



Measuring the magnetic birefringence of vacuum: the PVLAS experiment

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"Hands holding the void"
Alberto Giacometti



PVLAS collaboration

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Summary

- Introduction:
 - Aim of the PVLAS experiment
 - Experimental technique
- PVLAS - LNL
 - Overview of published results
- Development phases
 - Improvements with respect to PVLAS-LNL
 - Ferrara Test apparatus
 - Final experiment



Classical Electromagnetism in vacuum

Classical vacuum has no structure.

$$\operatorname{div} \vec{D} = 0; \quad \operatorname{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\operatorname{div} \vec{B} = 0; \quad \operatorname{rot} \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

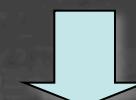
$$L_{EM} = \frac{1}{2\mu_0} \left(\frac{\mathbf{E}^2}{c^2} - \mathbf{B}^2 \right)$$

The superposition principle is valid



$$\vec{D} = \frac{\partial L_{EM}}{\partial \vec{E}}$$

$$\vec{H} = -\frac{\partial L_{EM}}{\partial \vec{B}}$$



$$\vec{D} = \epsilon_0 \vec{E}; \quad \mu_0 \vec{H} = \vec{B}$$

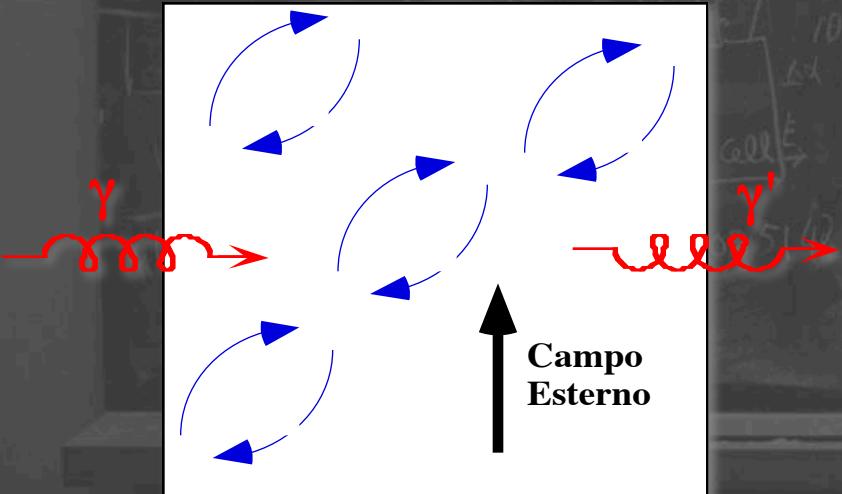


Heisenberg's Uncertainty Principle

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

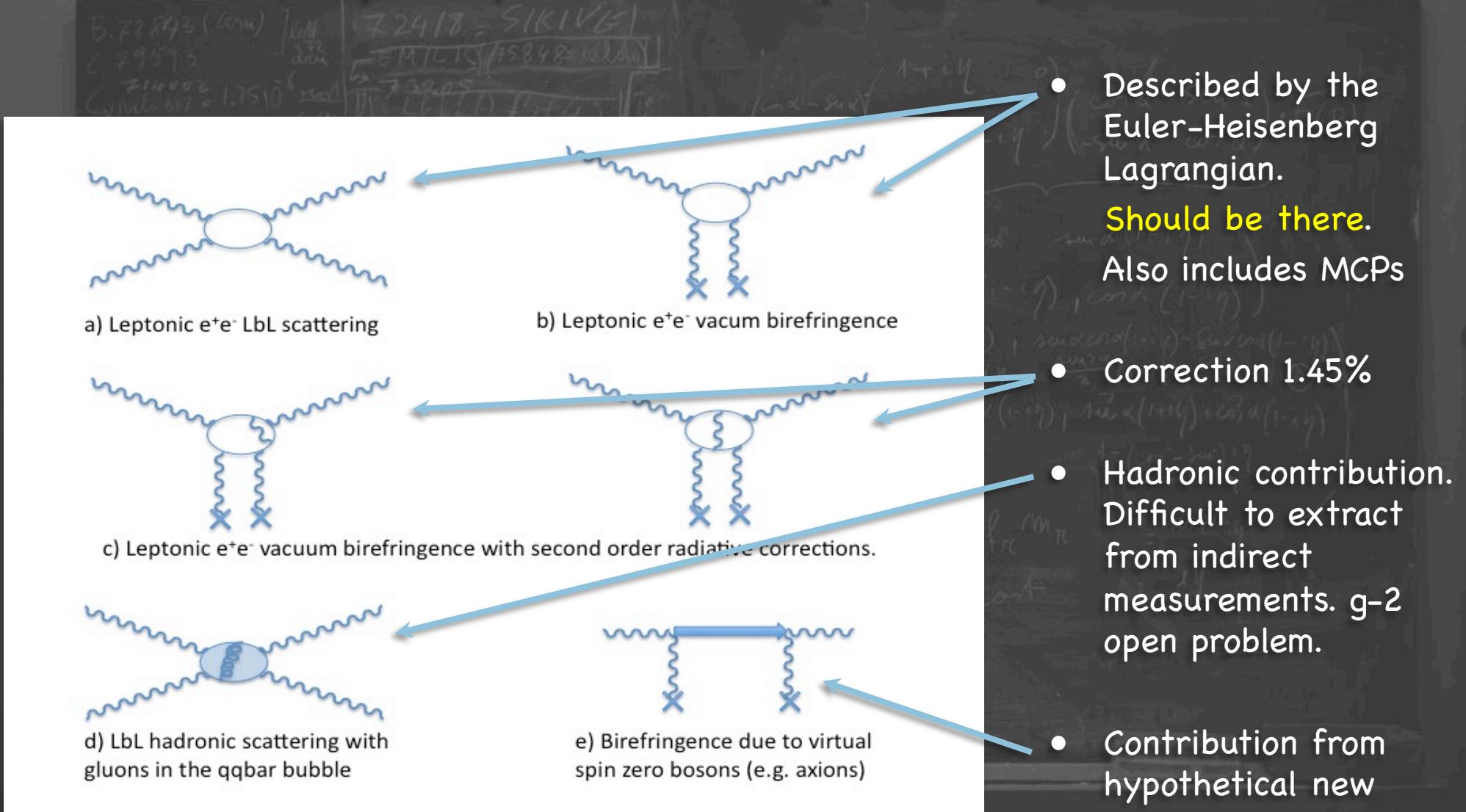
Vacuum is the minimum energy state and can fluctuate into anything compatible with vacuum

Vacuum has a structure which can be observed by perturbing it and probing it.



- QED tests in bound systems - Lamb shift
- QED tests in charged particles - ($g-2$)
- QED tests with photons is missing
- Macroscopically observable (small) non linear effects have been predicted since 1936 but have never been directly observed yet.

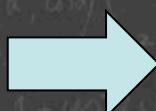
Summary of possible 4 photon processes





Index of refraction vs scattering

$$n = 1 + \frac{2\pi}{k^2} N f(0, E_\gamma)$$



Both related to the scattering amplitude

$$\frac{d\sigma_{\gamma\gamma}}{d\Omega}(\vartheta, E_\gamma) = |f(\vartheta, E_\gamma)|^2$$

Two principle methods for detecting light-light interaction:

- Direct scattering
- Birefringence measurements



Euler-Heisenberg Effective Lagrangian

For fields much smaller than the critical field
($B \ll 4.4 \cdot 10^9$ T; $E \ll 1.3 \cdot 10^{18}$ V/m) one can write

W Heisenberg and H Euler, *Z. Phys.* **98**, 714 (1936)
H Euler, *Ann. Phys.* **26**, 398 (1936)

$$L = L_{em} + L_{HE} = \frac{1}{2\mu_0} \left(\frac{E^2}{c^2} - B^2 \right) + \frac{A_e}{\mu_0} \left[\left(\frac{E^2}{c^2} - B^2 \right)^2 + 7 \left(\frac{\vec{E}}{c} \cdot \vec{B} \right)^2 \right]$$

$$A_e = \frac{2}{45\mu_0} \left(\frac{\alpha^2 \lambda_e^3}{m_e c^2} \right) = 1.32 \cdot 10^{-24} \text{ T}^{-2}$$

Are neglected:

- α^3 terms and higher
- virtual pairs with particles different from $e^+ e^-$

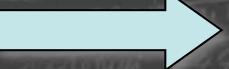


Induced Magnetic Birefringence of Vacuum

- By applying the constitutive relations to L_{EH} one finds

$$\vec{D} = \frac{\partial L_{EH}}{\partial \vec{E}}$$

$$\vec{H} = -\frac{\partial L_{EH}}{\partial \vec{B}}$$



$$\vec{D} = \epsilon_0 \vec{E} + \epsilon_0 A_e \left[4 \left(\frac{E^2}{c^2} - B^2 \right) \vec{E} + 14 (\vec{E} \cdot \vec{B}) \vec{B} \right]$$

$$\mu_0 \vec{H} = \vec{B} + A_e \left[4 \left(\frac{E^2}{c^2} - B^2 \right) \vec{B} - 14 \left(\frac{\vec{E} \cdot \vec{B}}{c^2} \right) \vec{E} \right]$$

- Light propagation is still described by Maxwell's equations in media but they no longer are linear due to E-H correction.



Photon propagating in an external field

Study the propagation of the photon in an external field

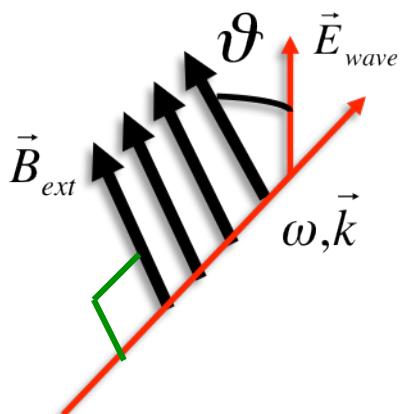
The index of refraction gives information on the nature of the “media”

ANNALS OF PHYSICS: 67, 599–647 (1971)

Photon Splitting and Photon Dispersion in a Strong Magnetic Field

STEPHEN L. ADLER

Considering linearly polarised light traversing an external magnetic field perpendicular to \vec{k}



$$\omega \ll m_e$$

$$B \ll B_{cr} = \frac{m_e^2 c^2}{\hbar e} = 4.42 \times 10^9 \text{ T}$$

$$\begin{cases} \vec{E} = \vec{E}_{wave} \\ \vec{B} = \vec{B}_{ext} + \vec{B}_{wave} \end{cases} \quad \text{and} \quad \left| \vec{B}_{ext} \right| \gg \left| \vec{B}_{wave} \right|$$



Linearly polarized light passing through a transverse external magnetic field perpendicular to \vec{k} .

$$\begin{cases} \epsilon_{\parallel} = 1 + 10A_e B_{Ext}^2 \\ \mu_{\parallel} = 1 + 4A_e B_{Ext}^2 \\ n_{\parallel} = 1 + 7A_e B_{Ext}^2 \end{cases}$$

$$\begin{cases} \epsilon_{\perp} = 1 - 4A_e B_{Ext}^2 \\ \mu_{\perp} = 1 + 12A_e B_{Ext}^2 \\ n_{\perp} = 1 + 4A_e B_{Ext}^2 \end{cases}$$

- $v \neq c$
- anisotropy

A_e can be determined by measuring the magnetic birefringence of vacuum.

$$\Delta n = 3A_e B_{Ext}^2$$

$$\Delta n = 2.5 \cdot 10^{-23} \text{ for } B_{Ext} = 2.5 \text{ T}$$



Photon - photon elastic scattering

From Euler-Heisenberg Lagrangian

$$f_{||}^{(\text{EH})}(\vartheta = 0, E_\gamma) = \frac{16\mu_0}{4\pi\hbar^2 c^2} A_e E_\gamma^3$$

$$f_{\perp}^{(\text{EH})}(\vartheta = 0, E_\gamma) = \frac{28\mu_0}{4\pi\hbar^2 c^2} A_e E_\gamma^3$$

J. Haissinski et al., Phys. Scr. 74, 678 (2006)

$f_{||}$ = parallel linear polarization of incident photons and output photons

f_{\perp} = perpendicular linear polarization of incident photons and output photons



Light-Light scattering

Very low energy photon-photon scattering is proportional to A_e^2 .

For non polarized light:

$$\sigma_{\gamma\gamma}^{[*]} = \frac{973\mu_0^2}{20\pi} \frac{E_\gamma^6}{\hbar^4 c^4} A_e^2$$

From Euler-Heisenberg Lagrangian (S.I. units)

- For light at 1064 nm this predicts a value of $\sigma_{\gamma\gamma} = 1.8 \cdot 10^{-65} \text{ cm}^2$
- Experimentally Bernard et al.^[**] have published $\sigma_{\gamma\gamma} < 1.5 \cdot 10^{-48} \text{ cm}^2$

*Duane et al., Phys Rev. D, vol 57 p. 2443 (1998)

**Bernard D. et al., The European Physical Journal D, vol 10, p. 141 (1999)



Post-Maxwellian models - 1

$$L_{pM} = \frac{\xi}{2\mu_0} \left[\eta_1 \left(\frac{E^2}{c^2} - B^2 \right)^2 + 4\eta_2 \left(\frac{\vec{E} \cdot \vec{B}}{c} \right)^2 \right]$$

$$\xi = \frac{1}{B_{crit}^2} = \left(\frac{e\hbar}{m^2 c^2} \right)^2 = 5 \cdot 10^{-20} \text{ T}^{-2}$$

$$\eta_2^{(QED)} = \frac{7}{4} \eta_1^{(QED)}; \quad \eta_1^{(QED)} = \frac{\alpha}{45\pi}$$

Generalized Lagrangian for which
QED is a particular case



Post-Maxwellian models - 2

Forward scattering

$$f_{\parallel}^{(pM)}(\vartheta = 0, E_{\gamma}) = \frac{8\mu_0}{4\pi\hbar^2 c^2} \xi \eta_1 E_{\gamma}^3$$

$$f_{\perp}^{(pM)}(\vartheta = 0, E_{\gamma}) = \frac{8\mu_0}{4\pi\hbar^2 c^2} \xi \eta_2 E_{\gamma}^3$$

The form of L_{pM} is defined by properties of invariance.

For $\vartheta = 0$

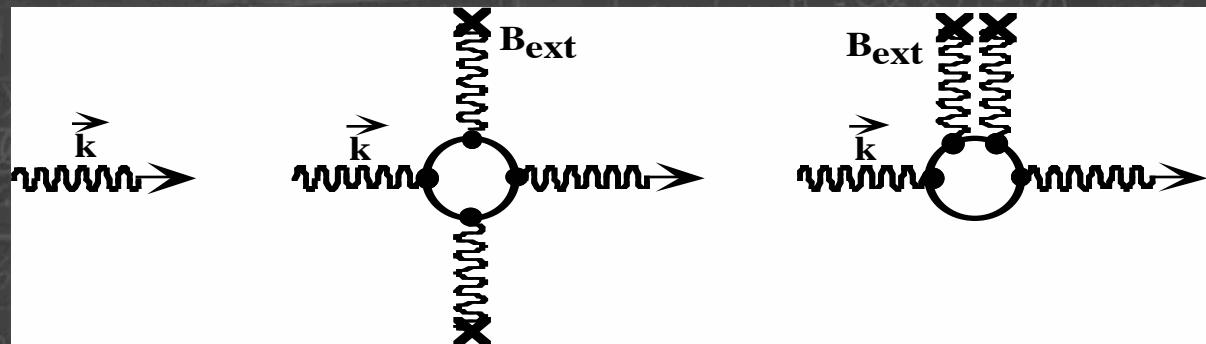
- η_1 represents the interaction between parallel fields
- η_2 represents the interaction between perpendicular fields

Birefringence

$$\Delta n^{(pM)} = 2\xi(\eta_2 - \eta_1)B_{Ext}^2$$

Vacuum as a Medium

- Scheme:
 - perturb the vacuum state with an external field
 - probe the perturbed vacuum state with a polarized laser beam
 - deduce information on the structure of the vacuum state



- The propagation of light will be affected by the polarized vacuum fluctuations.
- Although we consider vacuum, because of the fermion loops the propagation of photons in an external field is now described by Maxwell's equations like those in material media. Furthermore they are no longer linear



Aim of PVLAS

- We want to study the speed of light in the perturbed vacuum and therefore study changes in the refractive index

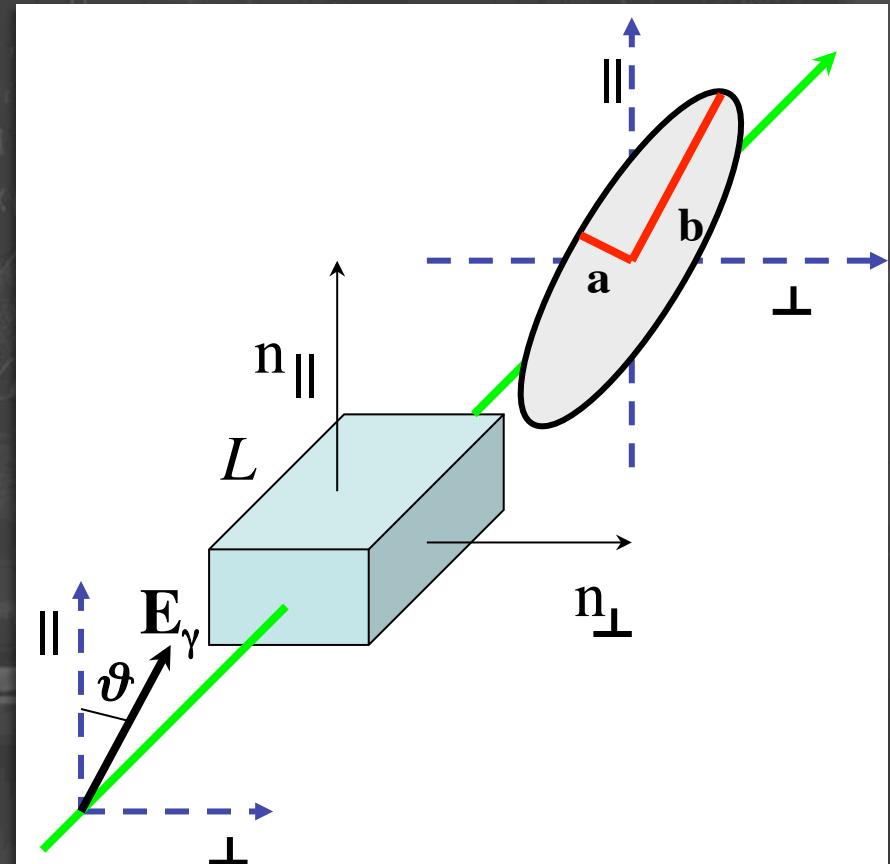
$$n_{\text{vacuum}} = 1 + (\delta n_r - i\kappa)_{\text{field}}$$

- Absolute changes of n_{vacuum} are too difficult to measure so we study anisotropies due to the perturbing field.
- Linear birefringence and linear dichroism
- Linear dichroism (photon splitting) results to be exceedingly small

Linear Birefringence

- A birefringent medium has $n_{||} \neq n_{\perp}$
- A linearly polarized light beam propagating through a birefringent medium will acquire an ellipticity ψ

$$\psi = \frac{a}{b} = \frac{\pi L(n_{||} - n_{\perp})}{\lambda} \sin 2\vartheta$$



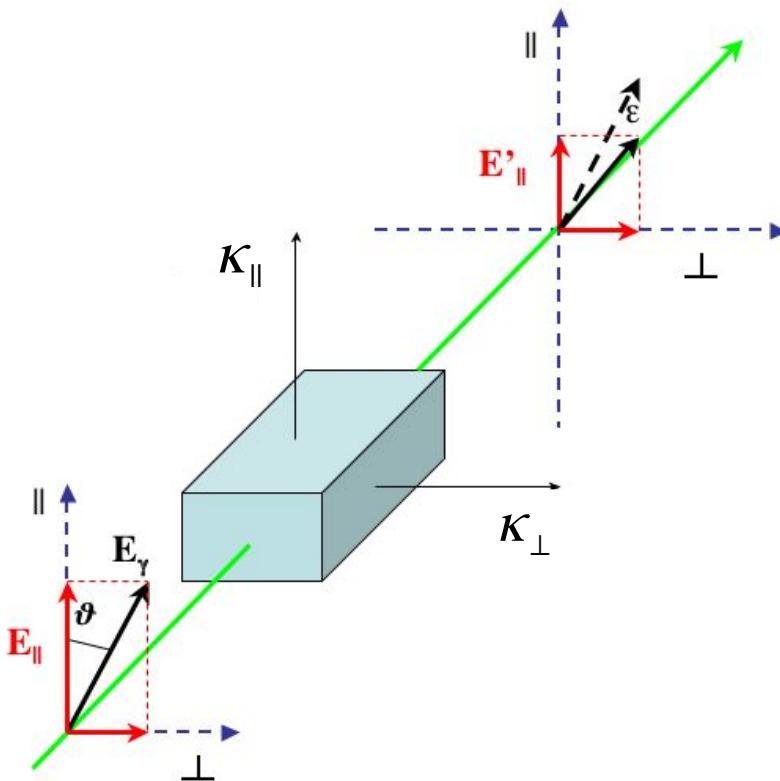
Linear Dichroism

- A dichroic medium has different extinction coefficients: $\kappa_{||} \neq \kappa_{\perp}$
- A linearly polarized light beam propagating through a dichroic medium will acquire an apparent **rotation** ε

$$\varepsilon = \frac{\pi L(\kappa_{||} - \kappa_{\perp})}{\lambda} \sin 2\vartheta$$

Absorption
coefficient

$$= \frac{2\pi}{\lambda} K$$





Cotton-Mouton Effect

Vacuum behaves like a gas: Cotton-Mouton effect

Gas	CM constant (atm Tesla ⁻²)	vacuum equiv. pressure (mbar)
N ₂	-2.45·10 ⁻¹³	1.6·10 ⁻⁸
Ar	6.8·10 ⁻¹⁵	5.8·10 ⁻⁷
Kr	9.9·10 ⁻¹⁵	4·10 ⁻⁷
Ne	2.8·10 ⁻¹⁶	1.4·10 ⁻⁵
He	1.8·10 ⁻¹⁶	2.2·10 ⁻⁵
H ₂	8.5·10 ⁻¹⁵	4.7·10 ⁻⁷

$$\Delta n_{CM} = CM \frac{P}{P_{atm}} B_0^2$$



For N₂:

Vacuum is 'equivalent' to
4.3·10⁸ molecules/cm³

For He:

Vacuum is 'equivalent' to
5.8·10¹¹ atoms/cm³



What else?

QED

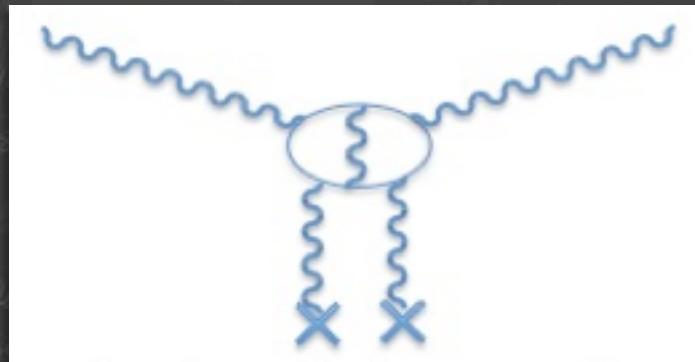
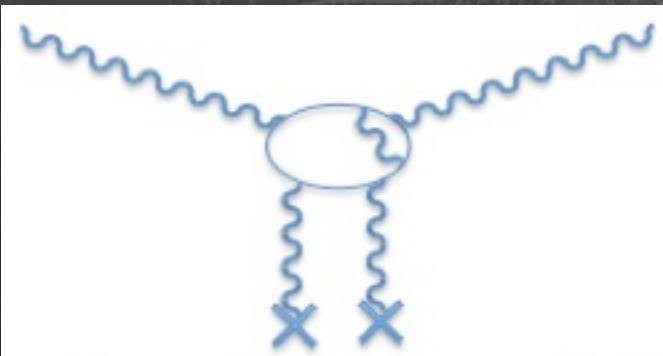
- photon splitting? Much smaller than birefringence.
- higher order corrections are $\sim 1\%$

OTHER

- *low mass, neutral particle search: axion-like*
- *millicharged particles*
- *hadronic contribution*



Higher order corrections



$$L_R = \frac{A_e}{\mu_0} \left(\frac{\alpha}{\pi} \right) \frac{90}{4} \left[\frac{16}{81} \left(\frac{\mathbf{E}^2}{c^2} - \mathbf{B}^2 \right)^2 + \frac{263}{162} \left(\frac{\vec{\mathbf{E}} \cdot \vec{\mathbf{B}}}{c} \right)^2 \right]$$

$$\Delta n_R = 3A_e \left[\frac{25\alpha}{4\pi} \right] B_0^2 = 0.0145 \times 3A_e B_0^2$$



Axion-like contribution

One can add extra terms [*] to the E-H effective lagrangian to include contributions from hypothetical neutral light particles interacting weakly with two photons

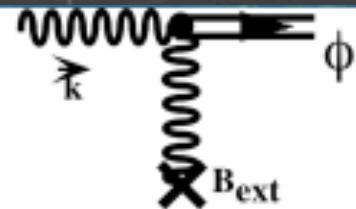
$$L_\phi = \frac{1}{M} \phi (\vec{E}_\gamma \cdot \vec{B}_{ext})$$

pseudoscalar case

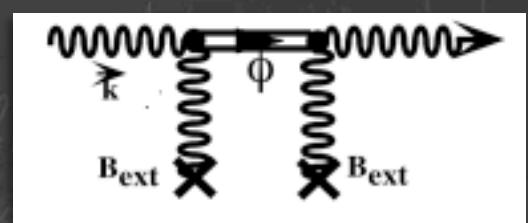
M, M_s are inverse coupling constants

Effects on photon propagation

Absorption



Dispersion`



$$L_\sigma = \frac{1}{M_s} \sigma (\vec{B}_\gamma \cdot \vec{B}_{ext})$$

scalar case

DICHROISM

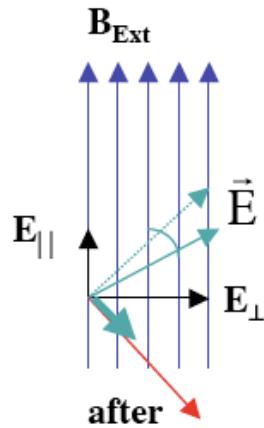
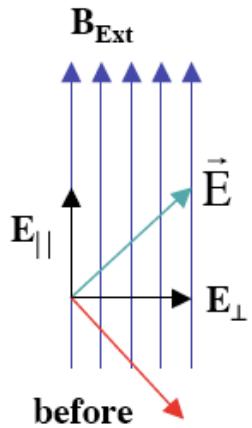
BIREFRINGENCE



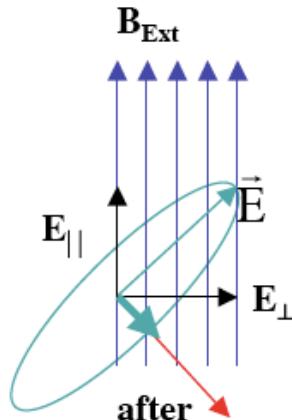
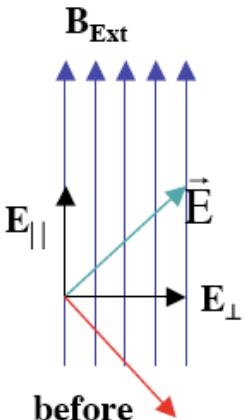
Propagation of the photon in an external field

Dichroism ΔK

- (Photon splitting)
- Real particle production



apparent
rotation ϵ



ellipticity ψ

Birefringence Δn

- QED dispersion
- Virtual particle production
- MCPs
- Hadrons

Both Δn and ΔK are defined with sign



Summing up

Experimental study of the quantum vacuum with:

- magnetic field perturbation
- linearly polarised light beam as a probe
- changes in the polarisation state are the expected signals

Key Ingredients

Ellipticity

$$\psi = \frac{\pi L}{\lambda} \Delta n \sin 2\vartheta$$

- **high magnetic field**
superconducting dipole magnet or high field permanent magnet
- **long optical path**
delay line cavity or very-high Q Fabry-Perot resonator
- **ellipsometer with heterodyne detection for best sensitivity**
periodic change of field amplitude/direction for signal modulation



Some numerical values

Main interest is the Euler-Heisenberg birefringence

- $B = 2.5 \text{ T}$
- $F = 4 \cdot 10^5 \rightarrow \Delta n = 2.5 \cdot 10^{-23} \rightarrow \psi = 3.7 \cdot 10^{-11}$
- $L = 2 \text{ m}$

If we assume a maximum integration time of 10^6 s (= 12 days)



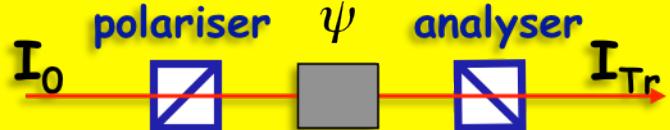
Ellipticity sensitivity of $< 3.7 \cdot 10^{-8} \text{ } 1/\sqrt{\text{Hz}}$
Birefringence sensitivity $< 2.5 \cdot 10^{-20} \text{ } 1/\sqrt{\text{Hz}}$

Present sensitivity in
 $\Delta n = 1.8 \cdot 10^{-18} \text{ } 1/\sqrt{\text{Hz}}$

$$\text{Shot noise limit} = \sqrt{\frac{e}{2I_0 q}} = 1 \cdot 10^{-9} \frac{1}{\sqrt{\text{Hz}}} \quad \text{for } I_0 = 100 \text{ mW}$$

(I_0 = output intensity reaching the analyzer)

Heterodyne detection

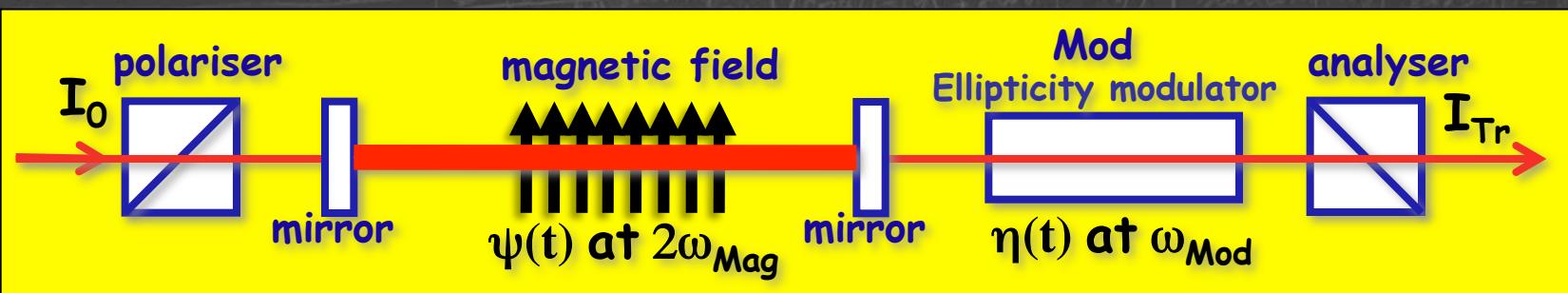


Static detection excluded

$$I_{tr} = I_0 [\sigma^2 + \psi^2]$$

In the heterodyne detection, using a beat with a calibrated effect, we have

- Signal linear in the birefringence
- Smaller 1/f noise



$$I_{Tr} = I_0 [\sigma^2 + (\psi(t) + \eta(t))^2] = I_0 [\sigma^2 + (\psi(t)^2 + \eta(t)^2 + 2\psi(t)\eta(t))]$$

Main frequency components at $\omega_{Mod} \pm 2\omega_{Mag}$ and $2\omega_{Mod}$

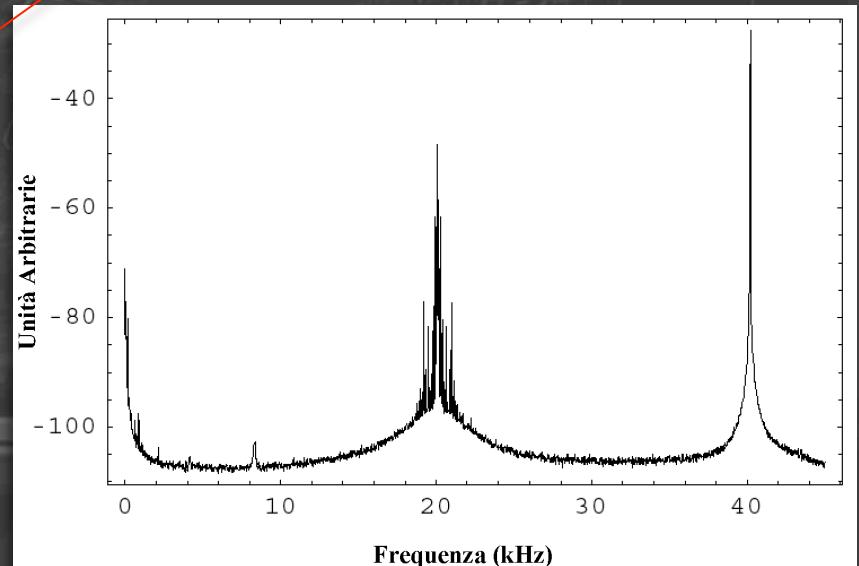
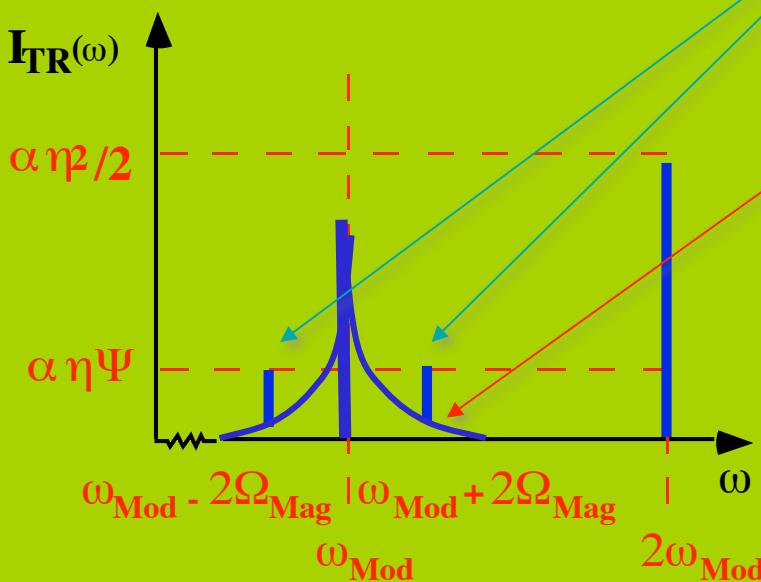


- Inserting a quarter wave plate before the modulator allows rotation measurements
 - Ellipticities and rotation do not mix and are independent
 - In practice, nearly static rotations/ellipticities α_s generate a $1/f$ noise around ω_{Mod}

$$I_{Tr} = I_0 \left[\sigma^2 + (\psi(t) + \eta(t) + \beta_s(t))^2 \right]$$

$$= I_0 \left[\sigma^2 + (\eta(t)^2 + 2\psi(t)\eta(t) + 2\beta_s(t)\eta(t) + \dots) \right]$$







Fabry Perot

Ferrara test apparatus - High finesse successful

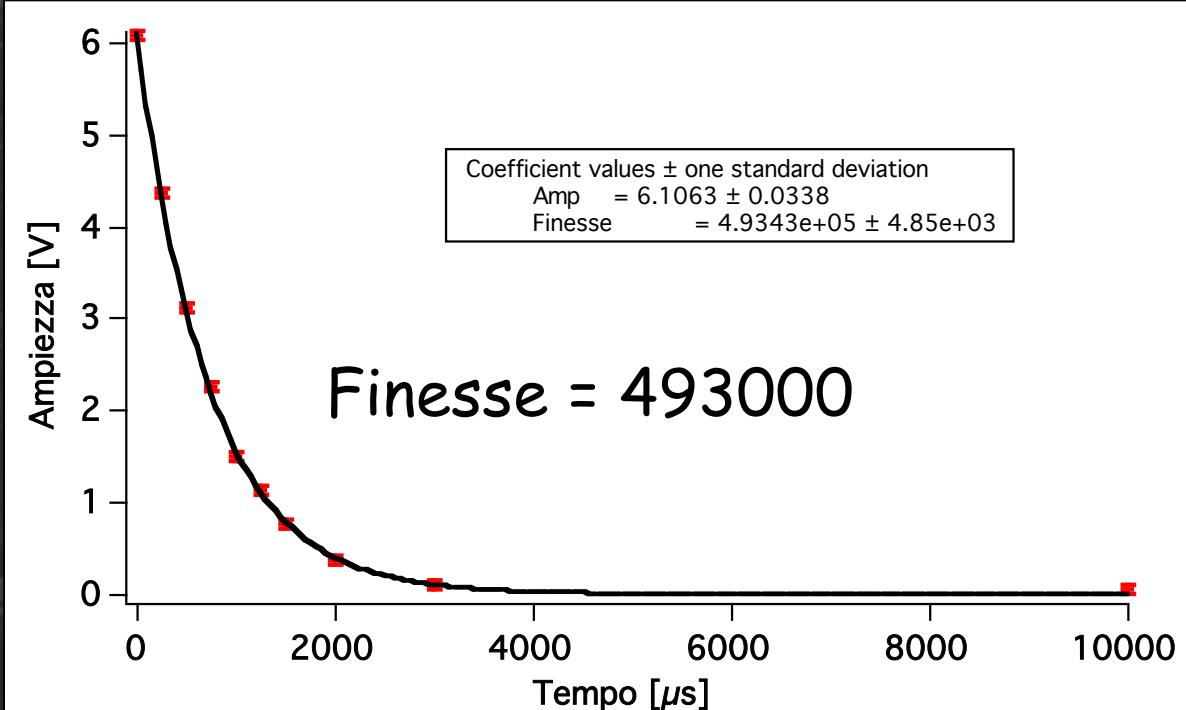
The **Fabry-Perot cavity** is a resonant optical cavity that **increases the effective optical path**. It is composed of **two mirrors placed at a separation d** which is an integer multiple of the light half wavelength. To obtain this condition a laser is phase locked to the cavity using a feedback circuit.

Amplification factor

$$N = \frac{2F}{\pi}$$

Finesse

$$F = \frac{\pi c \tau}{d}$$





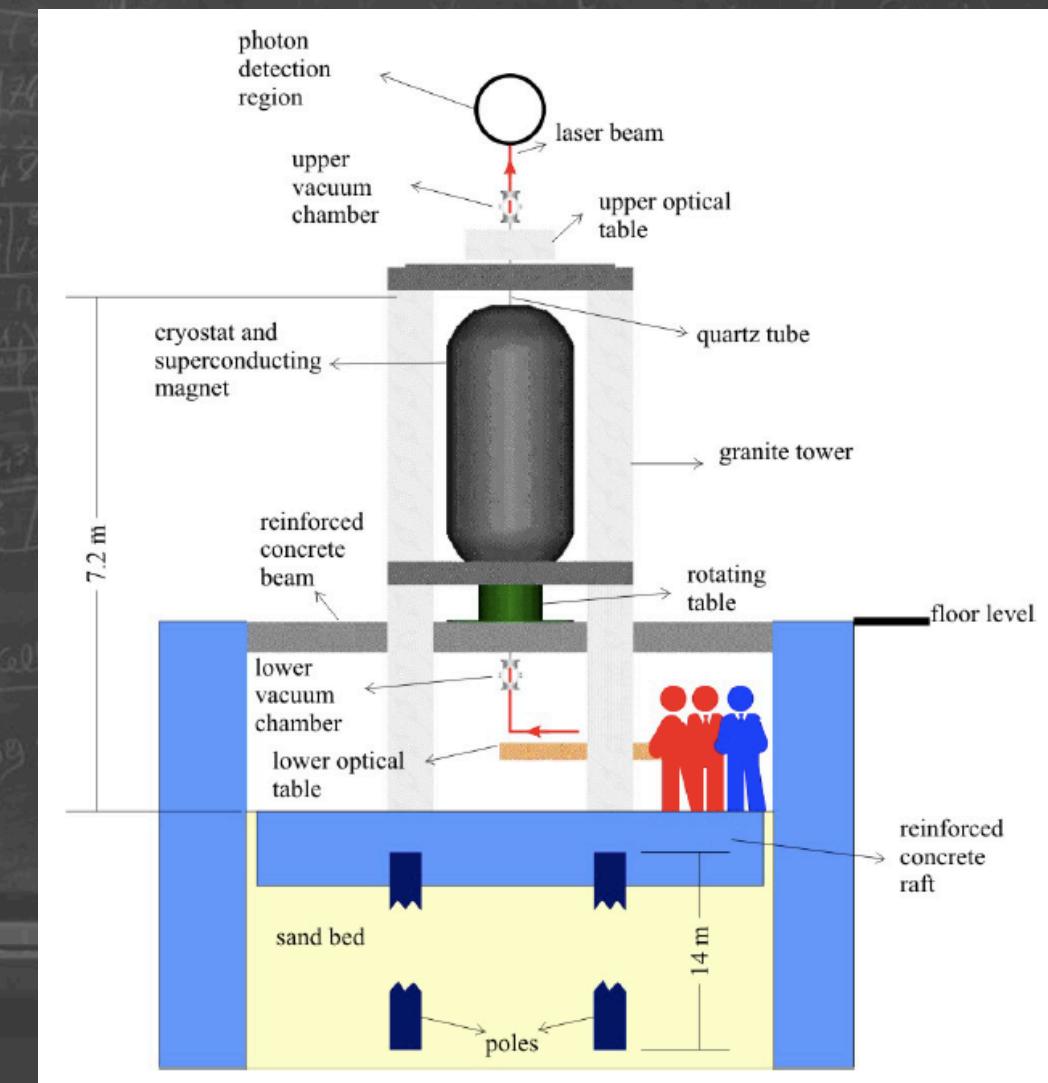
Past - PVLAS at Lab. Nazionali Legnaro

Focused on a general study of the vacuum in the presence of a magnetic field

Polarizzazione del Vuoto con LASer

Major improvements compared to previous efforts:

- Resonant FP cavity (6.4 m) for large amplification factor ($> 5 \cdot 10^4$)
- Rotating cryostat allows high modulation frequency (up to 0.4 Hz)
- Large magnetic field (magnet tested up to 7 T)
- Magnetic system mechanically decoupled from optical system





Past - PVLAS at Lab. Nazionali Legnaro





Present published results - QED

$$\Delta n = 3 A_e B_0^2$$

$$\Delta n_{1064} < 1.05 \cdot 10^{-19} \text{ @ } 1064 \text{ nm}$$

$$\Delta n_{532} < 1.0 \cdot 10^{-19} \text{ @ } 532 \text{ nm}$$

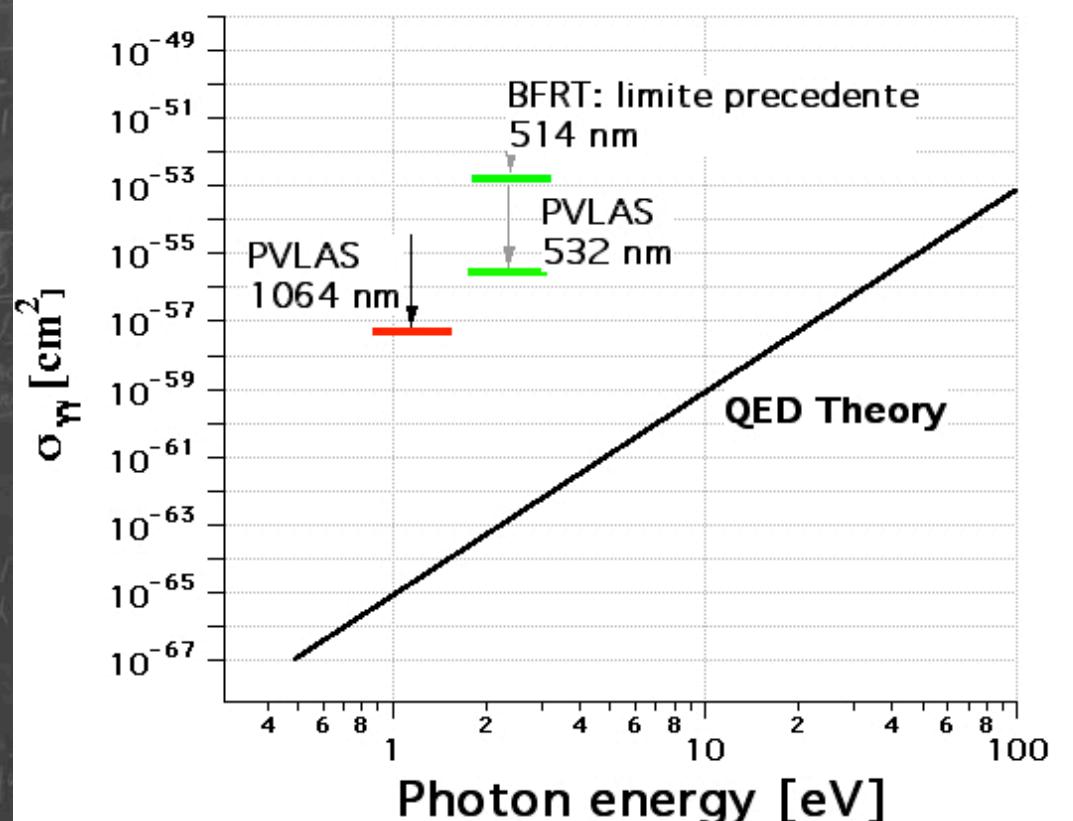


$$A_e^{(\text{LNL})} < 6.3 \cdot 10^{-21} \text{ T}^{-2}$$

$$A_e^{(\text{QED})} = 1.3 \cdot 10^{-24} \text{ T}^{-2}$$

$$\sigma_{\gamma\gamma} < 4.6 \cdot 10^{-58} \text{ cm}^2 \text{ @ } 1064 \text{ nm}$$

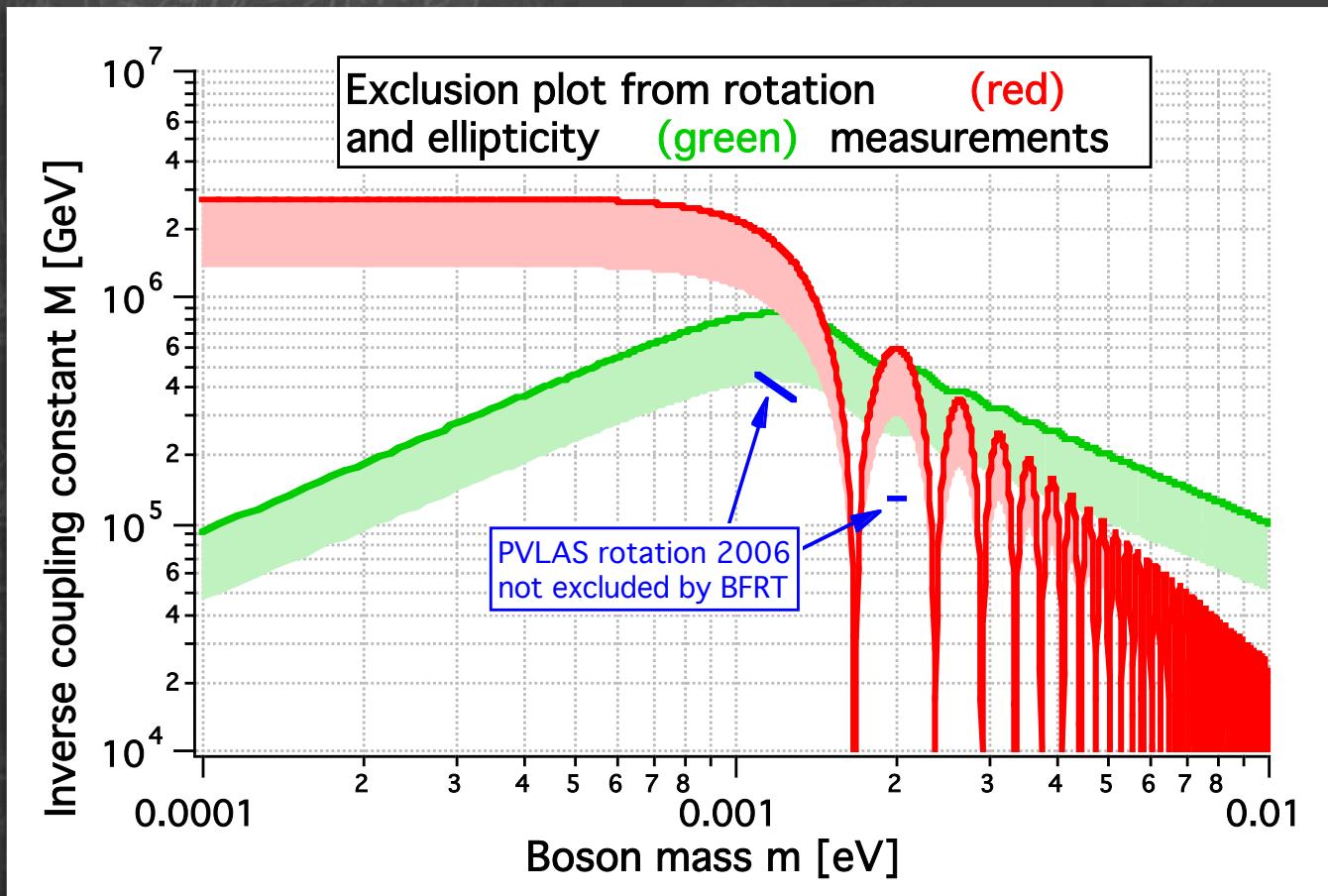
$$\sigma_{\gamma\gamma} < 2.7 \cdot 10^{-56} \text{ cm}^2 \text{ @ } 532 \text{ nm}$$



Bregant et al, PRD 78, 032006 (2008)



Present published results - ALP



The CAST experiment at CERN has excluded values of $M < 10^{10}$ GeV
Unreachable with present lab techniques



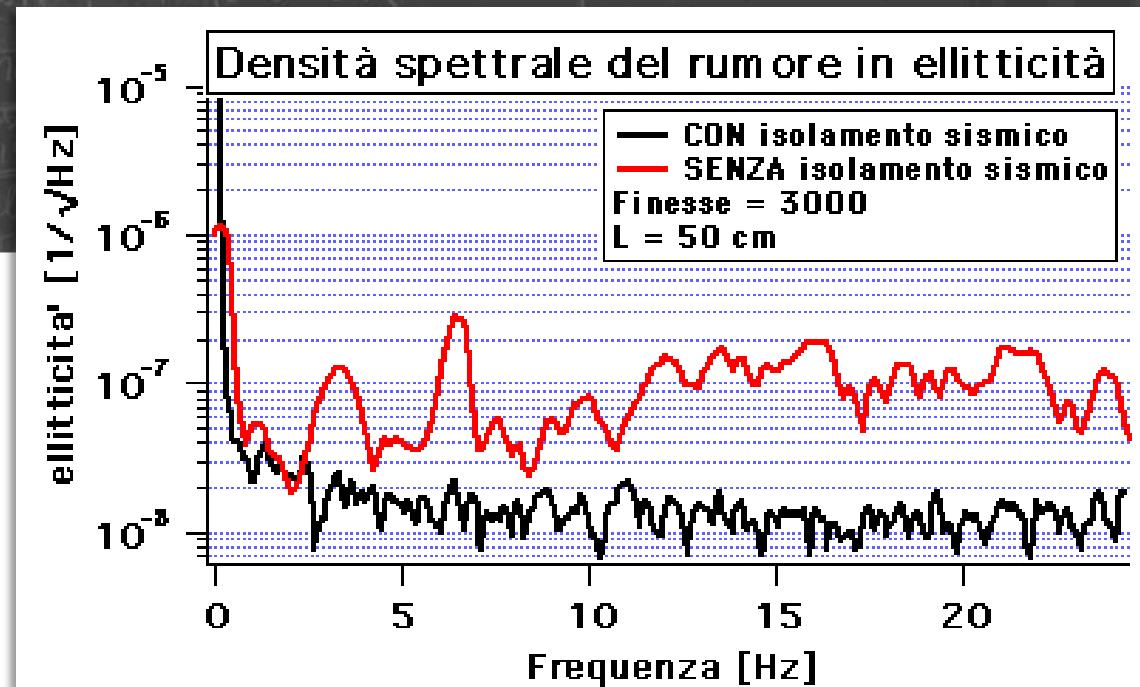
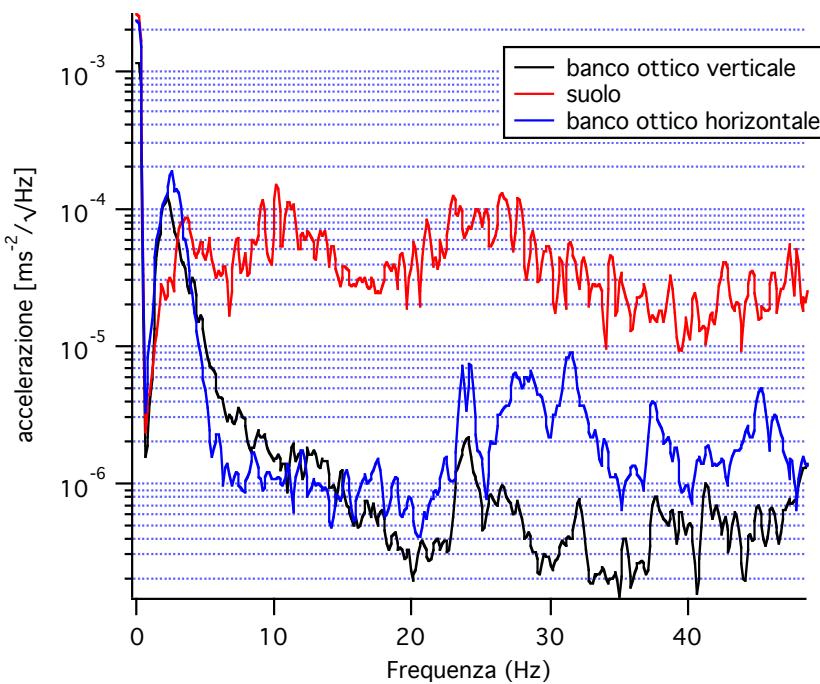
Limitations of the LNL apparatus

- Superconducting magnets produce **stray field** when operated at high fields (saturated iron)
- **Running time limited** due to liquid helium consumption
- Observed **correlation between seismic noise and ellipticity noise**. The Legnaro apparatus is large and therefore difficult to isolate seismically.
- **No zero measurement possible** with field turned ON.



Low finesse - seismic isolation

Compact 50 cm long ellipsometer without magnetic field



Flat noise spectrum above $\approx 5 \text{ Hz}$



High finesse - seismic isolation

High finesse: $F = 414000$

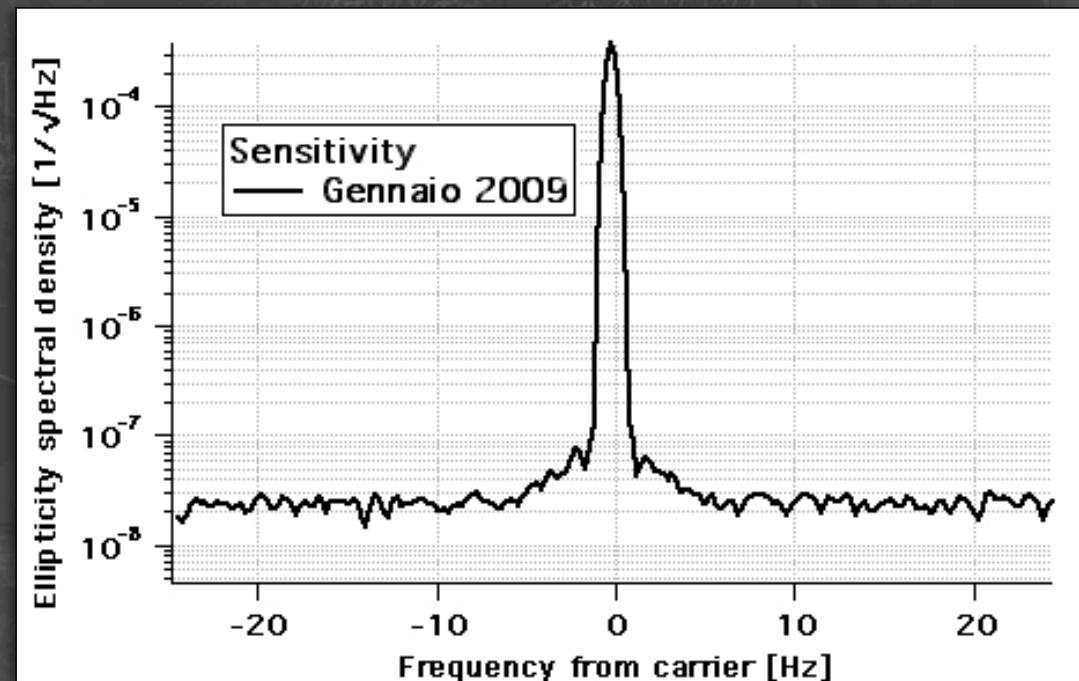
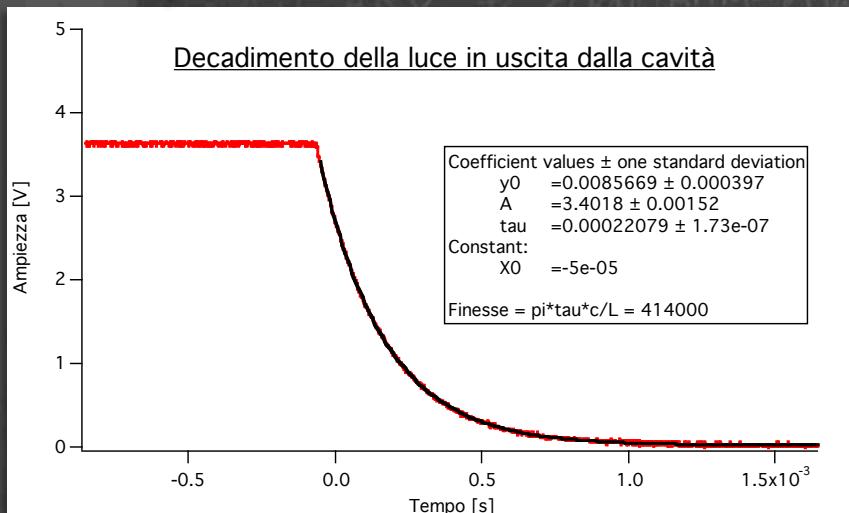
$$L_{\text{eff}} = (2F/\pi)L = 130 \text{ km}$$

Compact 50 cm long ellipsometer without magnetic field

Cavity output power = 25 mW

Laser-cavity coupling = 75%

Cavity transmission = 25%

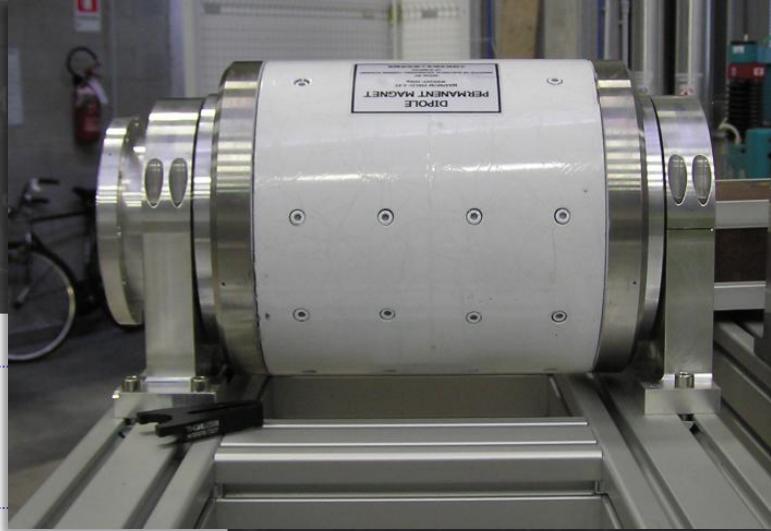
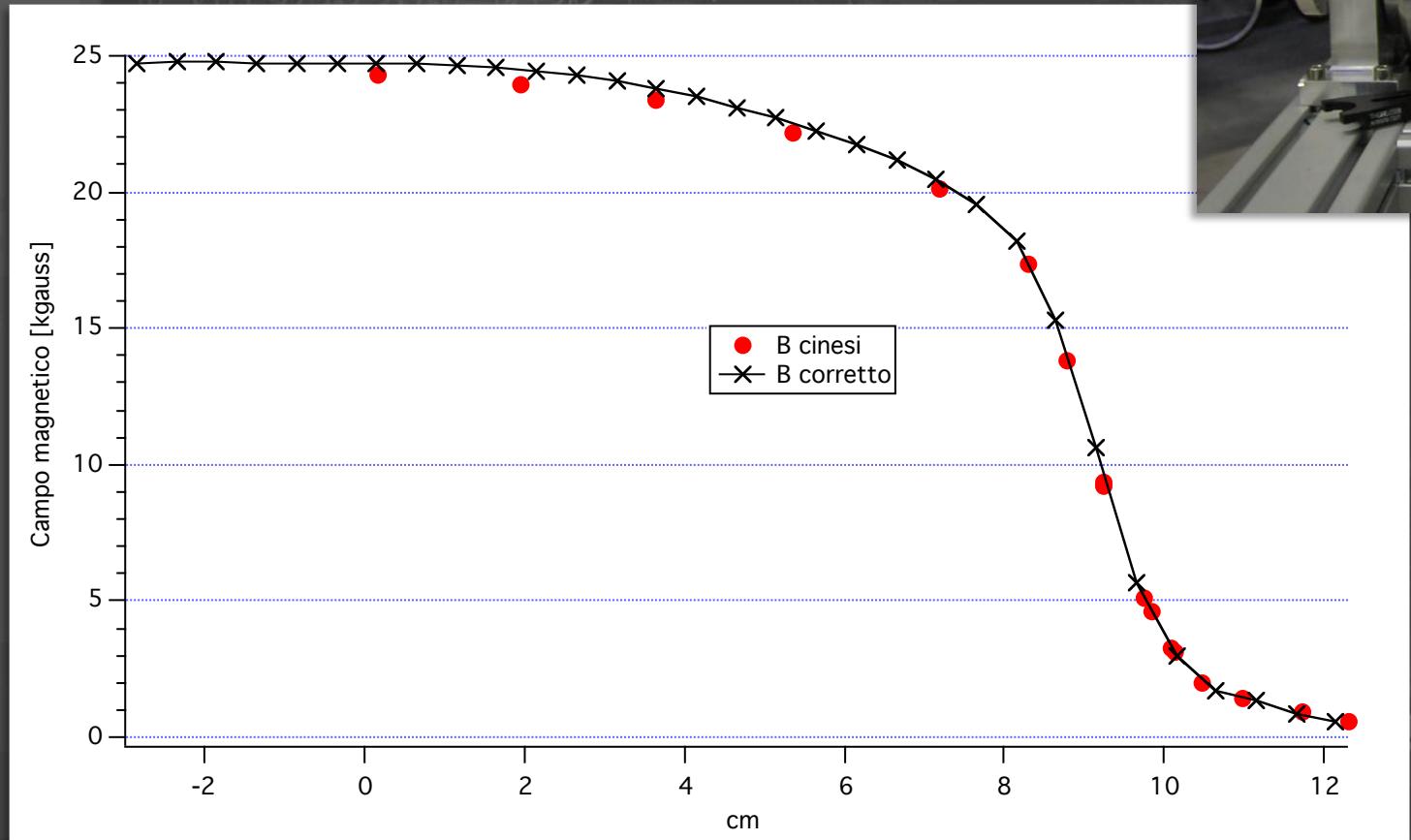


Record sensitivity with a cavity $\Psi = 3 \cdot 10^{-8} \text{ 1}/\sqrt{\text{Hz}}$

Assuming $B = 2.3 \text{ T}$: Sensitivity in $\Delta n / B^2 = 1.5 \cdot 10^{-20} \text{ T}^{-2} \text{ 1}/\sqrt{\text{Hz}}$



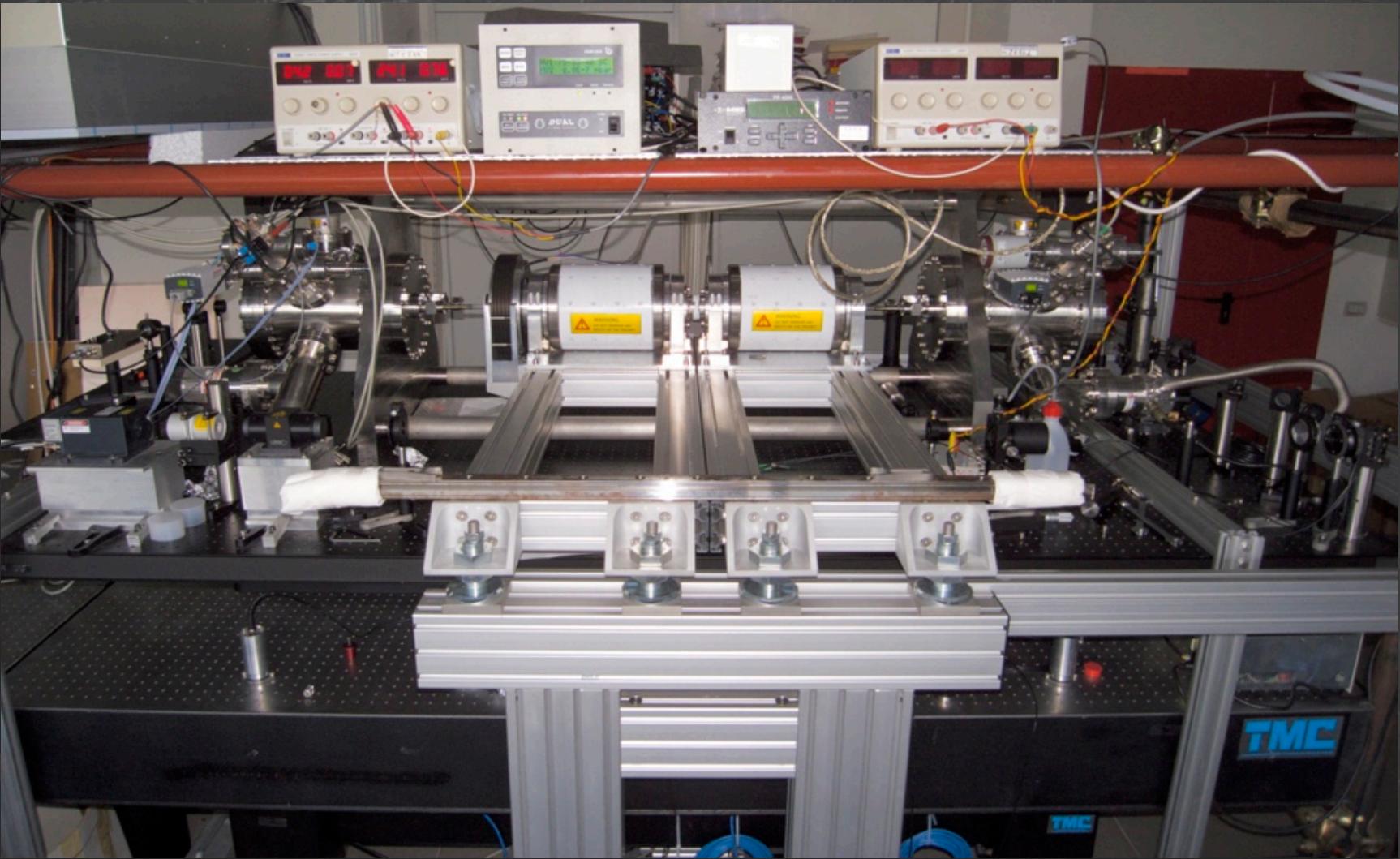
Permanent Test Magnets





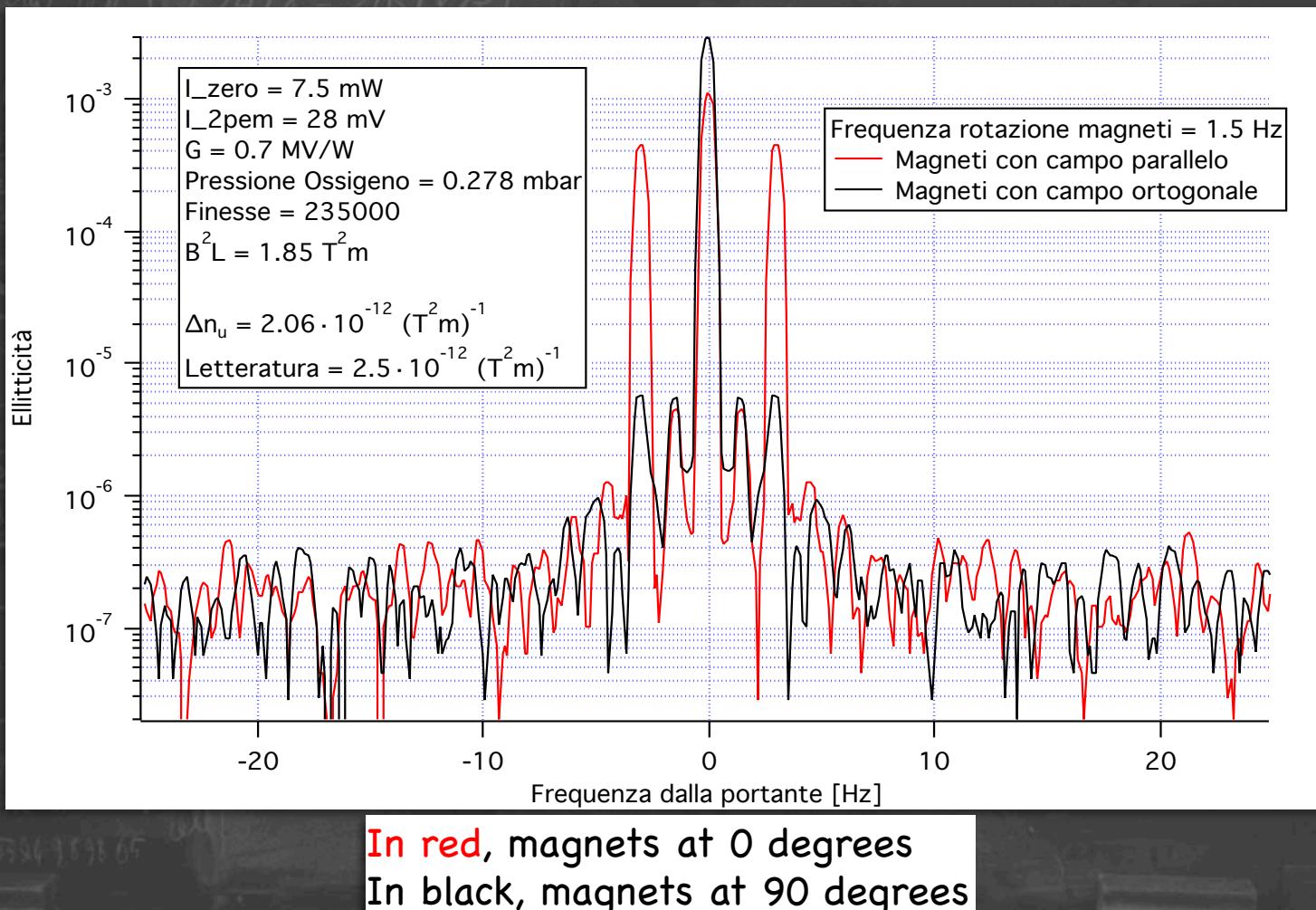
Test apparatus in Ferrara

Two permanent magnets allow a zero measurement





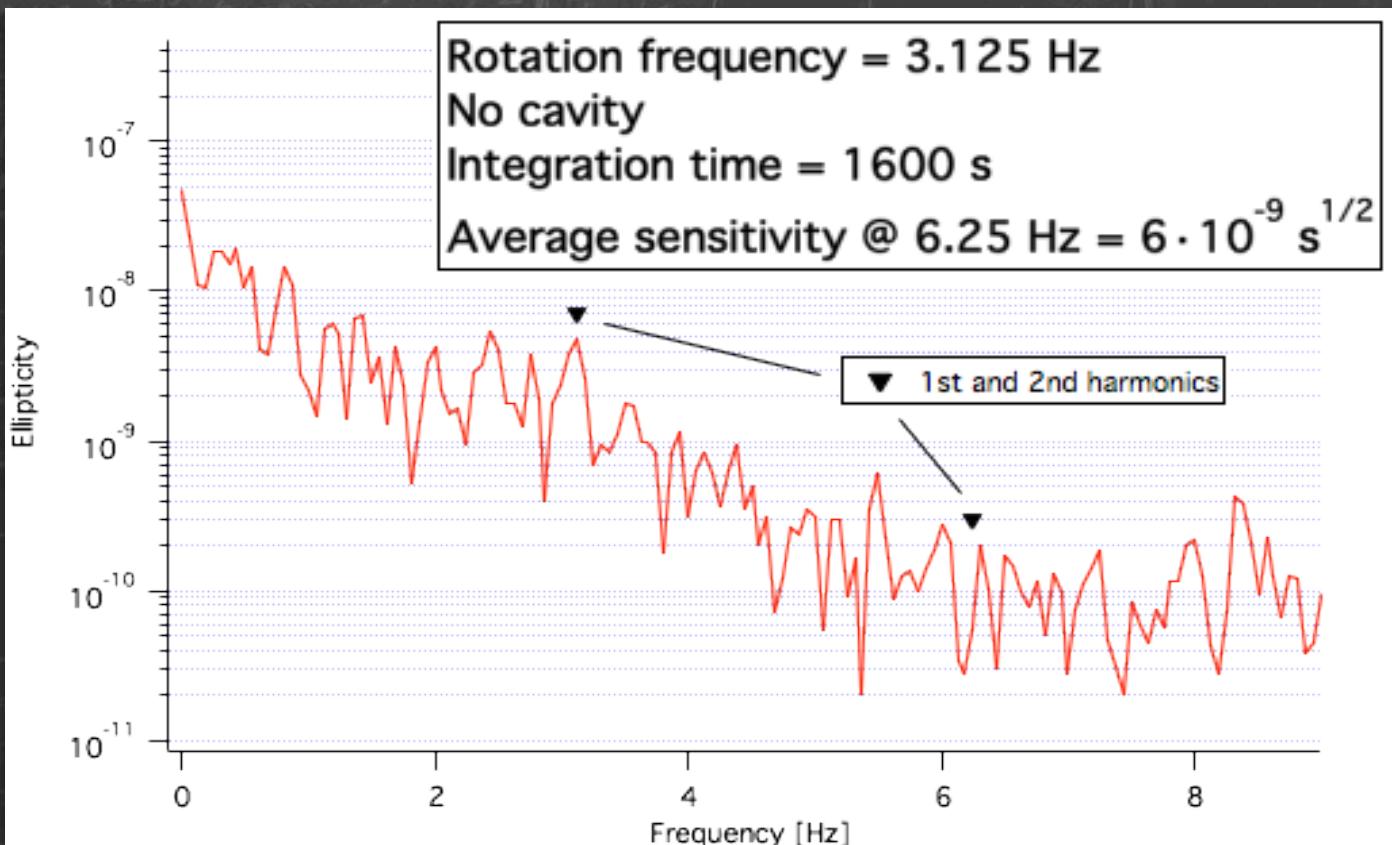
Two magnet configuration



With a finesse = 245000 and with the magnets perpendicular to each other we demonstrated a reduction of more than a factor 80 of the Cotton Mouton signal

Ferrara test apparatus - sensitivity

No cavity - reached expected noise level with rotating magnets



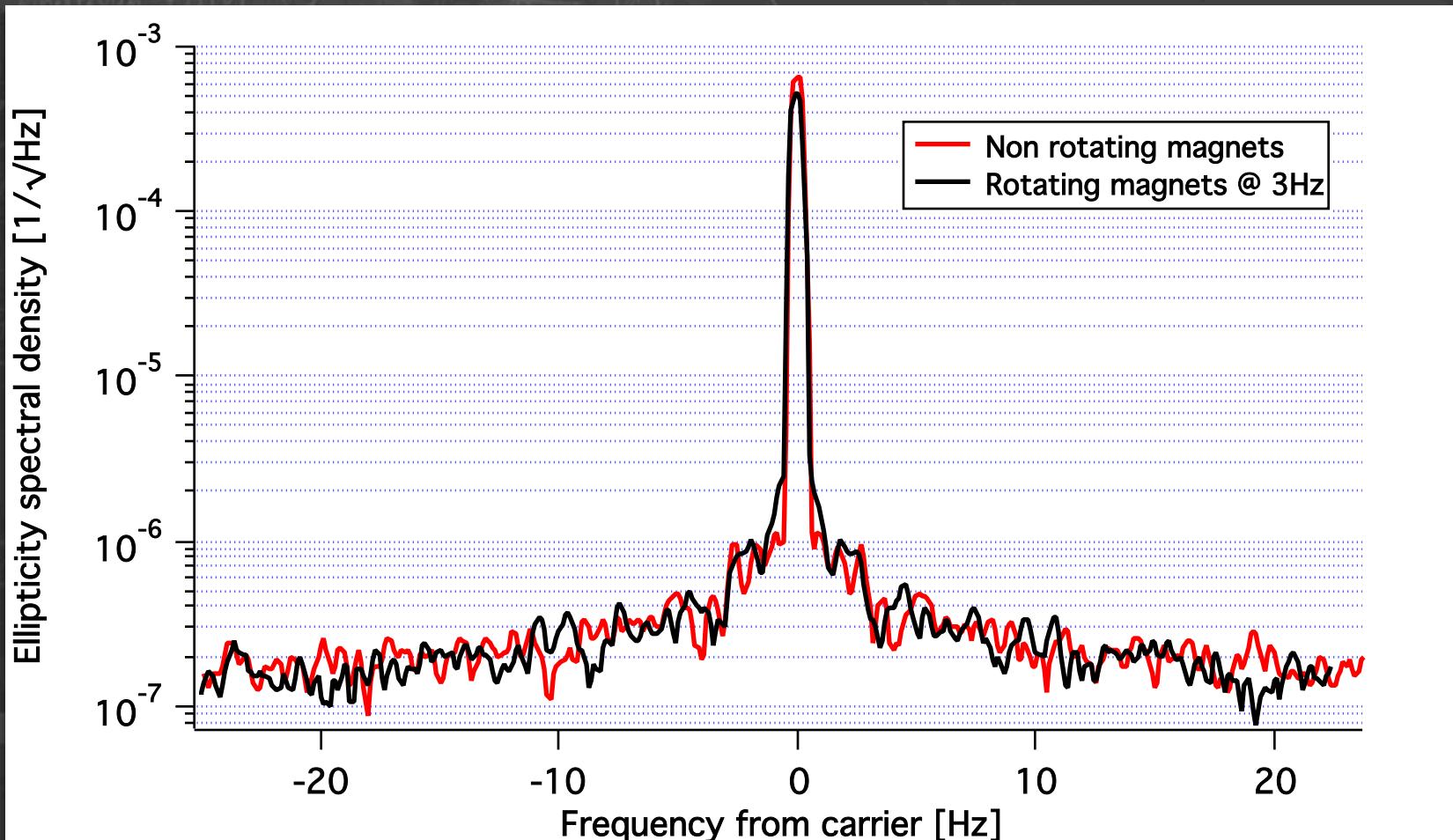
No electronically induced signals in the readout system



Ferrara test apparatus - sensitivity

With high-finesse cavity > 400000

Sensitivity worsened - still under study





Rotating vs non rotating magnets

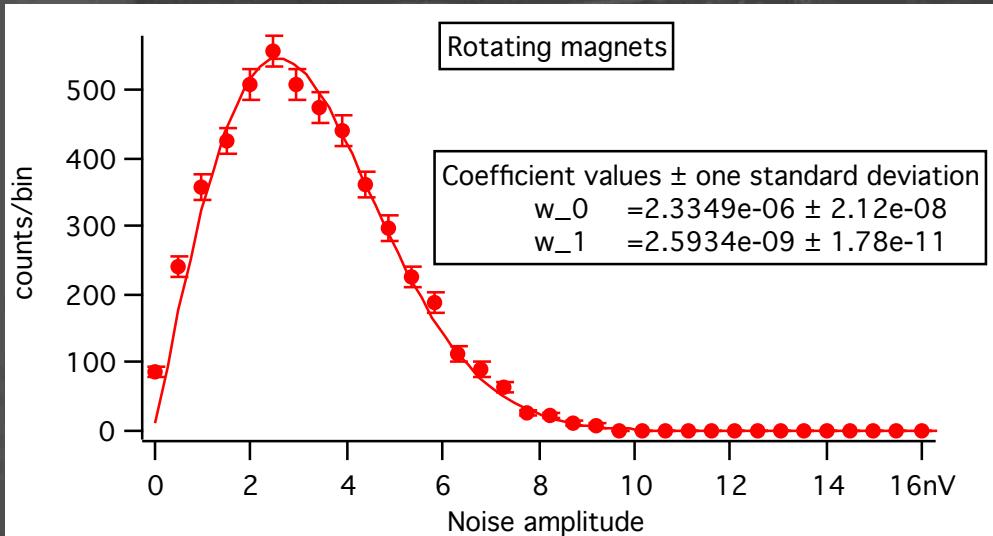
Rotating Magnets

Ellipticity peak = $4 \cdot 10^{-8}$

Sensitivity = $3.5 \cdot 10^{-7} \text{ } 1/\sqrt{\text{Hz}}$

Δn sensitivity = $1.8 \cdot 10^{-18} \text{ } 1/\sqrt{\text{Hz}}$

T = 5.3 ore



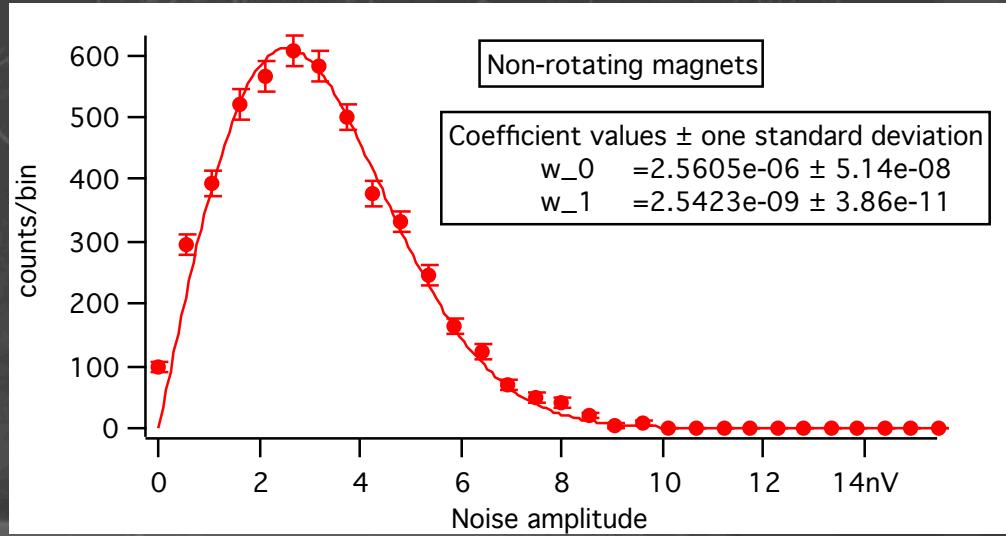
Non rotating Magnets

No peak

Sensitivity = $3.3 \cdot 10^{-7} \text{ } 1/\sqrt{\text{Hz}}$

Δn sensitivity = $1.8 \cdot 10^{-18} \text{ } 1/\sqrt{\text{Hz}}$

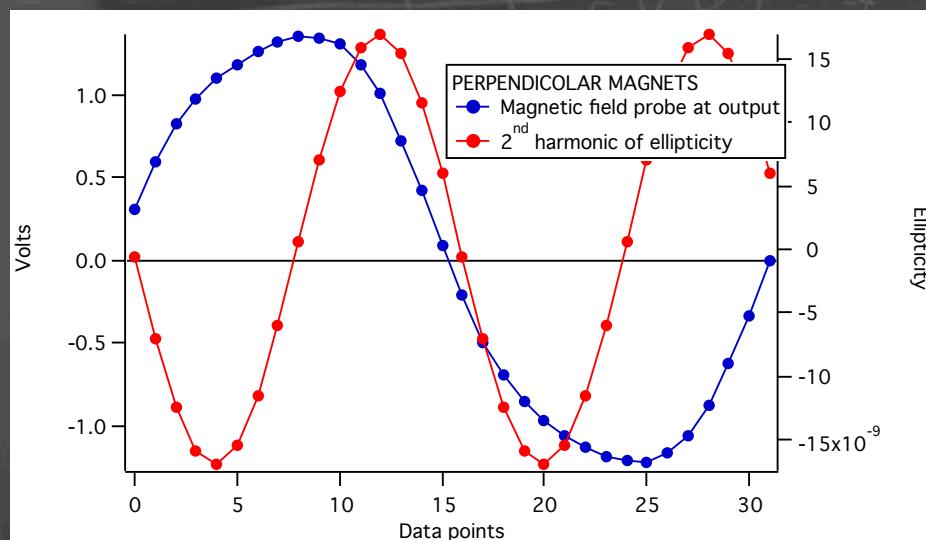
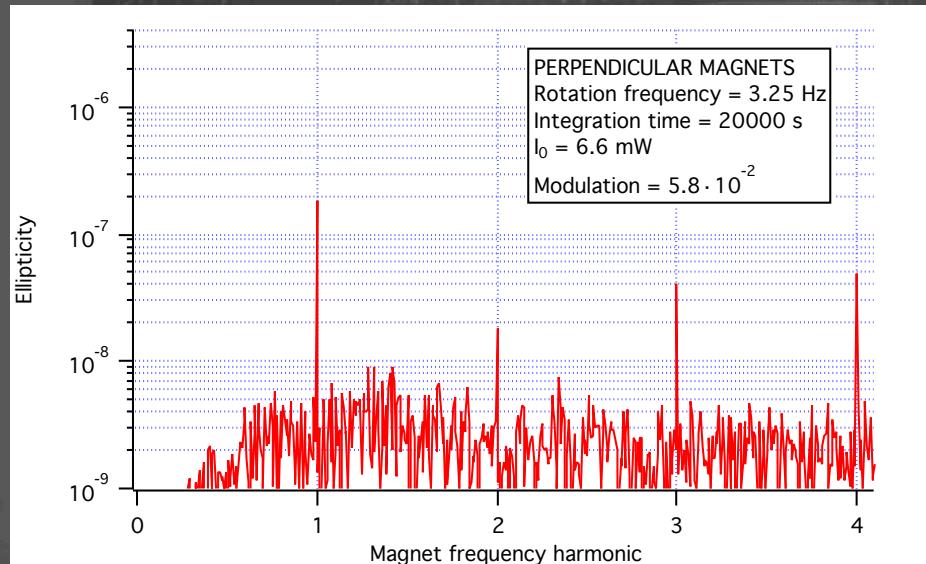
T = 4.8 ore



Noise amplitude histograms around $2 \Omega_{\text{Mag}}$



Vacuum - perpendicular magnets



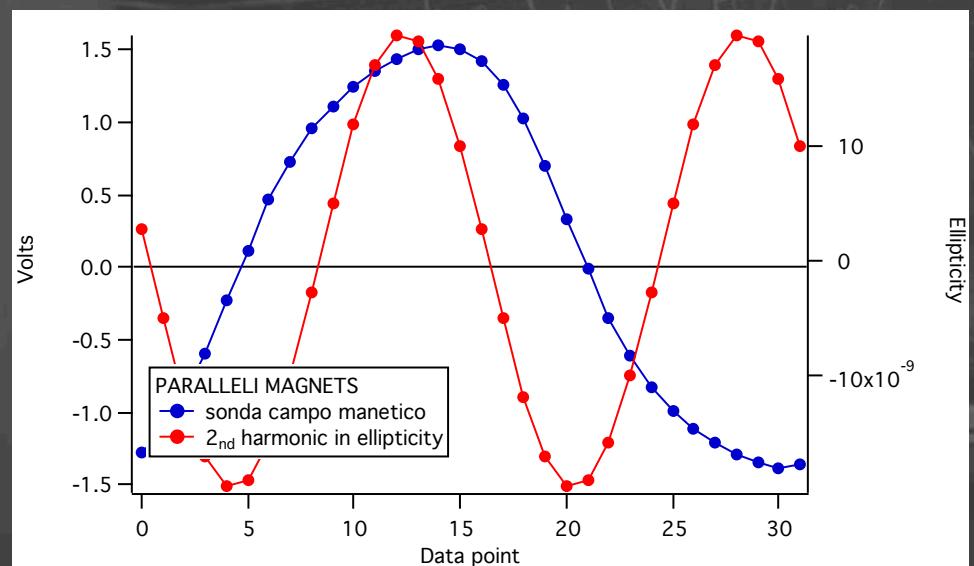
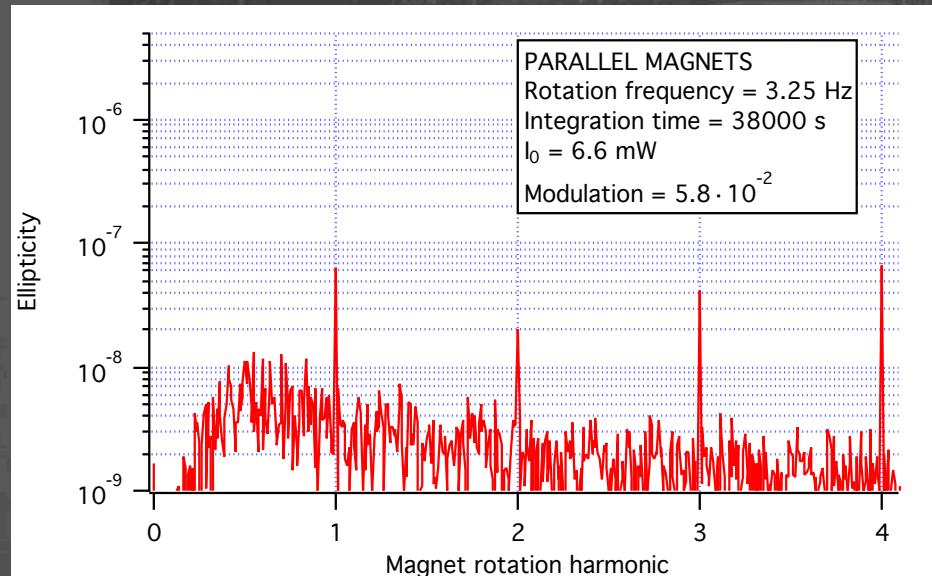
Finesse = 245000

Several harmonics at
moment under study.
Should NOT be there

Second harmonic (red)
compared to magnetic
probes near two optical
enclosures (black - output;
blue - input)



Vacuum - parallel magnets



Finesse = 245000

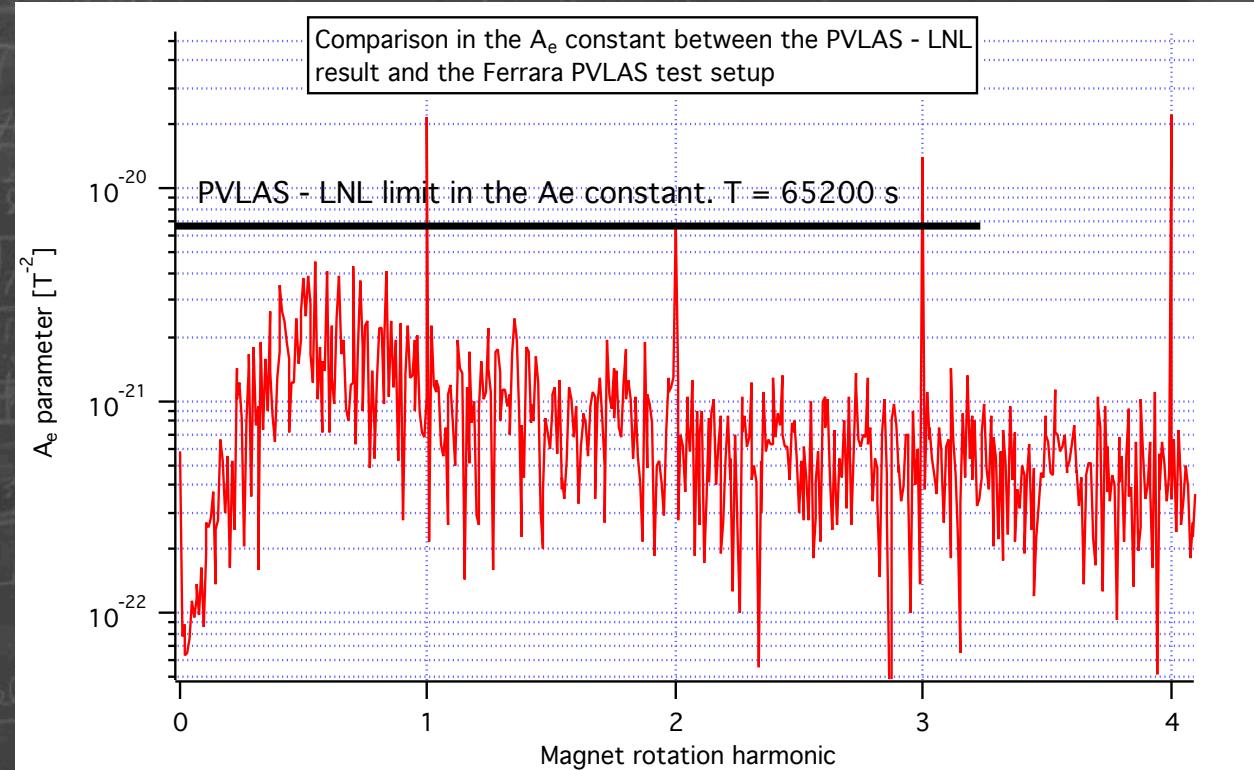
Several harmonics are again present. Different amplitudes with respect to the previous configuration.

Second harmonic (red) compared to magnetic probes near two optical enclosures (black - output; blue - input). Phase with respect to the probe has changed. Follows input magnet position



Vacuum - parallel magnets

- Integration time = 38000s
- Ellipticity = $1.9 \cdot 10^{-8}$
- Clearly systematic because also present with perpendicular magnets
- Assuming such peaks as a limit we confirm the PVLAS - LNL result



This limit confirms the published PVLAS value of
 $A_e^{(LNL)} < 6.3 \cdot 10^{-21} T^{-2}$



Present - Future

Currently building final apparatus in clean room in Ferrara.
Financed by INFN and MIUR

- Magnetic field: 2×1 m long magnets with 2.5 T (ordered)
- Optical bench with isolation system (installed)
- Optical enclosures designed and are in ordering phase
- New more powerful laser arrived (2 Watts, 1064 nm)
- All optical elements, supports and movements will be non magnetic (ordered)
- Getters will be used as vacuum pumps.



New granite optical bench

Installation in Ferrara clean room





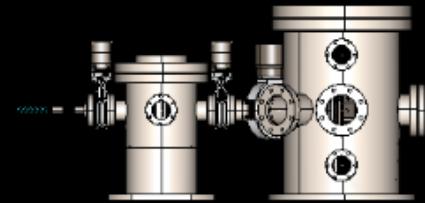
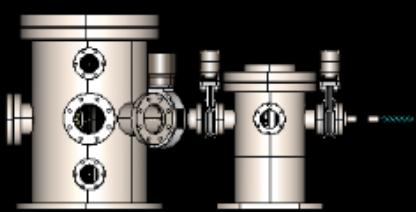
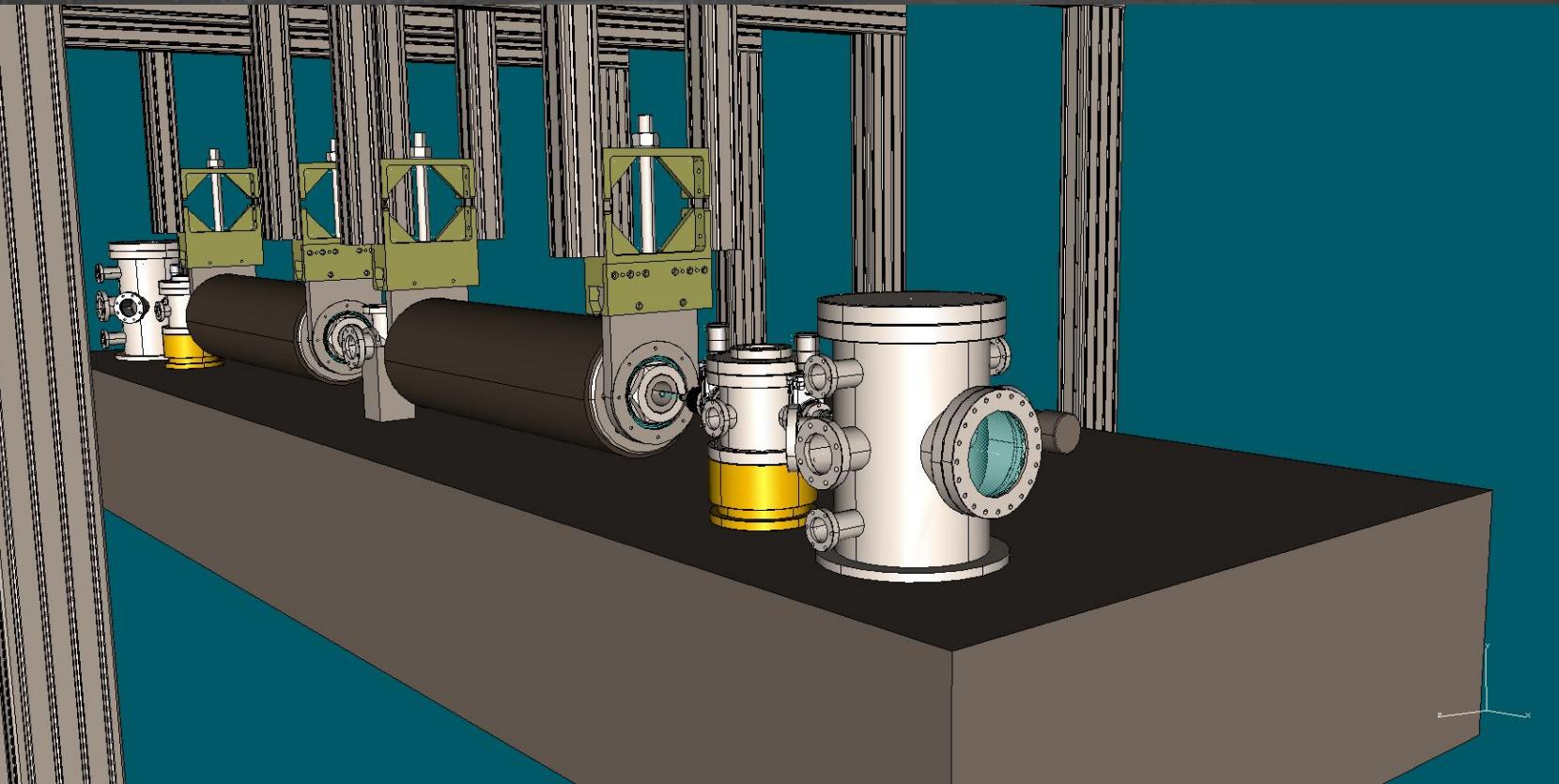
New granite optical bench

Installation in Ferrara clean room



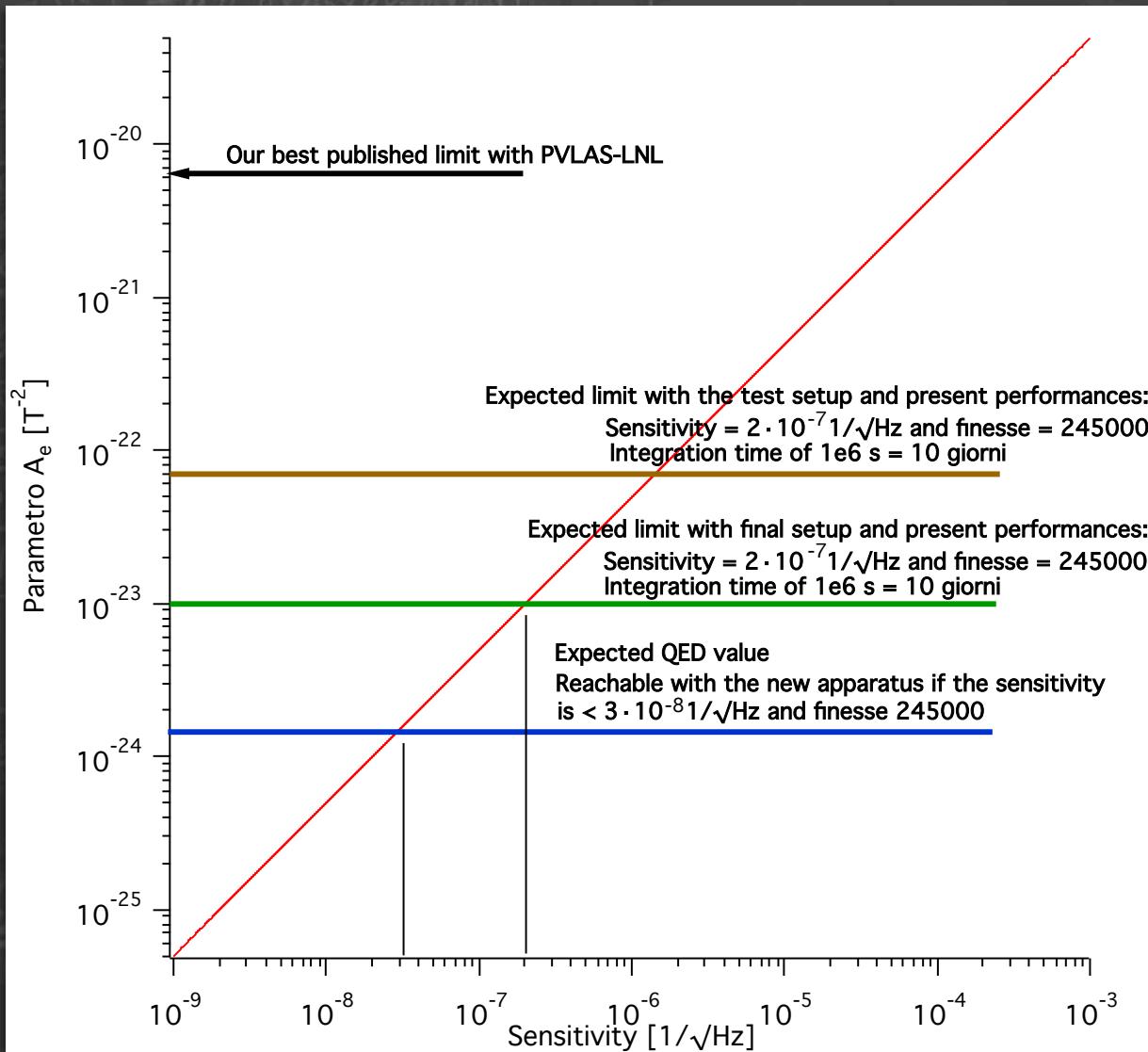


Mechanical Design





Parameter space





B. 72843 (cern)	$\frac{1}{\sin \alpha}$	$72418 = SIKIVG$
C. 79593	$\frac{1}{\sin \alpha}$	$\overline{EMILIO} / 75848 = \text{olday}$
$\sqrt{14002} = 1.75 \cdot 10^6$ rad	$\overline{3205}$	$\overline{LCL(A) + 5792}$
$\sqrt{\text{velocità}} = 1.75 \cdot 10^6$ rad	$\overline{(24592)}$	\overline{T}
B. INFN 3332/58	$\overline{3 cm 273/27}$	$\overline{74861}$
1. Appelvius 71516	$\overline{338155.0. p. min. w. i. m.}$	
3388 B. (72 ext)	$\overline{10 P. C. H. 17675716 C 164310}$	$\overline{75848 \text{ Mid}}$
$\overline{71865}$	$\overline{050 1430 P.}$	$\overline{17824788}$
$\overline{72419}$		
Centrale 76111 (nel cern) $\frac{\text{fuori} + 76}{76 - 76} =$	$\overline{2481}$	$\overline{\text{Rafelsk}}$
- MAIL BOX TS CERN	$\overline{= 581}$	$\overline{\text{TACOPIN}}$
$\overline{73891 Amila Beuelli}$	$\overline{q.s. / serum}$	$\overline{(cern) 79824}$
$\overline{TS212 4.91 \cdot 10^{13} G - B_c}$	$\overline{\text{Present (cern) 72843}}$	
$\overline{Zetas 164592}$	$\overline{L.S.V.P.}$	
$\overline{VACCINI 00390403756729}$	$\overline{10^{-7}}$	
$\overline{00393204232326 \text{ cell}}$	$\overline{\Delta x}$	
$\overline{GU, DO 00390532974340}$	$\overline{\text{TO CARDI}}$	
$\overline{00390532974299}$	$\overline{\text{uff}}$	$\overline{00390512095142 \#}$
$\overline{00393384989865}$	$\overline{\text{cell}}$	

$$E_F = \begin{pmatrix} \cos \alpha - \sin \alpha & 1 + iy & 0 \\ \sin \alpha \cos \alpha & 0 & 1 - iy \\ 0 & -\sin \alpha & \cos \alpha \end{pmatrix} \cdot \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot E(B)$$

$$\begin{pmatrix} \cos \alpha - \sin \alpha & ((1+iy)\cos \alpha & \sin \alpha (1+iy) \\ \sin \alpha \cos \alpha & -\sin \alpha (-iy), \cos \alpha (1-iy) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos^2(1-iy) + \sin^2(1-iy) & \sin \alpha \cos(1+iy) - \sin \alpha \cos(1-iy) \\ 1 + (\cos^2 \alpha - \sin^2 \alpha)iy - \sin \alpha \cos \alpha & \frac{\sin^2 \alpha}{2}(1+iy) - \frac{\sin^2 \alpha}{2}(1-iy) \end{pmatrix} \cdot \begin{pmatrix} \cos \alpha \sin \alpha (1-iy) - \sin \alpha \cos \alpha (1-iy) & \sin \alpha (1+iy) + \cos \alpha (1+iy) \\ 0 & 1 - \frac{(\cos^2 - \sin^2)iy}{1 - \cos 2\alpha} \end{pmatrix}$$

$$\begin{aligned} \cos A &= f_A \overline{m_A} = f_R \overline{m_R} \\ f_A \overline{m_A} &= \cos A \\ &= \sin 2\alpha iy \\ &= iy \end{aligned}$$

Thank you!