Hard and Soft Walls

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A Flat Hard Wall

We consider the scalar field, usually taking $\xi = \frac{1}{4}$.

$$\overline{T}(t, \mathbf{r}, \mathbf{r}') \equiv -\sum_{n=1}^{\infty} \frac{1}{\omega_n}\, \phi_n(\mathbf{r}) \phi_n(\mathbf{r}')^* e^{-t\omega_n}.$$

$$\mathcal{E} = T_{00} = -\lim_{\cdots} \frac{1}{2}\, \frac{\partial^2 \overline{T}}{\partial t^2}\,,$$

$$p_j = T_{jj} = \lim_{\cdots} \frac{1}{8} \left(\frac{\partial^2 \overline{T}}{\partial {x_j}^2} + \frac{\partial^2 \overline{T}}{\partial {x'_j}^2} - 2\, \frac{\partial^2 \overline{T}}{\partial x\, \partial x'} \right).$$

Empty space:

$$\overline{T}_0(\mathbf{r},t;\mathbf{r}',t') = -\,\frac{1}{2\pi^2}\,\frac{1}{(t-t')^2 + \|\mathbf{r}-\mathbf{r}'\|^2}\,.$$

Note dual role of t:

- Ultraviolet cutoff.
- Wick rotation: $t = -i(x^0 - x'^0)$.

Local point-splitting in direction u^μ (Christensen):

$$T_{\mu\nu} = \frac{1}{2\pi^2 t^4}\left(g_{\mu\nu} - 4\,\frac{u_\mu u_\nu}{u_\rho u^\rho}\right).$$

Thus $\quad T^{\mu\nu}_{\mathrm{ren}} = \Lambda g^{\mu\nu}$.

Dirichlet wall at $z = 0$: $\qquad (\mathbf{r}_\perp \equiv (x, y))$

$$\overline{T}_{\rm ren} = \frac{1}{2\pi^2}\,\frac{1}{t^2 + (\mathbf{r}_\perp - \mathbf{r}'_\perp)^2 + (z + z')^2}\,.$$

Set $\mathbf{r}'_\perp = 0$, $x'^0 = 0\,$;
t, $\mathbf{r}_\perp$, $z - z'$ are still available as cutoff parameters.

Recall $\mathcal{E} = -\,\frac{1}{2}\,\frac{\partial^2 \overline{T}}{\partial t^2}$, etc.

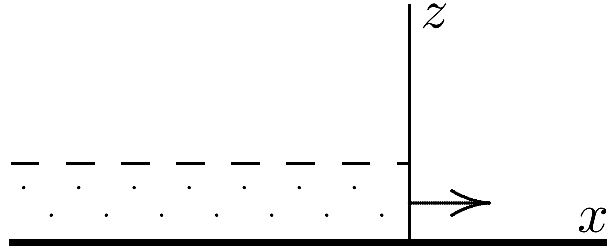
$$M \equiv t^2 + x^2 + y^2 + (z + z')^2.$$

$$2\pi^2 \mathcal{E} = M^{-3}[-3t^2 + x^2 + y^2 + (z + z')^2],$$

$$2\pi^2 p_1 = M^{-3}[-t^2 + 3x^2 - y^2 - (z + z')^2],$$

$$p_2 \text{ similar;} \qquad p_3 = 0.$$

(Rigid displacement of the wall does not change total energy. But ∃ layer of energy against wall.)



Imagine another planar boundary at $x = 0$; let's find pressure on it (from left side only). Volume of space occupied by boundary energy increases with x, so total energy does.

In accordance with the principle of energy balance (virtual work) one expects

$$F = \int_0^\infty T^{11}\, dz = -E = -\int_0^\infty T^{00}\, dz.$$

If *all cutoffs are removed*,

$$\mathcal{E} = \frac{1}{32\pi^2 z^4} = -p_1\,,$$

so energy balance is formally satisfied, but the integrals are divergent.

Ultraviolet cutoff ($t \neq 0$, $\mathbf{r}_\perp = 0$, $z' = z$):

$$F = +\tfrac{1}{2}E \quad (\text{not } (-1)E).$$

This E is negative and is the same one gets from expansion of

$$E = \frac{1}{2}\sum_n \omega_n e^{-t\omega_n}.$$

But I shall argue this E is wrong and this F is (relatively) correct.

Point-splitting $\perp$ *to movable wall* ($x \neq 0$, others 0):
$(t, \mathcal{E})$ exchange places with $(x, -p_1)$.

$$F = +2E > 0.$$

(This time E is "right" and F is wrong.)

Point-splitting in neutral direction ($y \neq 0$, others 0):

$$F = -E, \qquad \text{as should happen!}$$

$$2\pi^2\mathcal{E} = (y^2 + 4z^2)^{-2} > 0, \quad 2\pi^2 p_1 = -(y^2 + 4z^2)^{-2}.$$

Possible Responses to the Pressure Paradox

1. Divergent terms are so cutoff-dependent that they have no physical meaning whatsoever, and the only meaningful calculations are those in which these terms can be canceled out (e.g., forces between rigid bodies).

2. Expressions with finite cutoff, such as $2\pi^2\mathcal{E} = (y^2 + 4z^2)^{-2}$ (where y is now a cutoff parameter, not a coordinate) can be regarded as ad hoc models of real materials, more physical and instructive than their limiting values, such as $\mathcal{E} = 1/32\pi^2 z^4$.

The paradox casts some doubt on the viability of this point of view. It now appears that physically plausible results can be obtained only by using different cutoffs for different parts of the stress tensor. For the leading divergence (and higher-order divergences in the bulk that occur in curved space-time or external potentials) the preferred ansatz is "covariant point-splitting" based on the wave kernel, treating all directions in space-time equivalently, and removing the cutoff-dependent terms in such a way that the only ambiguity remaining can be regarded as a renormalization of the cosmological constant. For the divergences at boundaries, it appears that the points must be separated parallel to the boundary, but in a direction orthogonal to the component of the stress tensor being calculated. Moreover, if the separation has a time component, a Wick rotation seems mandatory.

This situation cannot be regarded as a logically sound, long-term solution; its sole justification is that, unlike less contrived alternatives, it does not immediately produce results that are obviously wrong.

3. Find a better model!

General ξ: Added terms

- do not exhibit the paradox: $\Delta p = -\Delta \mathcal{E}$ always;
- integrate to 0 anyway.

Curved Hard Walls

In flat case, isn't there an equal and opposite force from the other side of the wall?

The paradox was discovered (S.A.F. and M. Schaden) in calculations for a spherical boundary. Ultraviolet cutoff gave $F = +\frac{1}{2}E$. Inside and outside energy layers have the same sign; total energy proportional to surface area.

Cylindrical case is under investigation.

Flat Soft Walls

Precursors

[Plasma model: Barton (2004, 2005)]

Potential model: A. Actor and I. Bender, *Phys. Rev. D* **52** (1995) 3581.

[Also: Bordag (1995); Jaffe, Graham, et al. (2002– ; book of Graham–Quandt–Weigel, 2009)]

The Power Wall Model

$$\Box\varphi = v\varphi, \qquad v(\mathbf{r}) = \begin{cases} 0, & z < 0, \\ z^\alpha, & z > 0 \end{cases}$$

(increasingly steep wall near $z = 1$ as $\alpha \to \infty$).

Eigenfunctions $\quad \phi_{(\mathbf{k}_\perp, p)} = (2\pi)^{-1} e^{i\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \phi_p(z)$,

$$\left(-\frac{\partial^2}{\partial z^2} + v(z) - p^2 \right) \phi_p(z) = 0.$$

When $z < 0$, $\quad \phi_p(z) = \sqrt{\frac{2}{\pi}}\,\sin[pz - \delta(p)]$

for some real phase shift $\delta(p)$.

When $z > 0$,

$$\phi_p(z) \propto \begin{cases} \mathrm{Ai}(z - p^2), & \alpha = 1, \\ D_{\frac{1}{2}(p^2-1)}(\sqrt{2}\,z), & \alpha = 2, \end{cases} \qquad \cdots$$

$$\tan\big(\delta(p)\big) = -p\,\frac{\phi_p(0)}{\phi_p'(0)}\,.$$

The Texas Approach

Details in Proceedings of Dartmouth Conference on Spectral Geometry, *Proc. Symp. Pure Math.*, in press. [Bouas et al., `arXiv:1006.1162`]

Asymptotics of δ: E.g., for $\alpha = 1$ (the Airy function)

$$\delta(p) \sim \begin{cases} p\, 3^{2/3}\Gamma(\frac{4}{3})/\Gamma(\frac{2}{3}), & p \to 0, \\ \frac{2p^3}{3} + \frac{\pi}{4}\,, & p \to \infty. \end{cases}$$

(In general, $\delta \propto p^{1+2/\alpha}$ at ∞.) *Might walls be well parametrized by* $\delta(p)$ *instead of* $v(z)$?

After integration over transverse Fourier dimensions,

$$\overline{T}_{\text{ren}} = \frac{1}{2\pi^2} \int_0^\infty dp \, \frac{e^{-sp}}{s} \cos\bigl(p(z+z') - 2\delta(p)\bigr)$$

in potential-free region ($z < 0$); $s \equiv \sqrt{t^2 + |\mathbf{r}_\perp|^2}$. Components of $T^{\mu\nu}$ are second derivatives of this.

Convergence is extremely delicate when $s \to 0$, which is precisely where we need it. (In fact, the convergence is only in a distributional sense.)

A slight improvement from going to polar coordinates in Fourier space:

$$\overline{T}_{\text{ren}} = \frac{1}{\pi^3} \int_0^\infty d\rho \int_0^1 du\, s^{-1} \sin(s\rho\sqrt{1-u^2}\,) \\ \times \cos\bigl((z+z')\rho u - 2\delta(\rho u)\bigr).$$

These integrals are being investigated with Riesz–Cesàro summation and modern methods for oscillatory quadrature, and preliminary results (for $\alpha = 1$) look plausible.

The Oklahoma Approach

The Texas approach in effect did a generalized Fourier analysis in z to get to the problem of a *reduced Green function* in the $(t, \mathbf{r}_\perp)$ coordinates. (Note that e^{-sp}/s, $s \equiv \sqrt{t^2 + |\mathbf{r}_\perp|^2}$, is a Yukawa potential.) The oscillations in the eigenfunctions $\phi_p(z)$ are the source of the bad integral behavior.

Instead, let's do a Fourier analysis in the transverse dimensions to define a reduced Green function in the z direction. It will vanish at infinity, not oscillate. [Milton, *Phys. Rev. D*, in press; `arXiv:1107.4589`]

For $\alpha = 1$, in region $z, z' < 0$ $\quad (\kappa \equiv \sqrt{\mathbf{k}_\perp{}^2 - \omega^2}\,)$

$$g_{\omega,\mathbf{k}_\perp}(z,z') = \frac{1}{2\kappa}\, e^{-\kappa|z-z'|} + \frac{1}{2\kappa}\, e^{\kappa(z+z')}\, \frac{1 + \mathrm{Ai}'(\kappa^2)/\kappa\mathrm{Ai}(\kappa^2)}{1 - \mathrm{Ai}'(\kappa^2)/\kappa\mathrm{Ai}(\kappa^2)}\,.$$

$$\mathcal{E}_{\mathrm{ren}} = \frac{1-6\xi}{6\pi^2} \int_0^\infty d\kappa\, \kappa^3 e^{2\kappa z}\, \frac{1 + \mathrm{Ai}'(\kappa^2)/\kappa\mathrm{Ai}(\kappa^2)}{1 - \mathrm{Ai}'(\kappa^2)/\kappa\mathrm{Ai}(\kappa^2)}$$

(vanishes if $\xi = \frac{1}{6}$).

The integral for $\mathcal{E}$ can be computed without incident. It displays a weak divergence as $z \to 0^-$:

$$\mathcal{E} \sim -\frac{1}{192\pi^2}\frac{1}{z} \quad (\text{for } \xi = \tfrac{1}{4}).$$

It corresponds to a $z \ln z$ singularity in $\overline{T}$. This effect is attributable to diffraction off the singularity of the potential at $z = 0$; it goes away for larger α, as we'll see.

INSIDE THE WALL $(z, z' > 0)$:

$$g_{\omega,\mathbf{k}_\perp}(z,z') = \pi \mathrm{Ai}(\kappa^2 + z_>)\mathrm{Bi}(\kappa^2 + z_<) \\ - \frac{(\kappa \mathrm{Bi} - \mathrm{Bi}')(\kappa^2)}{(\kappa \mathrm{Ai} - \mathrm{Ai}')(\kappa^2)} \pi \mathrm{Ai}(\kappa^2 + z)\mathrm{Ai}(\kappa^2 + z').$$

Before renormalization, with ultraviolet cutoff,

$$\mathcal{E} \sim \frac{3}{2\pi^2}\,\frac{1}{t^4} - \frac{z}{8\pi^2 t^2} + \frac{z^2}{32\pi^2}\,\ln t,$$

showing the expected "Weyl" terms correlating with the heat kernel expansion in presence of a potential $v(z) = z$. Two new divergences, but not scary:

Renormalization

Removal of those terms has a physical interpretation. Include the dynamics of the v field:

$$-2\mathcal{L} = (\nabla\phi)^2 + m^2\phi^2 + \phi^2 v + (\nabla v)^2 + M^2 v^2 + Jv.$$

$$\Box\phi = m^2\phi + v\phi\,; \qquad \Box v = M^2 v + 2\phi^2 + 2J.$$

(J is whatever it takes to support our static v.)
T_{00} acquires new terms $\propto M^2 v^2$, Jv.

T_{00} acquires new terms $\propto M^2 v^2$, Jv. Now recall (from heat-kernel theory) that T_{00} contains $t^{-2} v$, $\ln t \, v^2$, $\ln t \, v''$. Thus $t^{-2} v$ and $\ln t \, v^2$ renormalize M and J. A v'' term in the action is formally a total divergence, so it doesn't contribute to the v equation of motion. But it will not integrate to 0 in the total energy, since v has noncompact support. When $\alpha = 1$ this term is a delta function that doesn't show up in the Oklahoma calculation.

General α and General ξ

$$(-\partial_z{}^2 + \kappa^2 + z^\alpha)F^\pm(z) = 0 \qquad (\kappa = \sqrt{\mathbf{k}_\perp{}^2 - \omega^2}).$$

$$F^\pm \sim Q^{-1/4} \exp\left[\pm \int dz \left(Q^{1/2} + \frac{v''}{8Q^{3/2}}\right)\right],$$

$$Q \equiv \kappa^2 + v(z), \qquad v(z) = z^\alpha \quad (\text{for } z > 0).$$

Inside the wall:

$$\mathcal{E} \approx \frac{3}{2\pi^2}\,\frac{1}{t^4} - \frac{v}{8\pi^2 t^2} + \frac{1}{32\pi^2}\left(v^2 + \tfrac{2}{3}(1-6\xi)v''\right)\ln t,$$

exhibiting the Weyl structure of divergences.

Outside (but near) the wall:

$$\mathcal{E}_{\mathrm{ren}}(z) \sim \frac{6\xi - 1}{96\pi^2}\,\Gamma(1+\alpha)\,|z|^{\alpha-2}\Gamma(2-\alpha, 2|z|).$$

The singularity at $z = 0$ disappears for $\alpha > 2$:

$$\mathcal{E}_{\mathrm{ren}}(0) \approx \frac{1-6\xi}{96\pi^2} \frac{\Gamma(1+\alpha)2^{2-\alpha}}{2-\alpha} .$$

For $\alpha < 2$,

$$\begin{aligned} \mathcal{E}_{\mathrm{ren}}(z) &\sim \frac{6\xi-1}{96\pi^2} \Gamma(1+\alpha) \left(|z|^{\alpha-2}\Gamma(2-\alpha) - \frac{2^{2-\alpha}}{2-\alpha} \right) \\ &\sim \frac{1-6\xi}{48\pi^2} (\gamma + \ln 2|z|) \quad \text{as } \alpha \uparrow 2. \end{aligned}$$

Conclusions

1. Understanding local energy density and pressure is essential for general relativity and clarifies the physics of global energy and force calculations.
2. For hard (Dirichlet) walls, an ultraviolet cutoff yields physically inconsistent results for energy and pressure.
3. Modifying the cutoff to point separation in a "neutral" direction yields physically plausible results, but logical justification is lacking.

4. We seek to model a wall by a soft but rapidly increasing potential barrier, such as the power wall.
5. Outside the potential, the effect of the soft wall is parametrized by the scattering phase shift, $\delta(p)$, whose asymptotics can be calculated at low and high frequency.
6. We have "exact" formulas for $\langle T^{\mu\nu} \rangle$ in terms of the phase shift, but evaluating them is numerically challenging.

7. Reorganizing the power-wall calculation gives rapidly converging integrals in terms of the eigenfunctions. Computations have been extended to the "inside" of the wall ($0 < z$).

8. "Bulk" divergences inside the wall renormalize the equation of motion of the potential itself.

9. Numerical computations have been done for $\alpha = 1$ (linear potential), but there are analytical results for general α. ($\alpha \to \infty$ best approximates a hard wall (at $z = 1$).)

10. Calculations are easily extended to general ξ. As usual, conformal coupling $\xi = \frac{1}{6}$ yields the least singular results.