad \text{in } Q, \quad z = Mu \quad \text{on } \Sigma_c,$$

$$+ \text{ Other B.C. on } \Sigma \setminus (\Sigma_e \cup \Sigma_c), \qquad z(0) = z_0 \quad \text{in } \Omega\,.$$

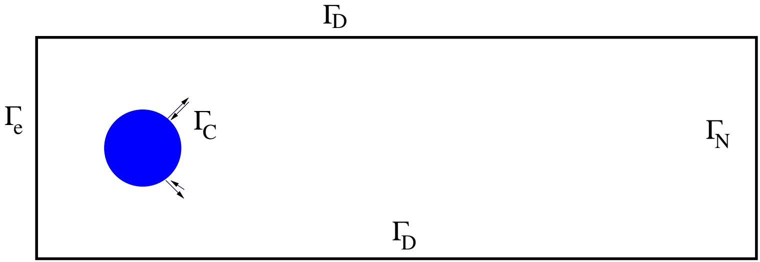
with $u(t) = Kz_e(t)$, z_e is an estimation of z based on the observation

$$y_{obs}(t) = Hz(t) + \eta(t),$$

and

$$\operatorname{supp} M \subset \Gamma_c.$$

The case of the flow around a cylinder with an outflow boundary condition – 2D domain



Boundary conditions

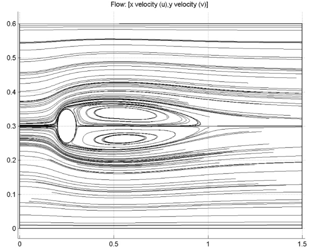
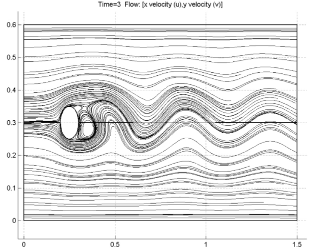
$$z = u_s \quad \text{on} \quad \Gamma_e \times (0,\infty), \qquad z = 0 \quad \text{on} \quad \Gamma_D \times (0,\infty),$$

$$z = Mu \quad \text{on} \quad \Gamma_C \times (0,\infty),$$

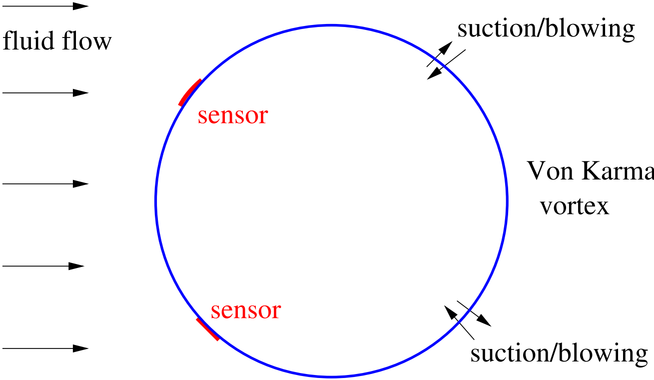
$$\nu \frac{\partial z}{\partial n} - pn = 0 \quad \text{or} \quad \sigma(z,p)n = 0 \quad \text{on} \quad \Gamma_N \times (0,\infty),$$

where $\sigma(z,p) = \nu(\nabla z + (\nabla z)^T) - p\,I.$

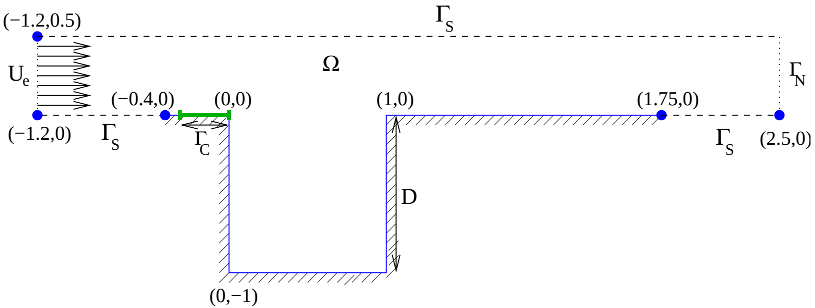
Control of the wake behind an obstacle – $Re = u_s Diam/\nu$

	$5 < Re < 50$	A fixed pair of vortices
	$50 < Re < 150$	Vortex street

The complete problem: estimation + feedback control



Control of a flow in an open cavity 2D domain

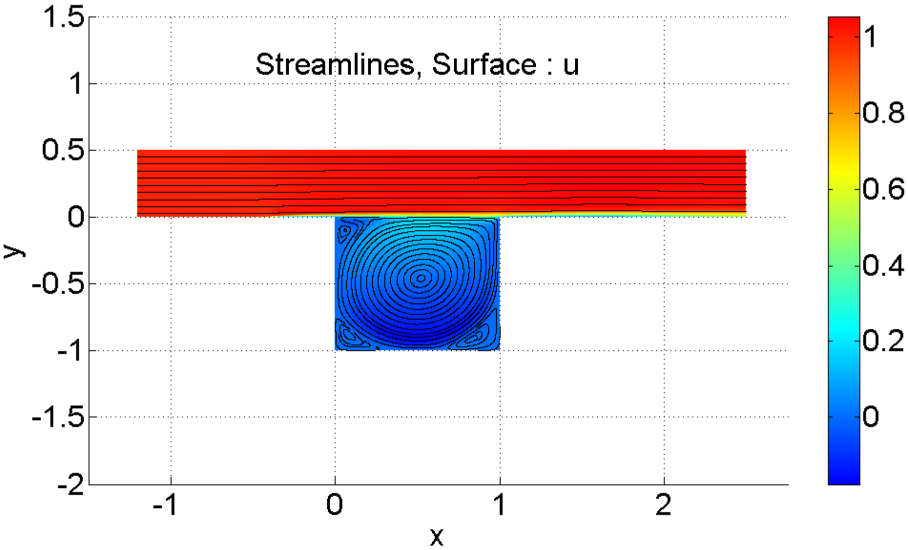


$$
\begin{aligned}
-\operatorname{div}\sigma(w_s, p_s) + (z_s\,.\,\nabla)w_s &= 0 \quad \text{in} \quad \Omega\\
\operatorname{div} w_s &= 0 \quad \text{in} \quad \Omega\\
w_s &= u_s \quad \text{on} \quad \Gamma_e\\
w_s &= 0 \quad \text{on} \quad \Gamma_D\\
\sigma(w_s, p_s)n &= 0 \quad \text{on} \quad \Gamma_N\\
w_s\,.\,n &= 0 \quad \text{on} \quad \Gamma_S\\
\big(\sigma(w_s, p_s)n\big)\cdot\tau &= 0 \quad \text{on} \quad \Gamma_S
\end{aligned}
$$

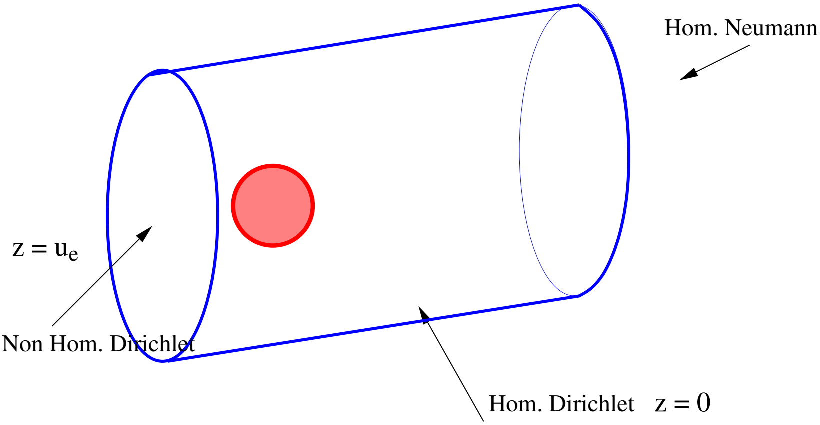
where

$$\sigma(w_s, p) = \nu\left(\nabla w_s + (\nabla w_s)^T\right) - pI.$$

Stationary solution of the open cavity – 2D



3D Domain with a right angle junction



2. Classical results from control system

The stabilization problem with full information. Consider a control system

$$z' = Az + Bu, \quad z(0) = z_0.$$

Assume that (A, B) is stabilizable. Find $K \in \mathcal{L}(Z, U)$ such that $A + BK$ is exponentially stable on Z. One way consists in solving an Algebraic Riccati Equation of the form

$$P = P^* \geq 0, \quad A^*P + PA - PBR^{-1}B^*P + Q = 0,$$

with $R = R^* > 0$ and $Q = Q^* \geq 0$. When the A.R.E. is well posed a convenient feedback is

$$K = -R^{-1}B^*P.$$

The estimation problem. Consider a noisy model and a noisy observation

$$z' = Az + f + \mu, \quad z(0) = z_0 + \mu_0, \quad y_{obs}(t) = Hz(t) + \eta(t) \in Y_o.$$

Find $L \in \mathcal{L}(Y_o, Z)$ such that $A + LH$ is exponentially stable. Such a filtering gain may be determined by solving

$$P_e = P_e^* \geq 0, \quad AP_e + P_eA^* - P_eH^*R_o^{-1}HP_e + Q_o = 0.$$

We choose $L = -P_e H^* R_o^{-1}$. We determine the control law by solving the system

$$z_e' = Az_e + BKz_e + L(Hz_e - y_{obs}), \quad z_e(0) = z_0.$$

Next prove that the original system with the feedback coming from the estimator

$$z' = Az + BKz_e + \mu, \quad z(0) = z_0 + \mu_0,$$

is stable.

Theorem. If $(e^{t(A+BK)})_{t\geq 0}$ is exponentially stable and if $(e^{t(A+LH)})_{t\geq 0}$ is exponentially stable, then the semigroup generated by

$$\mathcal{A} = \begin{bmatrix} A & BK \\ -LH & A + BK + LH \end{bmatrix}$$

is also exponentially stable on $Z \times Z$.

Indeed, if $e = z - z_e$, we have

$$\begin{pmatrix} z \\ e \end{pmatrix}' = \begin{pmatrix} A + BK & -BK \\ 0 & A + LH \end{pmatrix} \begin{pmatrix} z \\ e \end{pmatrix} + \mathcal{F}$$

with

$$\mathcal{F} = \left(\ \mu, L\eta \ \right)^T.$$

3. Rewriting the N.S.E. as a control system

We would like to have a classical of the form

$$z' = Az + Bu + F(z), \quad z(0) = z_0, \quad F(0) = F'(0) = 0.$$
$$B = (\lambda_0 I - A)DM.$$

• $(A, D(A))$ is the Oseen operator and Bu takes into account the non homogeneous Dirichlet B.C.

• We have to introduce the Leray projector Π and we shall obtain a system of the form

$$\Pi z' = A\Pi z + Bu + F(\Pi z + (I - \Pi)DMu), \quad \Pi z(0) = \Pi z_0,$$

$$(I - \Pi)z = (I - \Pi)DMu, \qquad F(0) = F'(0) = 0,$$

$$B = (\lambda_0 I - A)\Pi DM.$$

The Helmholtz decomposition in the case of mixed D/N boundary conditions

$$V^0_{n,\Gamma_D}(\Omega) = \Big\{ z \in L^2(\Omega;\mathbb{R}^d) \mid \ \text{div}\, z = 0,\ z\cdot n = 0 \text{ on } \Gamma_D \Big\},$$

$$L^2(\Omega;\mathbb{R}^d) = V^0_{n,\Gamma_D}(\Omega) \oplus \text{grad}\, H^1_{\Gamma_N}(\Omega),$$

$$\text{grad}\, H^1_{\Gamma_N}(\Omega) = \{ p \in H^1(\Omega) \mid p|_{\Gamma_N} = 0 \}.$$

$$\Pi \ : \ L^2(\Omega;\mathbb{R}^d) \longmapsto V^0_{n,\Gamma_D}(\Omega).$$

To define the Stokes operator, we need

$$V^1_{\Gamma_D}(\Omega) = \Big\{ y \in H^1(\Omega;\mathbb{R}^d) \cap V^0_{n,\Gamma_D}(\Omega) \mid z = 0 \text{ on } \Gamma_D \Big\},$$

$$V^1_{\Gamma_D}(\Omega) \hookrightarrow V^0_{n,\Gamma_D}(\Omega) \hookrightarrow V^{-1}_{\Gamma_D}(\Omega) = (V^1_{\Gamma_D}(\Omega))'.$$

The Helmholtz projector Π

$$\Pi f = f - \nabla p - \nabla q,$$

$$\Delta p = \operatorname{div} f \in H^{-1}(\Omega), \quad p \in H_0^1(\Omega),$$

$$\Delta q = 0, \quad \frac{\partial q}{\partial n} = (f - \nabla p) \cdot n \quad \text{on } \Gamma_D, \quad q = 0 \quad \text{on } \Gamma_N.$$

Characterization of Stokes operator $(A_0, D(A_0))$ in the case of Mixed D/N B.C. with a right angle junction

$$D(A_0) = \Big\{ z \in V^1_{\Gamma_D}(\Omega) \mid$$

$$\exists p \in L^2(\Omega) \text{ s. t. } \mathrm{Div}(\nu \nabla z - pI) \in L^2(\Omega; \mathbb{R}^d)$$

$$\text{and } \nu \frac{\partial z}{\partial n} - pn = 0 \quad \text{on} \quad \Gamma_N \Big\},$$

$$A_0 z = \Pi(\nu \Delta z - \nabla p) \quad (\text{does not depend on } p).$$

$$\text{Notice that } \gamma_\tau \frac{\partial z}{\partial n} = 0 \quad \text{on} \quad \Gamma_N.$$

In the 3D case with a right angle junction, we have

$$D(A_0) \subset H^{3/2+\varepsilon}(\Omega; \mathbb{R}^d) \quad \text{for some} \quad \varepsilon > 0.$$

(See Maz'ya and Rossmann, 2007.)

The Oseen operator $(A, D(A))$ is defined by

$$D(A) = D(A_0) \quad \text{and} \quad Az = A_0 z + \Pi\left((w_s \cdot \nabla) z + (z \cdot \nabla) w_s\right).$$

Theorem. The operator $(A, D(A))$ is the infinitesimal generator of an analytic semigroup on $V^0_{n,\Gamma_D}(\Omega)$. Its resolvent is compact.

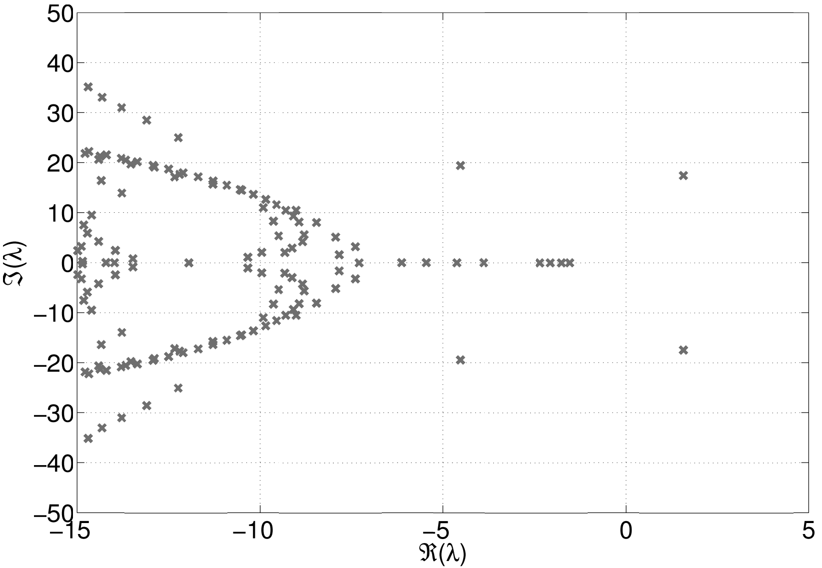
Proof.

$$((\lambda_0 I - A)z, z) \geq \frac{1}{2}\|z\|^2_{V^1_{\Gamma_D}(\Omega)} \quad \forall z \in D(A),$$

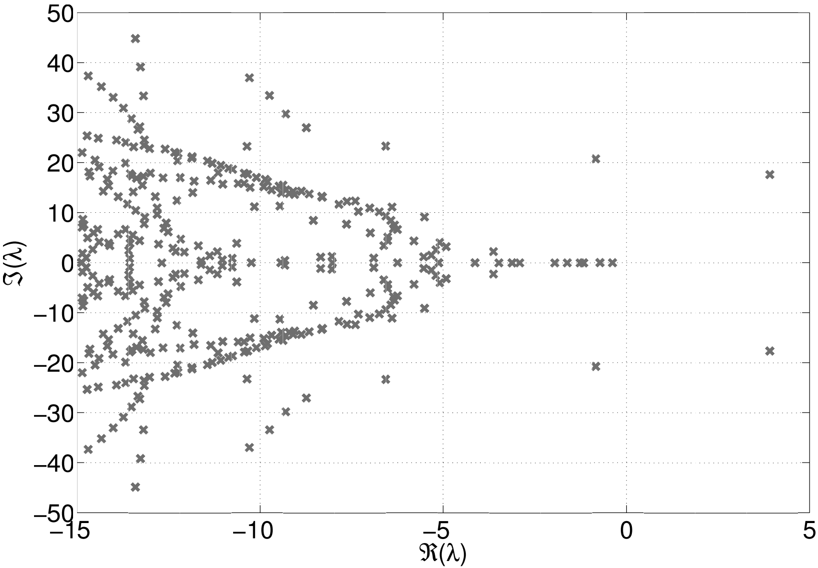
with $\lambda_0 > 0$ big enough.

Consequence. The spectrum of A is contained in a sector. The eigenvalues are isolated, pairwise conjugate when they are not real, and of finite multiplicity.

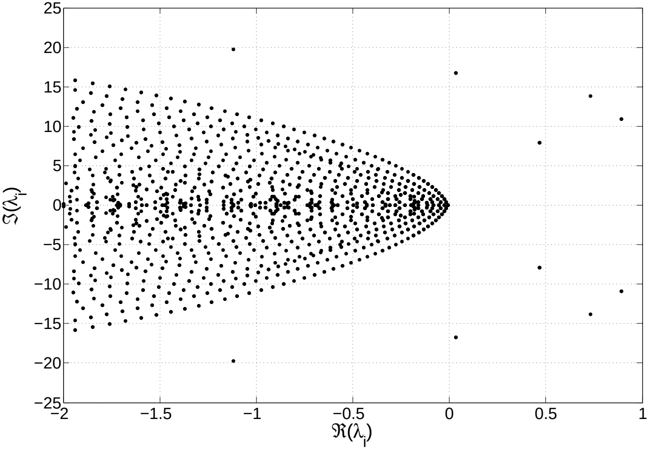
Spectrum of A. $Re = u_s\, Diam/\nu = 80$ (Cylinder)



Spectrum of A with $Re = 200$



Spectrum of A. $Re = u_s \times h_{cavity}/\nu = 7500$ (Cavity)



The linearized system with non homogeneous Dirichlet B.C.

$$\frac{\partial z}{\partial t} - \nu \Delta z + (w_s \cdot \nabla) z + (z \cdot \nabla) w_s + \nabla p = 0,$$

$$\mathrm{div}\, z = 0 \quad \text{in } Q, \quad z = Mu \quad \text{on } \Sigma_D,$$

$$\nu \frac{\partial z}{\partial n} - pn = 0 \quad \text{on} \quad \Sigma_N, \quad z(0) = z_0 \quad \text{in } \Omega.$$

We introduce the lifting operator D, defined by $DMu(t) = w(t)$ with

$$\lambda_0 w(t) - \nu \Delta w(t) + (w_s \cdot \nabla) w(t) + (w(t) \cdot \nabla) w_s + \nabla \pi(t) = 0,$$

$$\mathrm{div}\, w(t) = 0, \quad w(t) = Mu(t) \quad \text{on } \Gamma_D, \quad \nu \frac{\partial w(t)}{\partial n} - \pi(t) n = 0 \quad \text{on} \quad \Gamma_N.$$

We look for z in the form

$$z = w + y.$$

Writing the equation for y and with integration by parts, we show that the system satisfied by z is

$$\Pi z' = A\Pi z + (\lambda_0 I - A)\Pi DMu, \qquad \Pi z(0) = \Pi z_0,$$

$$(I - \Pi)z(t) = (I - \Pi)DMu(t) = (I - \Pi)DM(u(t) \cdot n\, n),$$

$$B = (\lambda_0 I - A)\Pi DM.$$

3. Stabilizability of the linearized N.S.E.

Theorem. Assume that the semigroup generated by $(A, D(A))$ is analytic on Y, the resolvent of A is compact, $(\lambda_0 I - A)^{\alpha-1}B \in \mathcal{L}(U, Y)$, and the spectrum of A obeys

$$\ldots < \mathrm{Re}\lambda_{N_u+1} < -\omega < \mathrm{Re}\lambda_{N_u} \leq \mathrm{Re}\lambda_{N_u-1} \leq \ldots \leq \mathrm{Re}\lambda_1 .$$

For $1 \leq j \leq N_u$, let $(\phi_j^k)_{1\leq k\leq \ell_j}$ be a basis of $\mathrm{Ker}(A^* - \lambda_j I)$.
The pair (A, B) is stabilizable with a decay rate $-\omega$ iff, for all $1 \leq j \leq N_u$, the family

$$(B^*\phi_j^k)_{1\leq k\leq \ell_j}$$

is linearly independent.

Proof of the stabilizability.

$$A^*\phi = \lambda\,\phi \quad \text{and} \quad B^*\phi = M\Big(\nu\frac{\partial\phi}{\partial n} + w_s\cdot n\,\phi - \psi n\Big) = 0,$$

implies that $\phi = 0$.

We can invoque the unique continuation results by Fabre-Lebeau.

If

$$\lambda\phi - \nu\Delta\phi - (w_s\cdot\nabla)\phi + (\nabla w_s)^T\phi + \nabla\psi = 0,$$

$$\text{div}\,\phi = 0 \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \Gamma_D,$$

and

$$\nu\frac{\partial\phi}{\partial n} + w_s\cdot n\,\phi - \psi n = 0 \quad \text{on} \quad \{x\in\Gamma_C \mid M(x) = 1\} \subset \Gamma_D,$$

then

$$\phi = 0 \quad \text{and} \quad \psi = 0.$$

Moreover, in that case the complexified system is also stabilizable with controls of the form

$$u = \sum_{j=1}^{N_u} \sum_{k=1}^{\ell_j} B^* \phi_j^k,$$

and the real system is stabilizable with controls of the form

$$u = \sum_{j=1}^{N_u} \sum_{k=1}^{\ell_j} (\mathrm{Re} B^* \phi_j^k + \mathrm{Im} B^* \phi_j^k).$$

We can find a control of finite dimension, in feedback form, by solving a Riccati equation of finite dimension.

3.3. Examples of boundary measure operators for the Stokes or the Oseen equations

Let us consider the Stokes (Oseen) equations with mixed boundary conditions

$$\frac{\partial z}{\partial t} - \nu \Delta z + (w_s \cdot \nabla) z + (z \cdot \nabla) w_s + \nabla p = \mu, \quad \operatorname{div} z = 0 \quad \text{in } Q,$$

$$z = M\, u \quad \text{on } \Sigma_c, \quad z = 0 \quad \text{on } \Sigma_D, \quad \sigma(z,p)n = 0 \quad \text{on } \Sigma_N,$$

$$z(0) = z_0 + \mu_0 \quad \text{in } \Omega.$$

Some boundary measure operators

$$H_1 z(t) = \sigma(z(t), p(t))n|_{\Gamma_1}, \quad H_2 z(t) = p(t)|_{\Gamma_1},$$

$$H_3 z(t) = \int_{\Gamma_1} \sigma(z(t), p(t))n, \quad H_4 z(t) = \left(\int_{\Gamma_1} p(t), \cdots, \int_{\Gamma_N} p(t) \right),$$

where $\Gamma_1, \cdots, \Gamma_N$ are N intervals of Γ_o and $\Gamma_o \subset \Gamma_D$.

Notice that the notation

$$H_2 z(t) = p(t)|_{\Gamma_1},$$

is meaningful. Indeed, $p(t)$ is the solution to

$$\Delta p(t) = 0 \quad \text{in } \Omega, \quad p(t) = \nu \frac{\partial z}{\partial n} \cdot n \quad \text{on } \Gamma_N,$$

$$\frac{\partial p(t)}{\partial n} = \nu \Delta z \cdot n - \frac{\partial z}{\partial t} \cdot n \quad \text{on } \Gamma \setminus \Gamma_N.$$

The same type of measure operator can be considered for the N.S.E. Let us set

$$p(t) = N\left(\nu \Delta z(t) \cdot n - \frac{\partial z}{\partial t} \cdot n\right)$$

and

$$p(t)|_{\Gamma_1} = N_1\left(\nu \Delta z(t) \cdot n - \frac{\partial z}{\partial t} \cdot n\right).$$

The measure operator may be decomposed as follows

$$N_1\left(\nu\Delta z(t)\cdot n - \frac{\partial z}{\partial t}\cdot n\right)$$
$$= N_1\left(\nu\Delta\Pi z(t)\cdot n\right) + N_1\left(\nu\Delta(I-\Pi)DMu(t)\cdot n - (I-\Pi)DMu'(t)\cdot n\right).$$

Thus we have

$$p(t)|_{\Gamma_1} = H_z\Pi z(t) + H_u\, u(t) + H_{u'}\, u'(t).$$

To estimate z, we have to assume that u' is well defined. In that case the estimator is of the form

$$\Pi z_e' = A\Pi z_e + Bu + L(H_z\Pi z_e + H_u\, u(t) + H_{u'}\, u'(t) - y_{obs}),$$
$$z_e(0) = z_0,$$
$$(I-\Pi)z_e(t) = (I-\Pi)DMu(t),$$

where $L = -P_e H_z^* R_o^{-1}$ and $P_e \in \mathcal{L}(Z)$ is the solution of

$$P_e = P_e^* \geq 0, \quad P_e A^* + AP_e - P_e H_z^*\, R_o^{-1}\, H_z P_e + Q_o = 0.$$

We have to study the stabilizability of the pair (A, H_z).

To determine the control

$$u(t) = -B^* P \Pi z_e,$$

we have to solve the equation

$$\Pi z_e' = A \Pi z_e + Bu + L(H_z \Pi z_e - H_u B^* P \Pi z_e - H_{u'} B^* P \Pi z_e' - y_{obs}),$$
$$z_e(0) = z_0.$$

This equation is not necessarily well posed because the time derivative z_e' appears on both sides of the equation.

Thus if we want to couple the estimator and the control law, we have to consider another system in which u will play the role of a new state variable and u' is the new control variable. It will be called 'extended system'.

The same boundary measure operators can be considered for the L.N.S.E.

- Detectability of $(A+\omega I, H_1)$

We have to show that if ζ is an eigenfunction of A and p_ζ the associated pressure, and if

$$H_1\zeta = \sigma(\zeta, p_\zeta)n|_{\Gamma_1} = 0 \quad \text{with e.g.} \quad \Gamma_1 = (1, 1.1),$$

then $\zeta \equiv 0$. This is a consequence of the unique continuation property for the Oseen operator (Fabre and Lebeau, 96).

Then $(A+\omega I, H_1)$ is detectable. Indeed for each eigenfunction ζ, the vector $\sigma(\zeta, p_\zeta)n|_{\Gamma_1}$ is non zero.

- Detectability of $(A+\omega I, H_2)$

We have to show that if λ_j is an unstable eigenvalue, e_j belongs to Ker$(A-\lambda_j)$, p_j is the pressure associated with e_j and if

$$H_2 e_j = p_j|_{\Gamma_1} = 0 \quad \text{with e.g.} \quad \Gamma_1 = (1, 1.1),$$

then $(e_j, p_j) \equiv 0$.

The only detectability result of this type is due to A. Osses and J.-P. Puel for Dirichlet B.C. with an angle at the boundary of the control zone.

There is no detectability result in the case of mixed B.C., but it can be checked numerically.

- Detectability of $(A + \omega I, H_3)$ and $(A + \omega I, H_4)$

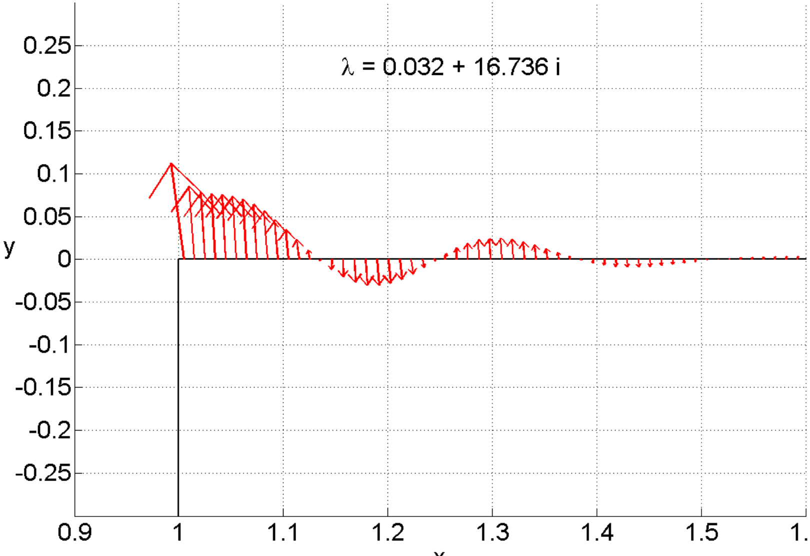
If the unstable eigenvalues are simple and if (e_j, p_j) is the corresponding eigenfunction, then

$$\int_{\Gamma_1} p_j \quad \text{with} \quad \Gamma_1 = (1, 1.1),$$

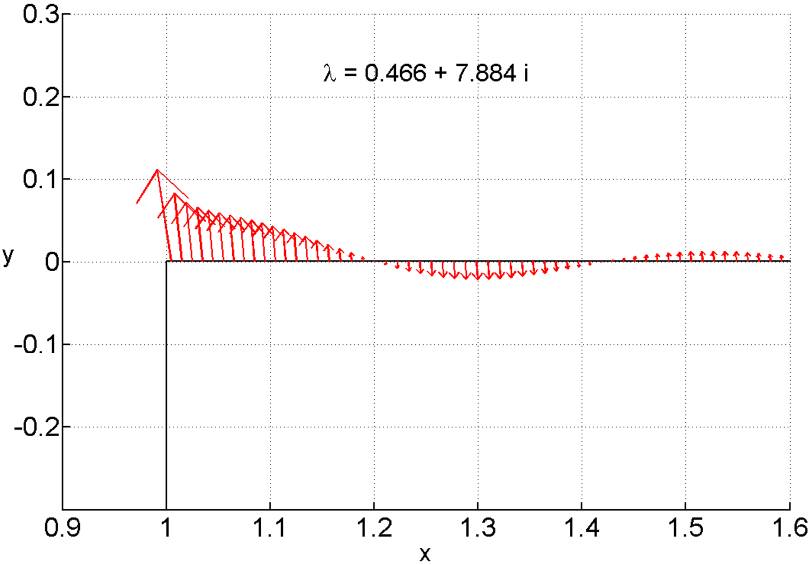
is non zero. Then $(A + \omega I, H_3)$ and $(A + \omega I, H_4)$ are detectable.

Numerical verification of the detectability condition for the L.N.E.

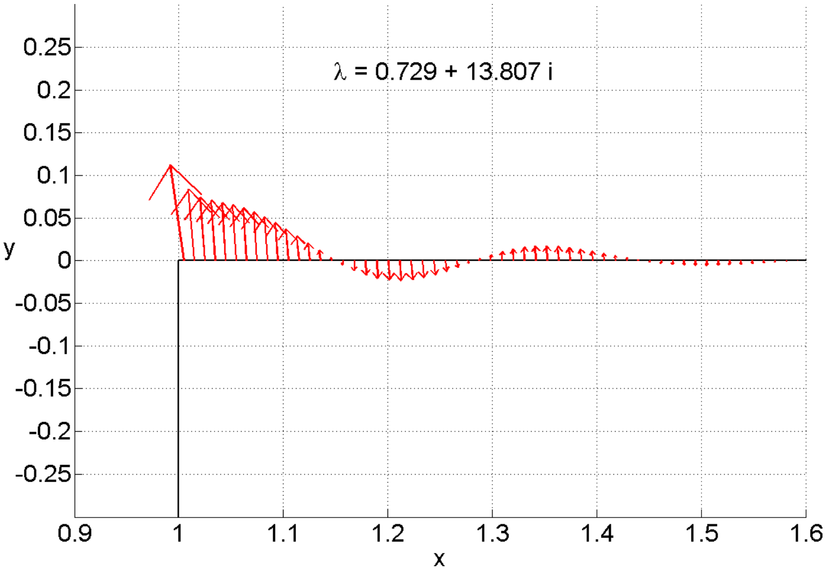
Stress tensor at the boundary for A – Cavity with $Re = 7500$



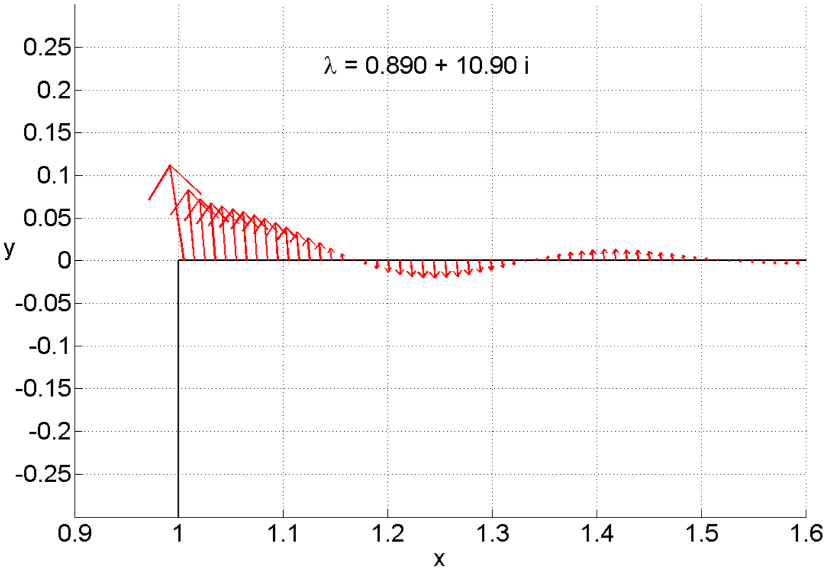
Stress tensor at the boundary for A – Cavity with $Re = 7500$



Stress tensor at the boundary for A – Cavity with $Re = 7500$



Stress tensor at the boundary for A – Cavity with $Re = 7500$



4. An extended system

If we look for a control u belonging to $H^1(0,\infty;U)$, we can consider u as an additional state variable, we can add the equation $u' = v$ and we can choose v as the new control variable

$$\widetilde{z}' = \begin{pmatrix} z \\ u \end{pmatrix}' = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ I_u \end{pmatrix} v, \quad \begin{pmatrix} z(0) \\ u(0) \end{pmatrix} = \begin{pmatrix} z_0 \\ u_0 \end{pmatrix}.$$

We set

$$\widetilde{A} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} 0 \\ I_u \end{pmatrix}.$$

We have chosen U is of finite dimension. The operator $\widetilde{A}$, with

$$D(\widetilde{A}) = \{(z,u) \in V^0_{n,\Gamma_D}(\Omega) \times U \mid Az + Bu \in V^0_{n,\Gamma_D}(\Omega)\},$$

is the generator of an analytic semigroup on $V^0_{n,\Gamma_D}(\Omega) \times U$. And $\widetilde{B} \in \mathcal{L}(U, Z \times U)$ with $Z = V^0_{n,\Gamma_D}(\Omega)$.

Thus, we have replaced the unbounded operator B by $\widetilde{B} \in \mathcal{L}(U, \widetilde{Z})$, with $\widetilde{Z} = V^0_{n,\Gamma_D}(\Omega) \times U$.

4.1. Stabilizability of the extended system

The adjoints of $\widetilde{A}$ and $\widetilde{B}$ are

$$\widetilde{A}^* = \begin{pmatrix} A^* & 0 \\ B^* & 0 \end{pmatrix}, \quad \widetilde{B}^* = (0 \quad I_u)\,,$$

with

$$D(\widetilde{A}^*) = \{(\phi,\gamma) \in V^0_{n,\Gamma_D}(\Omega)\times U \mid A\phi \in V^0_{n,\Gamma_D}(\Omega),\ B^*\phi \in U\} = D(A^*)\times U.$$

Now, we study the stabilisability of the extended system

$$\widetilde{z}' = \widetilde{A}\widetilde{z} + \widetilde{B}v, \quad \widetilde{z}(0) = \widetilde{z}_0.$$

Theorem

The system (A, B) is stabilizable by a control $u \in L^2(0,\infty; U)$ iff the extended system $(\widetilde{A}, \widetilde{B})$ is stabilizable by a control $v \in L^2(0,\infty; U)$.

Consequence. The system (A, B) is stabilizable by a control $u \in L^2(0,\infty; U)$ iff it is stabilizable by a control $u \in H^1(0,\infty; U)$.

Idea of the proof. We consider the case when there is no eigenvalue of A on the imaginary axis. Consider $(\phi, \gamma) \in D(A^*) \times U$, an eigenfunction of $\widetilde{A}^*$

$$A^* \phi = \lambda \phi, \quad B^* \phi = \lambda \gamma.$$

Thus ϕ is an eigenfunction of A^*, $\lambda \in \sigma(A^*)$, $\lambda \neq 0$ and $\gamma = B^* \phi / \lambda$. In particular the eigenvalues of A^* and $\widetilde{A}^*$ are the same ones.

The pair (A, B) is stabilizable iff, for all unstable eigenvalue λ_j, the family

$$(B^* e_j^k)_{1 \leq k \leq \ell_j}$$

is linearly independent.

The pair $(\widetilde{A}, \widetilde{B})$ is stabilizable iff, for all unstable eigenvalue λ_j, the family

$$(\widetilde{B}^* e_j^k)_{1 \leq k \leq \ell_j} = (\gamma_j^k)_{1 \leq k \leq \ell_j} = (\widetilde{B}^* e_j^k / \lambda_j)_{1 \leq k \leq \ell_j}$$

is linearly independent. The equivalence is obvious.

A Riccati equation for finding a feedback control law for the extended system.

A control can be found by solving a Riccati equation of the form

$$\widetilde{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \in \mathcal{L}(Z \times U), \quad \widetilde{P} = \widetilde{P}^* \geq 0,$$

$$\widetilde{P}\widetilde{A} + \widetilde{A}^*\widetilde{P} - \widetilde{P}\widetilde{B}\widetilde{B}^*\widetilde{P} + \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} = 0, \quad Q = C^*C.$$

4.2. Detectability of the extended system

The model to estimate is

$$\begin{aligned} &\Pi z' = A\Pi z + Bu + \mu, \\ &u' = v, \qquad \Pi z(0) = \Pi z_0, \quad u(0) = u_0, \\ &(I - \Pi)z(t) = (I - \Pi)DMu(t). \end{aligned}$$

The measure can be expressed as follows

$$y_{obs}(t) = Hz(t) + \eta(t) = H\Pi z(t) + H(I - \Pi)z(t) + \eta(t)$$
$$= H_z \Pi z(t) + H_u u(t) + H_v v(t) + \eta(t).$$

Since there is no model error in the equation for u and since v is assumed to be known, we have to estimate $\Pi z(t)$ from the noisy measure

$$\xi_{obs} = y_{obs} - H_u u(t) - H_v v(t) = H_z \Pi z(t) + \eta(t).$$

Thus, we have to look at the stabilizability of the pair (A, H_z), which has already been studied.

The filtering gain $L = -P_e H_y^* R_o^{-1}$ is the same one as for the L.N.S.E.

Later on v will be written in the form

$$v = -\widetilde{B}^* \widetilde{P} (\Pi z_e, u_e)^T = -P_{21} \Pi z_e - P_{22} u.$$

4.3. Coupling estimation and control for Linearized Navier-Stokes equations

The coupled system is

$$\begin{aligned}
&\Pi z_e' = A\Pi z_e + Bu + L(H_z\Pi z_e + H_u\, u(t) + H_v\, v(t) - y_{obs}),\\
&u' = -P_{21}\Pi z_e - P_{22}u = v,\\
&\Pi z_e(0) = \Pi z_0, \quad u(0) = u_0,\\
&(I-\Pi)z(t) = (I-\Pi)D\, u(t),\\
&\Pi z' = A\Pi z + Bu + \mu,\\
&\Pi z(0) = \Pi z_0 + \mu_0, \quad u_e(0) = u_0.
\end{aligned}$$

Theorem. The system

$$\begin{aligned}
&\Pi z_e' = A\Pi z_e + Bu + L(H_z\Pi z_e + H_u\, u + H_v\, v - y_{obs}),\\
&u' = -\widetilde{B}^*\widetilde{P}(\Pi z_e, u)^T = -P_{21}\Pi z_e - P_{22}u = v,\\
&\Pi z_e(0) = \Pi z_0, \quad u(0) = u_0,
\end{aligned}$$

is well posed. We use that U is of finite dimension.

Negative result.

If $(e^{t(\widetilde{A}-\widetilde{B}\widetilde{B}^*\widetilde{P})})_{t\geq 0}$ is exponentially stable on $Z \times U$ and if $(e^{t(A+LH)})_{t\geq 0}$ is exponentially stable on Z, then the semigroup generated by

$$\mathcal{A} = \begin{bmatrix} A & B & 0 \\ 0 & -P_{22} & -P_{21} \\ 0 & LH_u - LH_v P_{22} & A + LH \end{bmatrix}$$

is not necessarily exponentially stable.

Indeed, we have

$$\begin{pmatrix} z \\ u \\ e \end{pmatrix}' = \begin{pmatrix} A & B & 0 \\ 0 & -P_{22} & -P_{21} \\ 0 & 0 & A + LH \end{pmatrix} \begin{pmatrix} z \\ u \\ e \end{pmatrix} + \mathcal{F}$$

with

$$\mathcal{F} = \begin{pmatrix} \mu \\ 0 \\ L\eta \end{pmatrix}.$$

Remedy. The difficulty comes from the fact that the observation involves the control u and its derivative u'. Because of that the estimator does not take u into account.

If we want to construct an estimator for $(\Pi z, u)$, we have to extend the system as follows

$$
\begin{aligned}
&\Pi z' = A\Pi z + Bu, \quad \Pi z(0) = \Pi z_0, \\
&u' = \Lambda_u u + v, \quad u(0) = u_0, \\
&v' = \Lambda_v v + \xi, \quad v(0) = 0,
\end{aligned}
$$

and to construct an estimator for $(\Pi z, u, v)$. ξ is a new control variable.

The stabilizability for the extended system may be verified as before.

For the detectability, the measure operator is now $\widetilde{H} = (H_z \; H_u \; H_v)$. We can choose Λ_u and Λ_v to conclude.

Thank you for your attention