CASIMIR FRICTION FORCE FOR MOVING HARMONIC OSCILLATORS

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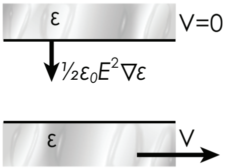

$$\mathbf{f} = -\frac{1}{2}\varepsilon_0 E^2 \nabla\varepsilon \quad \text{NORMAL FORCE}$$

Schwinger's source theory

$$E_i(x) = \int d^4x \Gamma_{ik}(x, x') P_k(x')$$

Stationarity: $\tau = t - t'$. Causality: $t' \leq t$.
From a statistical mechanical viewpoint: $\Gamma(x, x')$ is a generalized susceptibility.
Fourier transform

$$\Gamma_{ik}(x, x') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} \Gamma_{ik}(\mathbf{r}, \mathbf{r}', \omega)$$

Kubo:

$$\Gamma_{ik}(\mathbf{r}, \mathbf{r}', \omega) = i \int_{0}^{\infty} d\tau e^{i\omega\tau} \langle [E_i(x), E_k(x')] \rangle$$

Generalized susceptibility the same as the retarded Green function:

$$\Gamma_{ik}(x, x') = G^R_{ik}(x, x')$$

Fourier transform of the two-point function $\langle E_i(x) E_k(x') \rangle$:

$$\langle E_i(\mathbf{r}, \omega) E_k(\mathbf{r}', \omega') \rangle = 2\pi \langle E_i(\mathbf{r}) E_k(\mathbf{r}') \rangle_\omega \delta(\omega + \omega')$$

$\langle E_i(\mathbf{r}) E_k(\mathbf{r}') \rangle_\omega$: Spectral correlation tensor.

Fluctuation-dissipation theorem:

$$\langle E_i(\mathbf{r}) E_k(\mathbf{r}') \rangle_\omega = Im\, G^R_{ik}(\mathbf{r}, \mathbf{r}', \omega) \coth\left(\frac{1}{2}\beta\omega\right), \quad \beta = 1/(k_B T)$$

Absorption necessary.

Macroscopic approach

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MODEL:

TWO HARMONIC OSCILLATORS MOVE RELATIVE TO EACH OTHER WITH CONSTANT NONRELATIVISTIC VELOCITY

J. B. PENDRY (2010): QUANTUM FRICTION - FACT OR FICTION?

Statistical methods for moving harmonic oscillators (EPL 2010)

Hamiltonian H_0 for uncoupled motion.
Perturbation $-Aq(t)$, where
A: time-independent operator
$q(t)$: classical function of time

$$H = H_0 - Aq(t)$$

Put

$$-Aq(t) = \psi(\mathbf{r})x_1x_2,$$

where
$\psi(\mathbf{r})$: coupling strength,
x_1, x_2: internal vibrational coordinates.

Expand

$$-Aq(t) = [\psi(\mathbf{r}_0) + \nabla\psi(\mathbf{r}_0)\cdot\mathbf{v}t + ...]x_1x_2$$

Force between oscillators

$$\mathbf{B} = -(\nabla\psi(\mathbf{r}))x_1x_2$$

Restriction: First quantization only. Emission and absorption of photons neglected here.
Natural choice $q(t) = t$ implies the need of a convergence factor.
First term in above expansion: reversible equilibrium force. Friction force associated with the time dependent interaction.
Thermal average (Kubo)

$$\Delta\langle \mathbf{B}(t)\rangle = \int_{-\infty}^{t} \phi_{BA}(t-t')q(t')dt'$$

Heisenberg operator

$$\mathbf{B}(t) = e^{itH/\hbar}\mathbf{B}e^{-itH/\hbar}$$

Response function

$$\phi_{BA}(t) = \frac{1}{i\hbar}\mathrm{Tr}\{\rho[\mathrm{A}, \mathbf{B}(\mathrm{t})]\}.$$

With $q(t) = t$,

$$\phi_{BA}(t) = \mathbf{G}\phi(t),$$

$$\mathbf{G} = (\nabla\psi)(\mathbf{v}\cdot\nabla\psi),$$
$$\phi(t) = \mathrm{Tr}\{\rho\,\mathrm{C}(\mathrm{t})\},$$
$$C(t) = \frac{1}{i\hbar}[x_1x_2, x_1(t)x_2(t)].$$

The force can be written as

$$\mathbf{F} = \Delta\langle \mathbf{B}(t)\rangle = \mathbf{G}\int_{-\infty}^{t} \phi(t-t')t'dt' = \mathbf{F}_r + \mathbf{F}_f,$$

where

$$\mathbf{F}_r = \mathbf{G}t\int_{0}^{\infty} \phi(u)du$$

is the reversible force (it depends only upon position). $\mathbf{F}_r$ does not contribute to the net total dissipation.

Friction force:

$$\mathbf{F}_f = -\mathbf{G}\int_0^\infty \phi(u)u du,$$

in agreement with Høye and Brevik (Physica A, 1992).
[Observe that

$$\mathbf{G}t = (\nabla\psi)\psi(\mathbf{r}_0 + \mathbf{v}t), \quad \mathbf{r} = \mathbf{r}_0 + \mathbf{v}t,$$

where **v** here only represents a shift in position.]
Fourier transform of $\mathbf{F}_f$:

$$\mathbf{F}_f = -i\mathbf{G}\frac{\partial\tilde{\phi}(\omega)}{\partial\omega}\Big|_{\omega=0},$$

where

$$\tilde{\phi}(\omega) = \int_0^\infty \phi(t)e^{-i\omega t}dt \quad (\phi(t) = 0 \text{ for t} < 0)$$

. Introduce annihilation and creation operators

$$a_j(t) = a_j e^{-i\omega_j t}, \quad {a_j}^\dagger(t) = {a_j}^\dagger e^{i\omega_j t}$$

$$x_i = \left(\frac{\hbar}{2m_i\omega_i}\right)^{1/2}(a_i + a_i^\dagger), \quad i = 1, 2$$

$\Rightarrow$

$$\langle n_i|a_i^\dagger a_i(t) + a_i a_i^\dagger(t)|n_i\rangle = (2n_i + 1)\cos(\omega_i t) + i\sin(\omega_i t).$$

Thermal average

$$\phi(t) = \langle\langle n_1 n_2|C(t)|n_1 n_2\rangle\rangle$$
$$= D\left[(2\langle n_1\rangle + 1)\cos(\omega_1 t)\sin(\omega_2 t) + (2\langle n_2\rangle + 1)\cos(\omega_2 t)\sin(\omega_1 t)\right],$$

$$D = \frac{\hbar}{2m_1 m_2 \omega_1 \omega_2}$$

Energy levels

$$\varepsilon_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

$$2\langle n_i \rangle + 1 = \coth(\frac{1}{2}\beta\hbar\omega)$$

Insert $\phi(t)$ into $\mathbf{F}_f$ using convergence factor $e^{-\eta t}, \eta \to 0$:

$$\int_0^\infty t e^{-\eta t} \cos(\omega_1 t) \sin(\omega_2 t) dt$$

$$= \frac{\eta \Omega_1}{(\eta^2 + \Omega_1^2)^2} - \frac{\eta \Omega_2}{(\eta^2 + \Omega_2^2)^2} \to -\frac{\pi}{2\Omega_2}\delta(\Omega_2), \quad \eta \to 0$$

where $\Omega_1 = \omega_1 + \omega_2,\ \Omega_2 = \omega_1 - \omega_2$.

$\Rightarrow$

$$\mathbf{F}_f = -\frac{\pi\beta\hbar^2(\nabla\psi)(\mathbf{v}\cdot\nabla\psi)}{8m_1 m_2 \omega_1^2 \sinh^2(\frac{1}{2}\beta\hbar\omega_1)}\delta(\omega_1 - \omega_2),$$

as in Høye and Brevik 1992.

To get a finite result:

- Oscillators having the same frequency, $\omega_1 = \omega_2$,
- Finite temperature, $T > 0$.

If η is kept finite, the Ω_1- term contributes.

Same result obtained by means of a path integral formalism of quantum systems at thermal equilibrium (Høye and Stell, 1981). Analogy with classical polymer problem where imaginary time is a fourth dimension of length β. Closed loops of periodicity β.

Dissipation of energy

Dissipation associated with work done, during a finite time interval.
Motion starts at $t = 0$ with maximum velocity $\mathbf{v}$ when $\mathbf{r} = \mathbf{r}_0$. At $t \to \infty$ the motion dies out.

$$q(t) \to te^{-\eta t}$$

$$\mathbf{v} \to \mathbf{v}_1(t) = \mathbf{v}\dot{q}(t) = \mathbf{v}(1 - \eta t)e^{-\eta t}.$$

Total energy dissipated

$$\Delta E_d = \int_{-\infty}^{\infty} \mathbf{v}_1(t) \cdot \mathbf{F}_f \, \dot{q}(t) dt = \mathbf{v} \cdot \mathbf{F}_f \int_0^{\infty} |\dot{q}(t)|^2 dt = \frac{1}{4\eta} \mathbf{v} \cdot \mathbf{F}_f$$

Reversible force $\mathbf{F}_r \propto t \to q(t)$ does not contribute to the dissipation, since $\int_0^\infty \dot{q}(t) q(t) dt = 0$.

Extension of dissipation formula:

Force due to perturbation

$$F_f = \int_{-\infty}^{t} \phi_{AA}(t - t')q(t')dt',$$

where

$$\phi_{AA}(t) = \frac{1}{i\hbar}\mathrm{Tr}\left\{\rho[A, A(t)]\right\}.$$

$\Rightarrow$

$$\Delta E_d = \int_{-\infty}^{\infty} v(t)F_f dt = \int_{-\infty}^{\infty} \dot{q}(t)\left[\int_{-\infty}^{t} \phi_{AA}(t - t')q(t')dt'\right] dt.$$

This is consistent with expression for E_d above:

$$q(t') = q(t) - \dot{q}(t)(t - t') + ...,$$

$$q(t) = t \rightarrow te^{-\eta t},$$

$$\phi_{AA} = \mathbf{v} \cdot \mathbf{G}\psi,$$

$\Rightarrow$

$$\Delta E_d = -\int_{0}^{\infty} \phi_{AA}(u)udu \int_{0}^{\infty} \left[\dot{q}(t)\right]^2 dt + ...,$$

Energy dissipation, calculated from first order perturbation theory (EPJD, 2011)

Interaction effectively coupled in for a finite period of time.
Hamiltonian

$$H = H_0 - Aq(t)$$

$$-Aq(t) = \psi(\mathrm{r})x_1x_2$$

$$-Aq(t) = [\psi(\mathbf{r}_0) + \nabla\psi(\mathbf{r}_0)\cdot\mathbf{v}t + ...]x_1x_2$$

Thermal equilibrium:

$$\psi = \sum_n a_n\psi_n$$

Probability (Boltzmann factor):

$$P_n = |a_n|^2 = \frac{1}{Z}e^{-\beta E_n},$$

$$Z = \sum_n e^{-\beta E_n}.$$

Time-dependent interaction $V(t) = -Aq(t)$
$\Rightarrow$

$$\Delta a_n = b_{nm},$$

where

$$b_{nm} = \frac{1}{i\hbar}\int_{-\infty}^{t} V_{nm}(\tau)e^{i\omega_{nm}\tau}d\tau$$

$$V_{nm}(\tau) = \int \psi_n^* \, V(\tau) \psi_m dx = -A_{nm} \, q(\tau),$$

$$A_{nm} = \langle n | A | m \rangle = \int \psi_n^* \, A \, \psi_m dx,$$

Here $\omega_{nm} = \omega_n - \omega_m$, with $\omega_n = E_n/\hbar$.
Assume that the perturbation vanishes after some time. Then

$$b_{nm} = -\frac{1}{i\hbar} A_{nm} \, \hat{q}(-\omega_{nm}),$$

$$\hat{q}(\omega) = \int_{-\infty}^{\infty} q(t) e^{-i\omega t} dt,$$

Take into account many neighboring states:

$$\Delta a_n \to \sum_{m \not{n}} a_m b_{nm}$$

Perturbed coefficients

$$a_{1n} = a_n + \Delta a_n = a_n + \sum_{m \neq n} a_m b_{nm}$$

Thermal average uncorrelated coefficients:

$$\langle a_n^* a_m \rangle = 0$$

$\Rightarrow$ New probability of the state n

$$P_{1n} = \langle a_{1n}^* a_{1n} \rangle = |a_n|^2 + \sum_{m \neq n} |a_m|^2 B_{nm},$$

where

$$B_{nm} = b_{nm} b_{nm}^* = |b_{nm}|^2$$

Loss to other states is $\sum_{m\neq n} |a_n|^2 B_{mn}$.

Then

$$P_{1n} = |a_n|^2 + \sum_{m\neq n}(|a_m|^2 - |a_n|^2)B_{nm} = P_n + \sum_m (P_m - P_n)B_{nm}.$$

Change in energy

$$\Delta E = \sum_{nm}(E_n - E_m)P_m B_{nm} + \sum_{nm}(E_m P_m - E_n P_n)B_{nm} = \sum_{nm}(E_n - E_m)P_m B_{nm}.$$

$$\Delta E = \frac{1}{Z}\sum_{nm} e^{-\frac{1}{2}\beta(E_n+E_m)}\Delta_{nm}\sinh(\frac{1}{2}\beta\Delta_{nm})B_{nm},$$

with $\Delta_{nm} = E_n - E_m$, and

$$B_{nm} = \frac{1}{\hbar^2}A_{nm}A^*_{nm}\hat{q}(-\omega_{nm})\hat{q}(\omega_{nm}).$$

To second order, $\Delta E > 0$. The dissipation occurs to the second order in the perturbation. To first order, $\Delta E = 0$; the changes are adiabatic.

Energy dissipation from friction force

As before,

$$F_f = \int_{-\infty}^{t} \phi_{AA}(t - t')q(t')dt',$$

where

$$\phi_{AA}(t) = \frac{1}{i\hbar}\mathrm{Tr}\left\{\rho[\mathrm{A}, \mathrm{A(t)}]\right\}.$$

$$\rho = \frac{e^{-\beta H}}{Z}, \quad \text{with} \quad \mathrm{Z} = \mathrm{Tr}(\mathrm{e}^{-\beta \mathrm{H}}),$$

$$A(t) = e^{itH/\hbar} A\, e^{-itH/\hbar}.$$

Total dissipated energy

$$\Delta E = -\int_{-\infty}^{\infty} v(t) F_f\, dt = -\int_{-\infty}^{\infty} \left[\int_{-\infty}^{t} \dot{q}(t) \phi_{AA}(t-t') q(t') dt' \right] dt$$

With wave function representation

$$e^{-\beta H} \to \sum_n \psi_n(x) e^{-\beta E_n} \psi_n^*(x_1),$$

$$\rho A A(t) = \frac{1}{Z} \sum_{nmk} \int \psi_n(x) e^{-\beta E_n} \psi_n^*(x_1) A \psi_m(x_1) e^{i\omega_m t} \psi_m^*(x_2) A$$

$$\times \psi_k(x_2) e^{-i\omega_k t} \psi_k^*(x_3) dx_1\, dx_2.$$

Thus

$$\mathrm{Tr}(\rho A A(t)) = \frac{1}{Z} \sum_{nm} e^{-\beta E_n} A_{nm} e^{i\omega_m t} A_{mn} e^{-i\omega_n t},$$

with $\int \psi_k^*(x) \psi_n(x) dx = \delta_{kn}$.
Response function

$$\phi_{AA}(t) = \frac{1}{i\hbar} \mathrm{Tr}\left\{ \rho [A, A(t)] \right\} = \frac{1}{i\hbar} \sum_{nm} M_{nm} (e^{-i\omega_{nm} t} - e^{i\omega_{nm} t}),$$

$$M_{nm} = -\frac{1}{Z} e^{-\frac{1}{2}\beta(E_n + E_m)} \sinh(\frac{1}{2}\beta \Delta_{nm}) A_{nm} A_{nm}^*,$$

,

with $\Delta_{nm} = E_n - E_m = \hbar\omega_{nm}$.
By means of partial integrations and insertion into the expression for ΔE above, one gets

$$\Delta E = \frac{1}{\hbar}\sum_{nm} M_{nm}\,\omega\,\hat{q}\,(\omega)\,\hat{q}(-\omega).$$

In agreement with the result obtained above, from time dependent perturbation theory.

Friction between harmonic oscillators

Calculation directly from the last expression for ΔE.

Let $t \to te^{-\eta t}(\eta \to 0)$.

Introduce annihilation and creation operators

$$x_i = \left(\frac{\hbar}{2m_i\omega_i}\right)^{1/2}(a_i + a_i^\dagger)$$

Then the interaction becomes

$$-Aq(t) = \gamma(a_1 a_2 + a_1 a_2^\dagger + a_1^\dagger a_2 + a_1^\dagger a_2^\dagger)\, te^{-\eta t},$$

where

$$\gamma = (\frac{1}{2}D\hbar)^{1/2}(\mathbf{v}\cdot\nabla\psi), \quad D = \frac{\hbar}{2m_1 m_2\omega_1\omega_2}.$$

Since here only small η ($\to 0$) is considered,

$$A = a_1 a_2^\dagger + a_1^\dagger a_2, \quad \text{and} \quad q(t) = \gamma t e^{-\eta t}.$$

Matrix elements

$$A_{n_1,n_2,n_1+1,n_2-1} = \langle n_1 n_2 | a_1 a_2^\dagger | n_1+1, n_2-1\rangle = \sqrt{n_1+1}\,\sqrt{n_2},$$

$$A_{n_1,n_2,n_1-1,n_2+1} = \langle n_1 n_2 | a_1^\dagger a_2 | n_1-1, n_2+1\rangle = \sqrt{n_1}\,\sqrt{n_2+1},$$

while all other elements are zero. The Fourier transform of $q(t)$ is

$$\hat{q}(\omega) = \gamma \int_0^\infty t e^{-\eta t} e^{-i\omega t} = \frac{\gamma}{(\eta + i\omega)^2},$$

so that for $\eta \to 0$,

$$\hat{q}(\omega)\hat{q}(-\omega) = \frac{\gamma^2}{(\eta^2+\omega^2)^2} \to \frac{\pi\gamma^2}{2\eta\omega^2}\delta(\omega).$$

Here $\omega = \omega_1 - \omega_2$. With $\omega \to 0$ ($m = n \pm 1$)

$$\Delta_{nm}\sinh(\frac{1}{2}\beta\Delta_{nm}) \to \frac{1}{2}\beta\Delta_{nm}^2 = \frac{1}{2}\beta(\pm\hbar\omega)^2 = \frac{1}{2}\beta\hbar^2\omega^2.$$

One has $\langle n_1\rangle \approx \langle n_2\rangle \approx \langle n\rangle$, with $\omega_1 \to \omega_2$, and

$$\langle n\rangle = \frac{\sqrt{x}}{Z}\sum_{n=0}^\infty n x^n = \frac{x}{1-x}, \quad x = e^{-\beta\hbar\omega_1},$$

$$Z = \sqrt{x}\sum_{n=0}^\infty x^n = \frac{\sqrt{x}}{1-x}.$$

Then $\langle n\rangle + 1 = 1/(1-x)$, by which

$$\langle (n_1+1)n_2 + n_1(n_2+1)\rangle = 2(\langle n\rangle + 1)\langle n\rangle$$

$$= \frac{2x}{(1-x)^2} = \frac{1}{2\sinh^2(\frac{1}{2}\beta\hbar\omega_1)}.$$

Finally

$$\Delta E = \frac{\pi\beta\hbar^2\gamma^2}{8\eta\sinh^2(\frac{1}{2}\beta\omega_1)}\,\delta(\omega_1-\omega_2).$$

This is in agreement with the expression above.
At zero temperature, $\Delta E = 0$, due to the assumption slowly varying coupling, $\eta \to 0$.
With rapidly varying coupling or higher velocities, one would get a finite ΔE also at $T = 0$.

Basic assumptions and results:

• Initial thermal equilibrium at temperature T.
• Low velocities, and nonrelativistic mechanis. Photons not included. They were included in Brevik-Høye, 1993.
• First order perturbation theory sufficient to calculate the energy dissipation (second order effect), due to uncorrelated phases of eigenstates.

Comparison between different formulations (EPJD 2011)

G. Barton, New J. Phys. **12**, 113044 (2010).

Assume $T = 0$. Interaction Hamiltonian

$$H_{\rm int} = \frac{e^2}{s^3}\, y_1 y_2,$$

(Gaussian units assumed). Here , y_1 and y_2 are the oscillator coordinates, and $\mathbf{s} = \mathbf{s}(t)$ is the vectorial distance between the mass centers. Introduce new coordinates

$$y_\pm = \frac{y_1 \pm y_2}{\sqrt{2}}.$$

Then

$$H_{\rm int} = H_{\rm int+} + H_{\rm int-},$$

$$H_{\rm int\pm} = \pm\frac{1}{2} q\, y_\pm^2, \quad q = \frac{e^2}{s^3}.$$

Total dissipated energy

$$\Delta E = 2 \times 2\hbar\omega |c(\infty)|^2,$$

where

$$c(t) = -\frac{i}{2\hbar}\int_{-\infty}^{t} dt' q\, \langle 2_+ | y_+^2 | 0_+ \rangle\, e^{2i\omega t'}.$$

Evaluate the matrix elements in $c(t)$:

$$y_\pm = \sqrt{b}\,(a_\pm + a_\pm^\dagger), \quad b = \frac{\hbar}{2m\omega}.$$

Then,

$$\langle 2_+|y_+^2|0_+\rangle = b\,\langle 2_+|a_+^{\dagger\,2}|0_+\rangle = \sqrt{2}\,b.$$

$\Rightarrow$

$$\Delta E = 8\hbar\omega b^2|I(\infty)|^2,$$
$$I(t) = -\frac{i}{2\hbar}\int_{-\infty}^{t} dt'\,q\,e^{2i\omega t'}.$$

Comparison with formalism above: Interaction Hamiltonian $H_{\rm int} = -Aq(t)$, with

$$A = -y_1 y_2, \quad \text{and} \quad q(t) = q = \frac{e^2}{s^3}.$$

Change in energy

$$\Delta E = \sum_{nm}(E_n - E_m)P_m B_{nm},$$

Start from ground state: $P_m \to P_{00} = 1$. Also, $B_{mn} = |b_{nm}|^2$, so that

$$\Delta E = (E_{11} - E_{00})B_{1100} = 2\hbar\omega B_{1100}.$$

Calculate the transition coefficient B_{1100}:

$$\hat{q}(\omega) = \int_{-\infty}^{\infty} q(t)e^{-i\omega t}dt.$$

Then

$$\hat{q}(-\omega_{nm}) \to \hat{q}(-\omega_{1100}) = \hat{q}(-2\omega) = 2i\hbar I(\infty).$$

Further,

$$A_{nm} \to A_{1100} = \langle 11| - y_1 y_2 |00\rangle \to -b.$$

Altogether

$$B_{1100} = \frac{1}{\hbar^2}|A_{1100}|^2 \hat{q}(-2\omega)\hat{q}(2\omega) = 4b^2|I(\infty)|^2,$$

$$\Delta E = 8\hbar\omega b^2 |I(\infty)|^2,$$

in agreement with the expression above.