Numerical results for an optimal design problem with a non-linear cost in the gradient

Manuel Luna-Laynez
Dpto. Ecuaciones Diferenciales y Análisis Numérico
Universidad de Sevilla, Spain

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   J.Casado-Díaz, C.Castro, MLL, E.Zuazua

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   J.Casado-Díaz, J.Couce-Calvo, MLL, J.D.Martín-Gómez

1. Motivation and statement of the problem

Data: $\Omega \subset \mathbb{R}^N$ bounded,open, $\kappa > 0$, $F_1, F_2 : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$,
$f \in L^\infty(\Omega)$, A, B positive matrices

Control problem in the coefficients:

$$(P) \left\{ \begin{array}{l} \text{Find } \omega_0 \in \mathcal{U} \text{ such that} \\ \mathcal{J}(\omega_0) = \min_{\omega \in \mathcal{U}} \mathcal{J}(\omega) \end{array} \right.$$

where

- Set of controls:

$$\mathcal{U} = \{\omega \subset \Omega \, : \, \omega \text{ measurable }, \quad |\omega| \leq \kappa\}$$

- Cost functional:

$$\mathcal{J}(\omega) = \int_\omega F_1\left(x, u_\omega, \nabla u_\omega\right) dx + \int_{\Omega \setminus \omega} F_2\left(x, u_\omega, \nabla u_\omega\right) dx$$

with u_ω defined by

$$u_\omega \in H_0^1(\Omega), \quad -\text{div}\Big(\left(A\chi_\omega + B\chi_{\Omega\setminus\omega}\right) \nabla u_\omega\Big) = f \;\text{ in } \Omega$$

Optimal design of composite materials (mixing materials A and B).

$$(P)\begin{cases} \text{Find } \omega_0 \in \mathcal{U} \text{ such that} \\ \mathcal{J}(\omega_0) = \min\limits_{\omega \in \mathcal{U}} \mathcal{J}(\omega) \end{cases}$$

Existence of solution: Direct Method in the Calculus of Variations

If $\omega_n \in \mathcal{U}$ is a minimizing sequence, i.e. $\mathcal{J}(\omega_n) \to \inf\limits_{\omega \in \mathcal{U}} \mathcal{J}(\omega)$, then

$$\chi_{\omega_n} \overset{*}{\longrightarrow} \theta \ \text{ in } L^\infty(\Omega), \quad A\chi_{\omega_n} + B\chi_{\Omega\setminus\omega_n} \overset{H}{\longrightarrow} M \in L^\infty(\Omega)^{N\times N}$$

$$u_{\omega_n} \rightharpoonup u \text{ in } H_0^1(\Omega), \text{with} \quad -div\,(M\nabla u) = f \ \text{ in } \Omega$$

however

$$\nexists\ \omega \in \mathcal{U} \ \text{ such that } \ \theta = \chi_\omega \ \text{ and } \ M = A\chi_\omega + B\chi_{\Omega\setminus\omega}$$

(lack of compactness of the set of controls!)

$$\int F_i(x, u_{\omega_n}, \nabla u_{\omega_n})\, dx \not\longrightarrow \int F_i(x, u, \nabla u)\, dx$$

(lack of continuity in H_0^1-weak of the cost functional!)

Control problem (P) has not a solution in general (F.Murat).

A relaxation of (P):

$$(\hat{P}) \left\{ \begin{array}{l} \text{Find } (\theta_0, M_0) \in \hat{\mathcal{U}} \text{ such that} \\ \hat{\mathcal{J}}(\theta_0, M_0) = \min_{(\theta,M)\in\hat{\mathcal{U}}} \hat{\mathcal{J}}(\theta, M) \end{array} \right.$$

where

- Set of relaxed controls

$$\hat{\mathcal{U}} = \Big\{(\theta, M) \,:\, \exists \omega_n \in \mathcal{U} \text{ such that } \chi_{\omega_n} \xrightarrow{\ *\ } \theta \text{ in } L^\infty(\Omega),$$
$$A\chi_{\omega_n} + B\chi_{\Omega\setminus\omega_n} \xrightarrow{\ H\ } M \in L^\infty(\Omega)^{N\times N}\Big\}$$

- Relaxed cost functional

$$\hat{\mathcal{J}}(\theta, M) = \int_\Omega H(x, u_{\theta,M}, \nabla u_{\theta,M}, M\nabla u_{\theta,M}, \theta)\, dx$$

where $u_{\theta,M}$ is defined by

$$u_{\theta,M} \in H^1_0(\Omega), \quad -\mathrm{div}\Big(M\nabla u_{\theta,M}\Big) = f \ \text{ in } \Omega$$

(J.Casado-Diaz, J.Couce-Calvo,J.D.Martin-Gomez for the definition of H)

Remark: The function H is known explicitely only in few cases!
(L.Tartar; Y.Grabovsky; R.Lipton; P.Pedregal…)

We want to solve numerically the control problem (P):

For $h > 0$, find $\omega_h \in \mathcal{U}$ such that

$$\mathcal{J}(\omega_h) - \inf_{\omega \in \mathcal{U}} \mathcal{J}(\omega) \longrightarrow 0 \text{ as } h \to 0$$

and provide convergence rates.

We have two (main) strategies for discretization:

- Discretize directly the original problem (P).
- Discretize the relaxed formulation $(\hat{P})$.

Our goal: To study and compare both strategies.

To compute a "good" two-phase material,

is it worth working hard to obtain the relaxed problem for its discretization?

or on the contrary,

is it preferable to deal with a discretization of the original problem?

2. The one-dimensional design problem

(J.Casado-Díaz, C.Castro, MLL, E.Zuazua)

Dates: $\Omega = (0,1)$, α, β, $\kappa > 0$, $f \in L^\infty(0,1)$, $F_i \in W^{1,\infty}((0,1) \times \mathbb{R} \times \mathbb{R})$

Control problem in the coefficients:

$$(P) \begin{cases} \text{Find } \omega_0 \in \mathcal{U} \text{ such that} \\ \mathcal{J}(\omega_0) = \min_{\omega \in \mathcal{U}} \mathcal{J}(\omega) \end{cases}$$

where

- Set of controls:

$$\mathcal{U} = \{\omega \subset (0,1) \; : \; \omega \text{ measurable }, \quad |\omega| \leq \kappa\}$$

- Cost functional:

$$\mathcal{J}(\omega) = \int_\omega F_1 \left(x, u_\omega, \frac{du_\omega}{dx} \right) dx + \int_{(0,1)\setminus\omega} F_2 \left(x, u_\omega, \frac{du_\omega}{dx} \right) dx$$

with u_ω defined by

$$u_\omega \in H_0^1(0,1), \quad -\frac{d}{dx}\left((\alpha \chi_\omega + \beta(1 - \chi_\omega)) \frac{du_\omega}{dx} \right) = f \;\text{ in } (0,1)$$

Theorem [Relaxation]

A relaxation of (P) is given by:

$$(\hat{P}) \begin{cases} \text{Find } \theta_0 \in \hat{\mathcal{U}} \text{ such that} \\ \hat{\mathcal{J}}(\theta_0) = \min_{\theta \in \hat{\mathcal{U}}} \hat{\mathcal{J}}(\theta) \end{cases}$$

where

- Set of relaxed controls

$$\hat{\mathcal{U}} = \left\{\theta \,:\, \theta \in L^\infty(0,1;[0,1]) \text{ with } \int_0^1 \theta dx \leq \kappa\right\}$$

- Relaxed cost functional

$$\hat{\mathcal{J}}(\theta) = \int_0^1 \left(\theta F_1\left(x, u_\theta, \frac{\lambda_\theta}{\alpha}\frac{du_\theta}{dx}\right) + (1-\theta)F_2\left(x, u_\theta, \frac{\lambda_\theta}{\beta}\frac{du_\theta}{dx}\right)\right) dx$$

with u_θ defined by

$$u_\theta \in H_0^1(0,1), \quad -\frac{d}{dx}\left(\lambda_\theta \frac{du_\theta}{dx}\right) = f \text{ in } (0,1)$$

where $\lambda_\theta = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1}, \forall \theta \in \hat{\mathcal{U}}$ (harmonic mean)

To solve numerically the control problem we have two (main) strategies for discretization:

- Discretize directly the original problem (P).
- Discretize the relaxed formulation $(\hat{P})$.

Next we now approximate both optimal design problems, the original and the relaxed one, but we do it in a stratified manner, in two levels:

Level 1: Discrete set of controls but continuous ODE.

Level 2: Discrete set of controls and also discrete approximation of ODE (full discretization).

For $r>0$, we consider a partition $\mathcal{Q}^r=\{y_k\}_{k=0}^{m_r}$ of $[0,1]$, with

$$r=\max_{1\le k\le m_r}(y_k-y_{k-1})\,.$$

Discretization of (P), Level 1:

$$(P^r)\begin{cases}\text{Find } \omega_0^r\in\mathcal{U}^r \text{ such that}\\ \mathcal{J}(\omega_0^r)=\min_{\omega\in\mathcal{U}^r}\mathcal{J}(\omega)\end{cases}$$

where

$$\mathcal{U}^r=\{\omega\in\mathcal{U}\;:\;\exists J\subset\{1\dots,m_r\}\text{ such that }\omega=\cup_{k\in J}(y_{k-1},y_k)\}$$

$$\mathcal{J}(\omega)=\int_\omega F_1\left(x,u_\omega,\frac{du_\omega}{dx}\right)dx+\int_{(0,1)\setminus\omega}F_2\left(x,u_\omega,\frac{du_\omega}{dx}\right)dx$$

with $u_\omega\in H_0^1(0,1)$ satisfying

$$-\frac{d}{dx}\left((\alpha\chi_\omega+\beta(1-\chi_\omega))\frac{du_\omega}{dx}\right)=f\quad\text{in }(0,1)$$

Discretization of $(\hat{P})$, Level 1:

$$(\hat{P}^r)\begin{cases}\text{Find } \theta_0^r\in\hat{\mathcal{U}}^r \text{ such that}\\ \hat{\mathcal{J}}(\theta_0^r)=\min_{\theta\in\hat{\mathcal{U}}^r}\hat{\mathcal{J}}(\theta)\end{cases}$$

where

$$\hat{\mathcal{U}}^r=\{\theta\in\hat{\mathcal{U}}\;:\;\theta\text{ constant in every }(y_{k-1},y_k)\}$$

$$\hat{\mathcal{J}}(\theta)=\int_0^1\left(\theta F_1\left(x,u_\theta,\frac{\lambda_\theta}{\alpha}\frac{du_\theta}{dx}\right)+(1-\theta)F_2\left(x,u_\theta,\frac{\lambda_\theta}{\beta}\frac{du_\theta}{dx}\right)\right)dx$$

with $u_\theta\in H_0^1(0,1)$ satisfying

$$-\frac{d}{dx}\left(\lambda_\theta\frac{du_\theta}{dx}\right)=f\quad\text{in }(0,1)$$

Theorem 1 [Discretization of (P), Level 1]

Problem (P^r) has a solution for every $r > 0$, and we have

$$0 \leq \min_{\omega \in \mathcal{U}^r} \mathcal{J}(\omega) - \inf_{\omega \in \mathcal{U}} \mathcal{J}(\omega) \leq C r^{\frac{1}{2}}.$$

Moreover, if for some integer $\ell \geq 1$, we have that f belongs to $W^{\ell,1}(0,1)$ and $F_i(x, s, \xi)$ is independent of s and belong to $C^{\ell,1}_{loc}([0,1] \times \mathbb{R})$, then we have

$$0 \leq \min_{\omega \in \mathcal{U}^r} \mathcal{J}(\omega) - \inf_{\omega \in \mathcal{U}} \mathcal{J}(\omega) \leq C r^{\frac{\ell+1}{\ell+2}}.$$

Theorem 2 [Discretization of $(\hat{P})$, Level 1]

Problem $(\hat{P}^r)$ has a solution for every $r > 0$, and we have

$$0 \leq \min_{\theta \in \hat{\mathcal{U}}^r} \hat{\mathcal{J}}(\theta) - \inf_{\omega \in \mathcal{U}} \mathcal{J}(\omega) = o(r).$$

Moreover, if problem $(\hat{P})$ has a solution θ_0 in $BV(0,1)$, then

$$0 \leq \min_{\theta \in \hat{\mathcal{U}}^r} \hat{\mathcal{J}}(\theta) - \inf_{\omega \in \mathcal{U}} \mathcal{J}(\omega) \leq C r^2.$$

Remarks:

- The convergence rate for $(\hat{P}^r)$ is better than the one for (P^r).

$$\text{For } (P^r): \quad 0 \le \min_{\omega \in \mathcal{U}^r} \mathcal{J}(\omega) - \inf_{\omega \in \mathcal{U}} \mathcal{J}(\omega) \le C r^{\frac{1}{2}}$$

$$\text{For } (\hat{P}^r): \quad 0 \le \min_{\theta \in \hat{\mathcal{U}}^r} \hat{\mathcal{J}}(\theta) - \inf_{\omega \in \mathcal{U}} \mathcal{J}(\omega) = o(r)$$

- Problem $(\hat{P}^r)$ is simpler to solve because the set of controls $\hat{\mathcal{U}}^r$ is convex, and for example we can apply descent methods.

$$\mathcal{U}^r = \{\omega \subset (0,1) \,:\, |\omega| \le \kappa,\, \exists J \subset \{1 \ldots, m_r\} \text{ such that } \omega = \cup_{k \in J}(y_{k-1}, y_k)\}$$

$$\hat{\mathcal{U}}^r = \{\theta \in L^\infty(0,1;[0,1]) \,:\, \int_0^1 \theta dx \le \kappa,\ \theta \text{ constant in every } (y_{k-1}, y_k)\}$$

Discretizations of problems (P) and $(\hat{P})$: Level 2

Disctretization of the state and the cost: Finite Element Approximation

For $h > 0$, we take a partition $\mathcal{Q}^h = \{x_i\}_{i=0}^{n_h}$ of $[0,1]$, with

$$h = \max_{1 \le i \le n_h} (x_i - x_{i-1})$$

and we define

$$W^h = \{v \in C_0^0([0,1]) \,:\, v \text{ is affine on every } (x_{i-1}, x_i)\}$$

For $\theta \in \hat{\mathcal{U}}$, constant in every (x_{i-1}, x_i), we define $\tilde{u}_\theta \in W^h$ by

$$\int_0^1 \lambda_\theta \frac{d\tilde{u}_\theta}{dx}\frac{dv}{dx}dx = \int_0^1 f v dx, \quad \forall v \in W^h$$

and

$$\hat{\mathcal{J}}^h(\theta) = \int_0^1 \left(\theta F_1\left(x, \tilde{u}_\theta, \frac{\lambda_\theta}{\alpha}\frac{d\tilde{u}_\theta}{dx}\right) + (1-\theta)F_2\left(x, \tilde{u}_\theta, \frac{\lambda_\theta}{\beta}\frac{d\tilde{u}_\theta}{dx}\right)\right) dx$$

For $\omega \in \mathcal{U}$, with $\omega = \cup_{i \in J}(x_{i-1}, x_i)$, $J \subset \{1\ldots, n_h\}$, we denote $\tilde{u}_\omega = \tilde{u}_{\chi_\omega}$ and define

$$\mathcal{J}^h(\omega) = \int_\omega F_1\left(x, \tilde{u}_\omega, \frac{d\tilde{u}_\omega}{dx}\right) dx + \int_{(0,1)\setminus\omega} F_2\left(x, \tilde{u}_\omega, \frac{d\tilde{u}_\omega}{dx}\right) dx$$

Disctretization of the set of controls:

For $r > 0$, we take a partition $\mathcal{Q}^r = \{y_k\}_{k=0}^{m_r}$ of $[0,1]$, with $\mathcal{Q}^r \subset \mathcal{Q}^h$ and

$$r = \max_{1 \le k \le m_r} (y_k - y_{k-1}) \, .$$

and we define

$$\mathcal{U}^r = \{\omega \subset (0,1) \, : \, |\omega| \le \kappa, \, \exists J \subset \{1 \ldots, m_r\} \text{ such that } \omega = \cup_{k \in J} (y_{k-1}, y_k)\}$$

$$\hat{\mathcal{U}}^r = \{\theta \in L^\infty(0,1;[0,1]) \, : \, \int_0^1 \theta dx \le \kappa, \; \theta \text{ constant in every } (y_{k-1}, y_k)\}$$

Full discretization of (P):

We take $\mathcal{Q}^r = \mathcal{Q}^h = \{x_i\}_{i=0}^{n_h}$

$$(P_c^h) \begin{cases} \text{Find } \omega_0^h \in \mathcal{U}^h \text{ such that} \\ \mathcal{J}^h(\omega_0^h) = \min_{\omega \in \mathcal{U}^h} \mathcal{J}^h(\omega) \end{cases}$$

Full discretization of $(\hat{P})$:

We take $\mathcal{Q}^r = \{y_k\}_{k=0}^{m_r} \subset \mathcal{Q}^h = \{x_i\}_{i=0}^{n_h}$

with $r = \sqrt{h}$

$$(\hat{P}_c^h) \begin{cases} \text{Find } \theta_0 \in \hat{\mathcal{U}}^r \text{ such that} \\ \hat{\mathcal{J}}^h(\theta_0) = \min_{\theta \in \hat{\mathcal{U}}^r} \hat{\mathcal{J}}^h(\theta) \end{cases}$$

Theorem [Full discretization of (P)]

Problem (P_c^h) has a solution for every $h > 0$. Morover, every solution ω_0 satisfies

$$0 \leq \mathcal{J}(\omega_0) - \inf_{\omega \in \mathcal{U}} \mathcal{J}(\omega) \leq C h^{\frac{1}{2}}.$$

Moreover, if for some nonnegative integer ℓ, we have that f belongs to $W^{\ell,1}(0,1)$ and $F(x,s,\xi)$ is independent of s and belong to $C^{\ell,1}_{loc}([0,1]\times\mathbb{R})$, then we have

$$0 \leq \mathcal{J}(\omega_0) - \inf_{\omega \in \mathcal{U}} \mathcal{J}(\omega) \leq C h^{\frac{\ell+1}{\ell+2}}.$$

Theorem [Full discretization of $(\hat{P})$]

Problem $(\hat{P}_c^h)$ has a solution for every $h > 0$. Moreover, if we assume that exists an optimal control of bounded variation for $(\hat{P})$, then every θ_0 solution of $(\hat{P}_c^h)$ satisfies

$$0 \leq \mathcal{J}(\theta_0) - \inf_{\omega \in \mathcal{U}} \mathcal{J}(\omega) \leq C h.$$

Remarks:

- In the full discretization of the unrelaxed formulation, the PDE and the control are discretized in a same fine grid of size h.

- The full discretization of the relaxed problem, constitutes a bigrid strategy: The PDE is discretized in the fine grid of size h while the control is discretized in the coarser one of size $\sqrt{h}$. And it gives a faster convergence with a lower computacional cost!

- Previous results demonstrate that relaxation may be as important for calculation as it is for existence.

Postprocessing of relaxed controls:

A difficulty: Solutions of $(\hat{P}^r)$ are not physical solutions (they are not characteristics!)

Proposition. Given

$$\theta = \sum_{k=1}^{m_r} t_k \chi_{(y_{k-1},y_k)} \in \hat{\mathcal{U}}^r, \quad \text{with } t_k \in [0,1], \forall k \in \{1,\ldots,m_r\}.$$

Taking

$$j_k = \left[\frac{y_k - y_{k-1}}{r^2}\right] + 1, \; s_k = \frac{y_k - y_{k-1}}{j_k}, \quad \forall\, k \in \{1,\ldots,m_r\},$$

we define $\omega \subset (0,1)$ as

$$\omega = \bigcup_{k=1}^{m_r} \bigcup_{i=1}^{j_k} (y_{k-1} + (i-1)s_k, y_{k-1} + (i-1+t_k)s_k).$$

Then, we have

$$\left|\hat{\mathcal{J}}(\theta) - \mathcal{J}(\omega)\right| \leq Cr^2.$$

3. The N-dimensional design problem

(J.Casado-Díaz, J.Couce-Calvo, MLL, J.D.Martín-Gómez)

Data: $\Omega \subset \mathbb{R}^N$ bounded,open, $\kappa > 0$, $f \in L^\infty(\Omega)$,
A, B positive matrices, $\exists (B-A)^{-1}$ (for simplicity),
$F : \mathbb{R}^N \to \mathbb{R}$ satisfying

$$|F(\xi_1) - F(\xi_2)| \leq C(1 + |\xi_1| + |\xi_2|)\, |\xi_1 - \xi_2| \quad \forall \xi_1, \xi_2 \in \mathbb{R}^N.$$

Control problem in the coefficients:

$$(P) \begin{cases} \text{Find } \omega_0 \in \mathcal{U} \text{ such that} \\ \mathcal{J}(\omega_0) = \min_{\omega \in \mathcal{U}} \mathcal{J}(\omega) \end{cases}$$

where

- Set of controls:

$$\mathcal{U} = \{\omega \subset \Omega \,:\, \omega \text{ measurable} \,, \quad |\omega| \leq \kappa\}$$

- Cost functional:

$$\mathcal{J}(\omega) = \int_\Omega F\left(\nabla u_\omega\right) dx$$

with u_ω defined by

$$u_\omega \in H_0^1(\Omega), \quad -\mathrm{div}\Big(\big(A\chi_\omega + B\chi_{\Omega\setminus\omega}\big)\, \nabla u_\omega\Big) = f \;\text{ in } \Omega$$

A relaxation of (P):

$$(\hat{P}) \begin{cases} \text{Find } (\theta_0, M_0) \in \hat{\mathcal{U}} \text{ such that} \\ \hat{\mathcal{J}}(\theta_0, M_0) = \min\limits_{(\theta,M)\in\hat{\mathcal{U}}} \hat{\mathcal{J}}(\theta, M) \end{cases}$$

where

- Set of relaxed controls

$$\hat{\mathcal{U}} = \Big\{(\theta, M) \,:\, \exists \omega_n \in \mathcal{U} \text{ such that } \chi_{\omega_n} \xrightarrow{*} \theta \text{ in } L^\infty(\Omega), \\ A\chi_{\omega_n} + B\chi_{\Omega\setminus\omega_n} \xrightarrow{H} M \in L^\infty(\Omega)^{N\times N}\Big\}$$

- Relaxed cost functional

$$\hat{\mathcal{J}}(\theta, M) = \int_\Omega H(\nabla u_{\theta,M}, M\nabla u_{\theta,M}, \theta)\, dx$$

where $u_{\theta,M}$ is defined by

$$u_{\theta,M} \in H_0^1(\Omega), \quad -\mathrm{div}\Big(M\nabla u_{\theta,M}\Big) = f \ \text{ in } \Omega$$

About the relaxed set of controls:

$$\hat{\mathcal{U}} = \Big\{(\theta, M) \,:\, \exists \omega_n \in \mathcal{U} \text{ such that } \chi_{\omega_n} \overset{*}{\longrightarrow} \theta \text{ in } L^\infty(\Omega),$$
$$A\chi_{\omega_n} + B\chi_{\Omega\setminus\omega_n} \overset{H}{\longrightarrow} M \in L^\infty(\Omega)^{N\times N}\Big\}$$

The following theorem characterizes the set $\hat{\mathcal{U}}$

Definition: For $p \in [0, 1]$, we define $\mathcal{K}(p)$ by

$$\mathcal{K}(p) = \Big\{M \in \mathbb{R}^{N\times N} \;:\; \exists \chi_{\omega_n} \overset{*}{\rightharpoonup} p \text{ in } L^\infty(\Omega),\, A\chi_{\omega_n} + B\chi_{\Omega\setminus\omega_n} \overset{H}{\to} M\Big\}$$

Theorem: Given $(\theta, M) \in L^\infty(\Omega; [0, 1]) \times L^\infty(\Omega)^{N\times N}$, we have

$$(\theta, M) \in \hat{\mathcal{U}} \Longleftrightarrow \int_\Omega \theta \, dx \leq \kappa \text{ and}$$
$$M(x) \in \mathcal{K}(\theta(x)) \text{ a.e. } x \in \Omega.$$

Consequence:

$$\hat{\mathcal{U}} = \Big\{(\theta, M) \in L^\infty(\Omega; [0, 1]) \times L^\infty(\Omega)^{N\times N} : \int_\Omega \theta dx \leq \kappa,\; M \in \mathcal{K}(\theta)\Big\}$$

About the relaxed set of controls:

$$\hat{\mathcal{U}} = \left\{ (\theta, M) \in L^\infty(\Omega; [0,1]) \times L^\infty(\Omega)^{N\times N} : \int_\Omega \theta dx \leq \kappa, \;\; M \in \mathcal{K}(\theta) \right\}$$

The sets $\mathcal{K}(p)$, $p \in [0,1]$, are explicitely known only when (L.Tartar; K.A.Lurie, A.V.Cherkaev)

$$A = \alpha I, \; B = \beta I.$$

To deal with $(\hat{P})$ we only need to know the sets $\mathcal{K}(p)\xi$, $\forall \xi \in \mathbb{R}^N$. Indeed we can reformulate $(\hat{P})$ as

$$\begin{cases} \displaystyle \min_{\theta, u, \sigma} \int_\Omega H(\nabla u, \sigma, \theta)\, dx \\ \displaystyle \theta \in L^\infty(\Omega; [0,1]), \;\; \int_\Omega \theta dx \leq \kappa, \quad u \in H^1_0(\Omega) \\ -div\, \sigma = f \;\; \text{in } \Omega, \;\; \sigma \in \mathcal{K}(\theta)\nabla u \text{ a.e. } x \in \Omega, \;\; \sigma \text{ measurable.} \end{cases}$$

Theorem: For every $p \in [0,1]$ an $\xi \in \mathbb{R}^N$, we have

$$\mathcal{K}(p)\xi = \Big\{ \eta \in \mathbb{R}^N : (1-p)(B-A)^{-1}A(B-A)^{-1}(B\xi - \eta)\cdot(B\xi - \eta)$$
$$+p(B-A)^{-1}B(B-A)^{-1}(\eta - A\xi)\cdot(\eta - A\xi) \leq p(1-p)\eta\cdot\xi \Big\}.$$

About the relaxed cost functional:

$$\hat{\mathcal{J}}(\theta, M) = \int_\Omega H(\nabla u_{\theta,M}, M\nabla u_{\theta,M}, \theta)\, dx$$

where

$$H(\xi,\eta,p) = \sup_{\delta>0} H_\delta(\xi,\eta,p), \quad \forall(\xi,\eta,p) \in \mathbb{R}^N \times \mathbb{R}^N \times [0,1].$$

with $H_\delta : \mathbb{R}^N \times \mathbb{R}^N \times [0,1] \longrightarrow \mathbb{R}$, $\delta > 0$, given by

$$\begin{aligned} H_\delta(\xi,\eta,p) = \inf\Big\{ & \int_Y F(\xi + \nabla w)\, dy : \\ & -div\left(\left(A\chi_Z + B\chi_{Y\setminus Z}\right)\right)(\xi + \nabla w) = 0 \ \text{ in } \mathbb{R}^N, \quad w \in H^1_\sharp(Y), \\ & \Big|\int_Y \left(\left(A\chi_Z + B\chi_{Y\setminus Z}\right)\right)(\xi + \nabla w)\ dy - \eta\Big| < \delta, \\ & Z \subset Y \text{ measurable, } |Z| = p \Big\}, \qquad \forall \delta > 0. \end{aligned}$$

In general we do not have an explicit representation of H!

(L.Tartar; Y.Grabovsky; R.Lipton; P.Pedregal…)

Some properties of H:

- $$\begin{aligned} Dom(H) &= \{(\xi,\eta,p)\in\mathbb{R}^N\times\mathbb{R}^N\times[0,1] \;:\; H(\xi,\eta,p)<+\infty\} \\ &= \{(\xi,\eta,p)\in\mathbb{R}^N\times\mathbb{R}^N\times[0,1] \;:\; \eta\in\mathcal{K}(p)\xi\} \end{aligned}$$

- The value of H on the boundary of $Dom(H)$ is given by

$$H(\xi,\eta,p) = pF\left((B-A)^{-1}(\frac{B\xi-\eta}{p})\right) + (1-p)F\left((B-A)^{-1}(\frac{\eta-A\xi}{1-p})\right),$$

$$\forall\,(\xi,\eta,p)\in\partial Dom(H),\;\; p\neq 0,1,$$

$$H(\xi,A\xi,1) = H(\xi,B\xi,0) = F(\xi), \quad \forall\,\xi\in\mathbb{R}^N.$$

- $$\liminf_{n\to\infty}\int_\Omega H(\nabla u_n,\sigma_n,\theta_n)\,dx \geq \int_\Omega H(\nabla u,\sigma,\theta)\,dx,$$

$$\forall\,(u_n,\sigma_n,\theta_n) \text{ such that } \quad u_n \rightharpoonup u \text{ in } H_0^1(\Omega)$$

$$\theta_n \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(\Omega), \quad \sigma_n \rightharpoonup \sigma \text{ in } L^2(\Omega)^N, \quad div\,\sigma_n \to div\,\sigma \text{ in } H^{-1}(\Omega).$$

$$(\hat{P}) \begin{cases} \text{Find } (\theta_0, M_0) \in \widehat{\mathcal{U}} \text{ such that} \\ \widehat{\mathcal{J}}(\theta_0, M_0) = \min\limits_{(\theta,M)\in\widehat{\mathcal{U}}} \widehat{\mathcal{J}}(\theta, M) \end{cases}$$

where

$$\widehat{\mathcal{U}} = \Big\{(\theta, M) \in L^\infty(\Omega; [0,1]) \times L^\infty(\Omega)^{N\times N} : \int_\Omega \theta dx \le \kappa,\ M \in \mathcal{K}(\theta)\Big\}$$

$$\widehat{J}(\theta, M) = \int_\Omega H(\nabla u_{\theta,M}, M\nabla u_{\theta,M}, \theta)\, dx$$

$$u_{\theta,M} \in H^1_0(\Omega), \quad -\text{div}\Big(M\nabla u_{\theta,M}\Big) = f \ \text{ in } \Omega$$

How compute the solutions of $(\hat{P})$ when H is not known explicitely?

We have an explicit formula for $H(\xi, \eta, p)$ when $(\xi, \eta, p) \in \partial Dom(H)$. On the other hand, under additional conditions we have that if (θ, M) is a solution of $(\hat{P})$, then $(\nabla u_{\theta,M}, M\nabla u_{\theta,M}, \theta) \in \partial Dom(H)$.

We develop numeric strategies based on the approximation of H by a larger function $\overline{H}$ or a lower function $\underline{H}$, which coincide with H on some subset of $\partial Dom(H)$.

Analogously to the one-dimensinal case, we use two grids, one for the set of controls and other for the state equation:

Discretization of $\hat{\mathcal{U}}$:

For every $h > 0$, we consider a partition of Ω given by $T_{j,h}$ measurable, $1 \leq j \leq n_h$, such that

$$\Omega = \bigcup_{j=1}^{n_h} T_{j,h}, \ |T_{j,h}| > 0, \ diam(T_{j,h}) < h, \ |T_{j,h} \cap T_{k,h}| = 0, \ j \neq k.$$

Then we define

$$\widehat{\mathcal{U}}^h = \left\{ (\theta, M) \in \widehat{\mathcal{U}} : \theta, M \text{ are constant in every } T_{j,h} \right\}$$

Discretization of the PDE:

For every $h>0$, we consider a closed subspace $V_h\subset H^1_0(\Omega)$ satisfying

$$i)\quad \lim_{h\to 0}\min_{v_h\in V_h}\|v_h-v\|_{H^1_0(\Omega)}=0,\quad \forall v\in H^1_0(\Omega)$$

$$ii)\quad \lim_{h\to 0}\min_{v_h\in V_h}\|v_h-w_h\varphi\|_{H^1_0(\Omega)}=0,\quad \forall \varphi\in C^\infty_c(\Omega),$$

$$\forall w_h\in V_h \text{ bounded in } H^1_0(\Omega)$$

$$iii)\quad \liminf_{h\to 0}\int_\Omega H(\nabla u_h,\sigma_h,\theta_h)\,dx\geq \int_\Omega H(\nabla u,\sigma,\theta)\,dx,$$

$$\forall (u_h,\sigma_h,\theta_h)\in V_h\times L^2(\Omega)^N\times L^\infty(\Omega) \text{ such that}$$

$$\begin{cases} u_h\rightharpoonup u \text{ in } H^1_0(\Omega),\quad \theta_h\overset{*}{\rightharpoonup}\theta \text{ in } L^\infty(\Omega),\quad \sigma_h\rightharpoonup\sigma \text{ in } L^2(\Omega)^N,\\ \displaystyle\lim_{h\to 0}\max_{v_h\in V_h\setminus\{0\}}\frac{1}{\|v_h\|_{H^1_0(\Omega)}}\int_\Omega(\sigma_h-\sigma)\cdot\nabla v_h\,dx=0.\end{cases}$$

Then, for every $(\theta,M)\in\widehat{\mathcal{U}}_h$ we define $u^h_{\theta,M}$ by

$$u^h_{\theta,M}\in V_h,\qquad \int_\Omega M\nabla u^h_{\theta,M}\cdot\nabla v^h\,dx=\langle f,v^h\rangle,\quad \forall v^h\in V_h$$

Are properties $i)$, $ii)$ and $iii)$ too restrictive?

- $V_h = H_0^1(\Omega)$, $\forall h > 0$, satisfies $i)$, $ii)$, $iii)$.

  The choice $V_h = H_0^1(\Omega)$ implies to solve exactly the state equation (i.e., continuous PDE!).

- If V_h is the usual finite elements, then it satisfies $i)$, $ii)$.

- In examples where H is known, every closed subspace V_h satisfies $iii)$.

Take $\overline{H} : \mathbb{R}^N \times \mathbb{R}^N \times [0,1] \to \mathbb{R}$ lower semicontinuous, with

$$\begin{cases} \overline{H}(\xi,\eta,p) \geq H(\xi,\eta,p), \quad \text{in } (\mathbb{R}^N \times \mathbb{R}^N \times [0,1]) \\ \overline{H}(\xi,A\xi,1) = \overline{H}(\xi,B\xi,0) = H(\xi,A\xi,1) = H(\xi,B\xi,0) = F(\xi) \end{cases}$$

and consider the discrete control problem

$$(\overline{P}_c^h) \begin{cases} \text{Find } (\theta_0, M_0) \in \widehat{\mathcal{U}}_h \text{ such that} \\ \bar{\mathcal{J}}_h(\theta_0, M_0) = \min\limits_{(\theta,M)\in\widehat{\mathcal{U}}_h} \bar{\mathcal{J}}_h(\theta, M) \end{cases}$$

where

$$\widehat{\mathcal{U}}^h = \Big\{(\theta, M) \in \widehat{\mathcal{U}} : \theta, M \text{ are constant in every } T_{j,h}\Big\}$$

$$\bar{\mathcal{J}}^h(\theta, M) = \int_\Omega \overline{H}(\nabla u^h_{\theta,M}, M\nabla u^h_{\theta,M}, \theta)\, dx$$

$$u^h_{\theta,M} \in V_h, \qquad \int_\Omega M\nabla u^h_{\theta,M} \cdot \nabla v^h \, dx = \int_\Omega f v^h \, dx, \quad \forall v^h \in V_h$$

Theorem: Problem $(\overline{P}^h)$ has a solution (θ_h, M_h). Besides, denoting $u_h = u^h_{\theta,M}$, for a subsequence we have

$$\begin{cases} u_h \rightharpoonup u \text{ in } H^1_0(\Omega) \\ M_h \nabla u_h \rightharpoonup \sigma \text{ in } L^2(\Omega)^N \\ \theta_h \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(\Omega) \end{cases}$$

The functions u, σ, θ satisfy there exists $M \in L^\infty(\Omega)^{N\times N}$ such that $\sigma = M\nabla u$ and

$$(\theta, M) \text{ is a solution of } (\hat{P}).$$

Moreover

$$\lim_{h\to 0} \int_\Omega \overline{H}(\nabla u_h, M_h \nabla u_h, \theta_h)\, dx = \int_\Omega H(\nabla u, M\nabla u, \theta)\, dx.$$

Remark: There is an analogous result replacing H by a lower approximation $\underline{H}$.

Example 1:

$$\overline{H}(\xi, A\xi, 1) = \overline{H}(\xi, B\xi, 0) = F(\xi)$$
$$\overline{H}(\xi, \eta, p) = +\infty \text{ otherwise}$$

Example 2:

$$\overline{H}(\xi, \eta, p) = pF\left((B-A)^{-1}(\frac{B\xi - \eta}{p})\right) +$$
$$(1-p)F\left((B-A)^{-1}(\frac{\eta - A\xi}{1-p})\right), \quad \forall(\xi, \eta, p) \in \partial Dom(H),$$
$$\overline{H}(\xi, \eta, p) = +\infty \text{ otherwise}$$

Remark: The choice of $\overline{H}$ given in Example 1 proves the convergence of the dicretization of the original problem, whereas the choice $\overline{H} = H$ proves the convergence of the dicretization of the relaxed one.

Open problems:

- Convergence rates for the N-dimensional case.

- Smoothness properties for the optimal controls (even for the one-dimensional case!).

- Explicit representation of H.

Numerical experiments

1) $\Omega = (0,1)^2$, $A = \alpha I$, $B = \beta I$ with $0 < \alpha < \beta$,

$$(P) \begin{cases} \inf \displaystyle\int_{(0,1)^2} F(\nabla u)dx \\ -div\left((\alpha\chi_\omega + \beta\chi_{(0,1)^2\setminus\omega})\nabla u\right) = f \text{ in } (0,1)^2, \\ u = 0 \text{ in } \partial(0,1)^2, \quad |(0,1)^2 \setminus \omega| \leq \kappa. \end{cases}$$

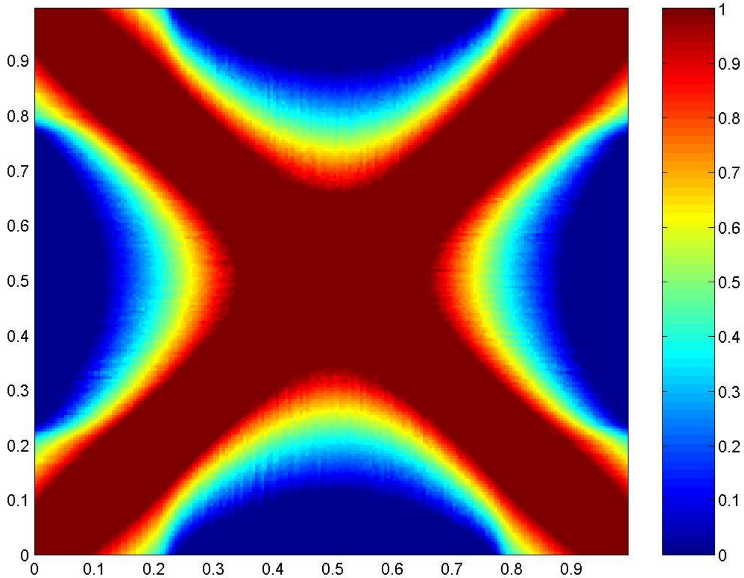
Case 1:

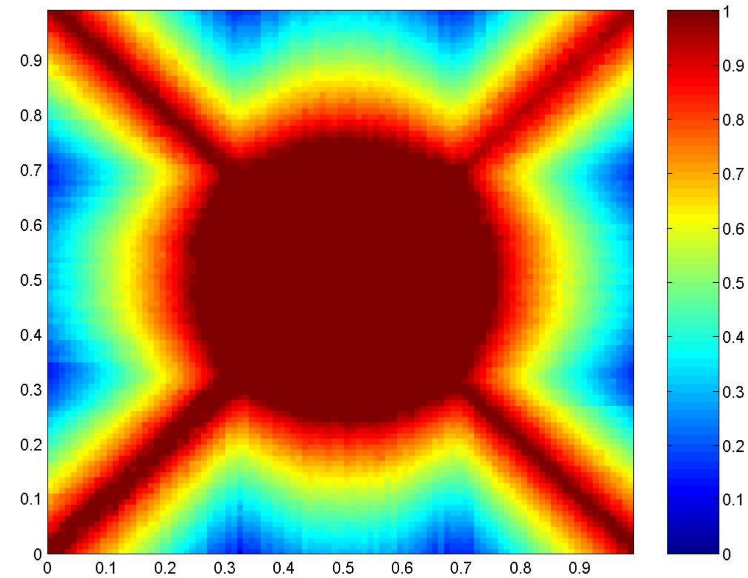
$F(\nabla u) = |\nabla u|^2 \quad \Rightarrow H$ is explicitely known

Case 2:

$F(\nabla u) = |\partial_{x_1} u|^2 + 20|\partial_{x_2} u|^2 \quad \Rightarrow H$ is not explicitely known

Case 1: $F(\nabla u) = |\nabla u|^2$

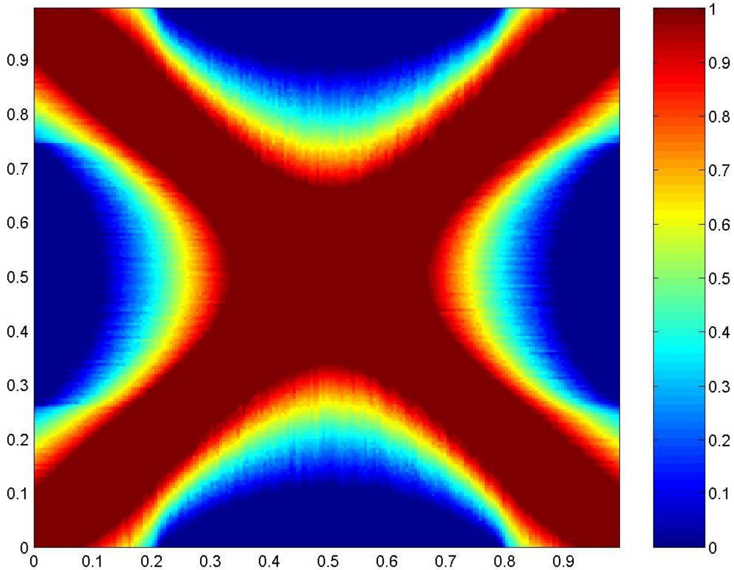


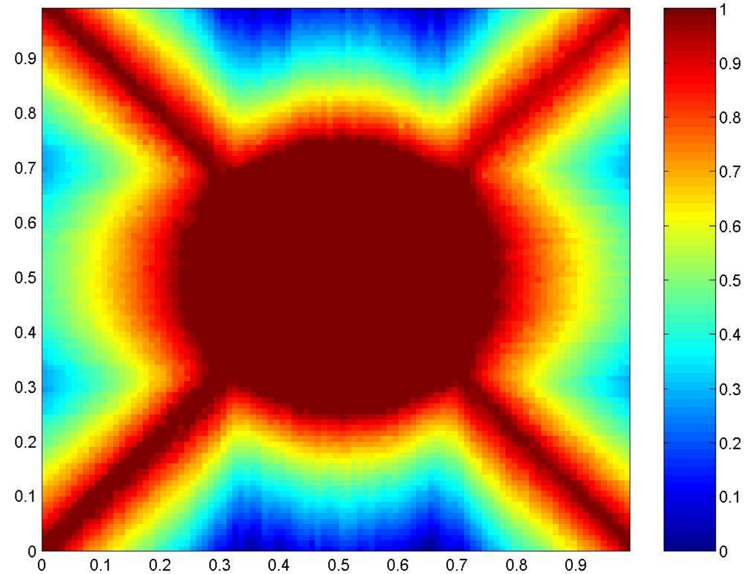


$\alpha = 1,\ \beta = 2,\ \kappa = 0.4,\ f = 1$

$\alpha = 1,\ \beta = 20,\ \kappa = 0.3,\ f = \chi_{\{|x-(0.5,0.5)|>0.25\}}$

Case 2: $F(\nabla u) = |\partial_{x_1} u|^2 + 20|\partial_{x_2} u|^2$





$\alpha = 1,\ \beta = 2,\ \kappa = 0.4,\ f = 1$

$\alpha = 1,\ \beta = 20,\ \kappa = 0.3,\ f = \chi_{\{|x-(0.5,0.5)|>0.25\}}$

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