

Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier-Stokes system

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joint work with

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Incompressible Navier-Stokes system

Caffarelli, Kohn, and Nirenberg [1982]

$$\operatorname{div}_x \mathbf{u} = 0$$

$$\partial_t \mathbf{u} + \operatorname{div}_x (\mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \Delta \mathbf{u}$$

Energy inequality

$$p \in L^{3/2}(0, T) \times \Omega$$

$$\partial_t |\mathbf{u}|^2 + \nabla_x |\mathbf{u}|^2 \cdot \mathbf{u} + 2 \operatorname{div}_x (p \mathbf{u}) + 2 |\nabla_x \mathbf{u}|^2 \leq \Delta |\mathbf{u}|^2$$

Compressible Navier-Stokes system

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$\mathbf{u}|_{\partial\Omega} = 0$$

$$\mathbb{S} = \mu(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I}) + \eta \operatorname{div}_x \mathbf{u} \mathbb{I}$$

Germain [2010]:

- weak-strong results in the framework of “better” weak solutions
- suitable conditions formulated in terms of hypothetical smooth solutions
- periodic boundary conditions

Finite-energy weak solutions

EQUATION OF CONTINUITY

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

MOMENTUM EQUATION

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Energy inequality

$$\begin{aligned} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right) (\tau, \cdot) \, dx + \int_0^\tau \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, dx \, dt \\ \leq \int_{\Omega} \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + H(\varrho_0) \right) \, dx \end{aligned}$$

$$H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, dz$$

Relative entropy

Germain [2010], Berthelin and Vasseur [2005]

$$E(\varrho, r) = H(\varrho) - H'(r)(\varrho - r) - H(r)$$

$$H(\varrho) \equiv \varrho \int_1^\varrho \frac{p(z)}{z^2} \, dz, \quad P \equiv H'$$

$$\int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r) \right) \, dx$$

Suitable weak solutions

RENORMALIZED EQUATION OF CONTINUITY

$$\partial_t b(\varrho) + \operatorname{div}_x(b(\varrho)\mathbf{u}) + \left(b'(\varrho)\varrho - b(\varrho)\right)\operatorname{div}_x\mathbf{u} = 0$$

MOMENTUM EQUATION

$$\partial_t(\varrho\mathbf{u}) + \operatorname{div}_x(\varrho\mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Generalized energy inequality

$$\begin{aligned}
 & \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + E(\varrho, r) \right) (\tau, \cdot) \, dx \\
 & + \int_0^\tau \int_{\Omega} [\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})] : \nabla_x (\mathbf{u} - \mathbf{U}) \, dx \, dt \\
 & \leq \int_{\Omega} \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0 - \mathbf{U}(0, \cdot)|^2 + E(\varrho_0, r(0, \cdot)) \right) \, dx \\
 & + \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) \, dt \text{ for a.a. } \tau \in (0, T),
 \end{aligned}$$

for any smooth $r > 0$, $\mathbf{U}|_{\partial\Omega} = 0$

Remainder term

$$\begin{aligned}
 & \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) \\
 = & \int_{\Omega} \left(\varrho \left(\partial_t \mathbf{U} + \mathbf{u} \nabla_x \mathbf{U} \right) \cdot (\mathbf{U} - \mathbf{u}) + \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{U})(\mathbf{u} - \mathbf{U}) \right) dx \\
 & + \int_{\Omega} \left((r - \varrho) \partial_t P(r) + \nabla_x P(r) \cdot (r \mathbf{U} - \varrho \mathbf{u}) - \right. \\
 & \left. \operatorname{div}_x \mathbf{U} \left(\varrho (P(\varrho) - P(r)) - E(\varrho, r) \right) \right) dx
 \end{aligned}$$

Global existence

[E.F., A. Novotný, Y. Sun, Indiana Univ. Math. J., to appear]

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain. Let the pressure p be a continuously differentiable function satisfying

$$p(0) = 0, \quad p'(\varrho) > 0 \text{ for all } \varrho > 0, \quad \lim_{\varrho \rightarrow \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = a > 0$$

for a certain $\gamma > 3/2$. Assume that the initial data satisfy

$$\varrho_0 \geq 0, \quad \varrho_0 \not\equiv 0, \quad \varrho_0 \in L^\gamma(\Omega), \quad \varrho_0 |\mathbf{u}_0|^2 \in L^1(\Omega).$$

Then the compressible Navier-Stokes system possesses a suitable weak solution on $(0, T) \times \Omega$.

Weak-strong uniqueness

Strong solutions:

$$0 < \underline{\varrho} \leq \tilde{\varrho}(t, x) \leq \bar{\varrho}, \quad |\tilde{\mathbf{u}}(t, x)| \leq \bar{u} \quad (1)$$

$$\nabla_x \tilde{\varrho} \in L^2(0, T; L^q(\Omega; R^3)), \quad \nabla_x^2 \tilde{\mathbf{u}} \in L^2(0, T; L^q(\Omega; R^{3 \times 3 \times 3})) \quad (2)$$

$$q > \max\left\{3, \frac{3}{\gamma - 1}\right\}.$$

Theorem

Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$. In addition to hypotheses of existence theorem, suppose that p is twice continuously differentiable on the open interval $(0, \infty)$. Assume that the Navier-Stokes system admits a weak solution $\tilde{q}, \tilde{\mathbf{u}}$ in $(0, T) \times \Omega$ belonging to the regularity class specified through (1), (2).

Then $\tilde{q} \equiv q$, $\tilde{\mathbf{u}} \equiv \mathbf{u}$, where $q, \mathbf{u}$ is the suitable weak solution of the Navier-Stokes system emanating from the same initial data.

Conditional regularity

Using the result of Sun, Wang and Zhang [2010] we have:

Theorem

Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$. Let $\varrho_0, \mathbf{u}_0$ be given such that

$$\varrho_0 \in W^{1,6}(\Omega), \quad 0 < \underline{\varrho} \leq \varrho_0(x) \leq \bar{\varrho} \text{ for all } x \in \Omega,$$

$$\mathbf{u}_0 \in W^{2,2}(\Omega; R^3) \cap W_0^{1,2}(\Omega; R^3).$$

Suppose that the pressure p satisfies the hypotheses of existence theorem, and that

$$\mu > 0, \quad \eta = 0.$$

Let $\varrho, \mathbf{u}$ be a suitable weak solution of the Navier-Stokes system in $(0, T) \times \Omega$.

If, in addition,

$$\operatorname{ess\,sup}_{(0,T) \times \Omega} \varrho < \infty,$$

then $\varrho, \mathbf{u}$ is the unique (strong) solution of the Navier-Stokes.

Corollary

Let Ω and the initial data $\varrho_0, \mathbf{u}_0$ be the same as in the previous theorem. Assume that $\varrho, \mathbf{u}$ is a suitable weak solution of the Navier-Stokes system such that

$$\operatorname{ess\,inf}_{x \in \Omega} \varrho(\tau, x) = 0 \text{ for a certain } \tau \in (0, T).$$

Then there exists $0 < \tau_0 \leq \tau$ such that

$$\limsup_{t \rightarrow \tau_0} [\operatorname{ess\,sup}_{x \in \Omega} \varrho(t, x)] = \infty.$$

Stability

Theorem

Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$. In addition to hypotheses of existence theorem, suppose that p is twice continuously differentiable on the open interval $(0, \infty)$. Assume that the Navier-Stokes system admits a (strong) solution $\tilde{\varrho}, \tilde{\mathbf{u}}$. In addition, let

$$\varrho_{0,\varepsilon} \rightarrow \tilde{\varrho}_0 \text{ in } L^\gamma(\Omega), \quad \varrho_{0,\varepsilon} \geq 0, \quad \int_{\Omega} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon} - \tilde{\mathbf{u}}_0|^2 \, dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Then

$$\sup_{\tau \in [0, T]} \|\varrho_\varepsilon(\tau, \cdot) - \tilde{\varrho}(\tau, \cdot)\|_{L^\gamma(\Omega)} \rightarrow 0$$

$$\sup_{\tau \in [0, T]} \|\varrho_\varepsilon \mathbf{u}_\varepsilon(\tau, \cdot) - \tilde{\varrho} \tilde{\mathbf{u}}(\tau, \cdot)\|_{L^1(\Omega; \mathbb{R}^3)} \rightarrow 0,$$

and

$$\mathbf{u}_\varepsilon \rightarrow \tilde{\mathbf{u}} \text{ in } L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^3)),$$

where $\varrho_\varepsilon, \mathbf{u}_\varepsilon$ is a suitable weak solution of the Navier-Stokes system emanating from the initial data $\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}$.

Finite energy solutions are suitable

[E.F., A. Novotný, Bum Ja Jin, Preprint 2011]

Finite energy weak solutions are suitable weak solutions

- general domains (bounded, unbounded, irregular)
- general boundary conditions (no-slip, complete slip, Navier's slip)
- general driving force