VAN DER WAALS FRICTION: A HAMILTONIAN TEST-BED

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Abstract

In the van der Waals regime (neglecting relativity and retardation), we find the power P generated by friction between two Drude-modelled dissipative half-spaces, at fixed separation and relative speed u, admitting only low u and low temperatures. This requires only elementary quantum mechanics; but the results can serve as partial checks on calculations in the fully retarded Casimir regime. They also raise questions regarding (i) the frequency-distribution of P; (ii) the status of predictions about Casimir forces generally, insofar as they feature parameters like conductivities with their empirical temperature-dependence; and (iii) calculations of heat transfer, insofar as they assume fluctuations in the two bodies to be uncorrelated.

1. Introduction and preview

Since this is not a research paper but a workshop talk, the reader is asked to take most technicalities on trust: they are spelled out elsewhere (Barton 2011, cited as as JPC). At this stage, the writer's own interest is focussed mainly on the open questions outlined in section 5 here.

We consider friction between two parallel Drude-modelled half-spaces, call them L, R, with response functions

$$\varepsilon(\omega) = 1 + \frac{\omega_{pl}^2}{\omega_0^2 - \omega^2 - i\omega\Gamma} \, . \tag{1.1}$$

We shall choose a Hamiltonian that reproduces ε: in this sense ε is not input but output. The interaction V between the half-spaces depends only on their surface plasmons: bulk plasmons generate no exterior fields, whence we ignore them. The simple *nondissipative (nd) model,* with $\Gamma = 0$ from the start, yields the same results as the *nondissipative limit* $\Gamma \to 0$. Thus it can serve as a check; moreover there are simple rules, on which we shall lean heavily, for obtaining certain crucial dissipative from the corresponding nondissipative expressions.

The frequency ω_S of surface plasmons on a single nondissipative half-space, convenient dimensionless parameters β and γ, and the Ohmic conductivity $\bar{\sigma}$, are given by

$$\omega_S = \sqrt{\omega_0^2 + \frac{\omega_{pl}^2}{2}} \, , \qquad \beta^2 \equiv \frac{\omega_{pl}^2}{2\omega_S^2} \leq 1, \qquad \gamma \equiv \frac{\Gamma}{\omega_S} = \frac{\beta^2 \omega_S}{2\pi\bar{\sigma}} \, . \tag{1.2}$$

In metals, $\gamma \ll 1$ is typically 10^{-3} to 10^{-2}.

The separation ζ is in the z direction, and fixed; and the relative velocity $\mathbf{u} = u\hat{\mathbf{x}}$ is constrained, eventually to constant u, by an externally applied force counteracting the friction. Position coordinates are written as $\mathbf{r} = (\mathbf{s}, z)$, with $\mathbf{s} = (s_1, s_2)$. The half-spaces are taken to be displaced from their reference positions at time zero through

$$\boldsymbol{\sigma}_{L,R} = \mp\boldsymbol{\sigma}/2 = \mp\, \hat{\mathbf{x}}\sigma/2, \qquad \boldsymbol{\sigma}_R - \boldsymbol{\sigma}_L \equiv \boldsymbol{\sigma} = \mathbf{u}t = (u, 0, 0)\, t. \tag{1.3}$$

Let $\mathbf{F} = -F\hat{\mathbf{x}}$ be the frictional resistance and $P = uF$ the frictionally generated power per unit area. Our aim is to calculate P subject to two highly simplifying restrictions:

(i) *We examine only the van der Waals (vdW) regime*, shorthand for the nonrelativistic/nonretarded approximation

$$u/c \ll 1, \quad \zeta \ll c/\omega_S, \quad \zeta \ll c\hbar/k_B T, \quad \textit{but} \quad \zeta \gg (a_B \equiv \textit{Bohr radius}), \tag{1.4}$$

where there are only instantaneous Coulomb forces, but no Maxwell equations, no QED, and no photons. Often this regime is identified by taking the formal limit $c \to \infty$.

(ii) *We consider only low u and low initial temperatures T_i*, in the sense that

$$v \equiv u/\zeta\omega_S \ll 1, \qquad \tau \equiv k_B T/\hbar\omega_S \ll 1, \qquad B \equiv 1/\tau \gg 1. \tag{1.5}$$

Typically, $\hbar\omega_S$ is of the order of atomic ionization energies, whence $\tau \gtrsim 1$ would entail very disturbed solids; while $\zeta \gg a_B$ ensures that $v \ll 1$ for u not larger than the atomic unit $\simeq c/137$. We shall treat v and τ as comparable, and focus on the asymptotics. Note that $\tau/v = k_B T\zeta/\hbar u$ does not depend on the properties of the material.

It is of the essence that, given these restrictions, we need and shall use only textbook-level NR quantum mechanics with simple Hamiltonians, to the exclusion of Lifshitz-derived formulae and of Maxwell stress-tensors. (To accommodate nonzero Γ, section 4 below will adopt a nonretarded version of the now-standard Huttner-Barnett theory.) The results can serve as partial checks on fully retarded Casimir-regime calculations. The point is that most (though not all) such theories proceed from assumptions which under nonretarded conditions are satisfied by our model: therefore in the vdW limit their results must reduce to ours , irrespective of any interest one might take in our model for its own sake. For instance, the rather elaborate recent controversy, insofar as it turned on the mere existence of P, might have been resolved simply by observing that the vdW limit is manifestly nonzero; JPC cites other examples where candidate theories fail the vdW test on sight, even though the reasons for the failure of the supposedly general expressions are well hidden from view.

The Hamiltonian reads

$$H_0 = H_L + H_R, \qquad H = H_0 + V. \tag{1.6}$$

Unfortunately V cannot be treated perturbatively: for instance, the attractive and the frictional forces calculated only to order V^2 turn out to be wrong by 10% and 20% respectively.

The rest of this paper is organized as follows. We start, in section 2, with nondissipative ε (ie $\gamma = 0$) and zero temperature (ie $\tau = 0$), which allows motion to be introduced as simply as possible. Section 3 admits nonzero γ, which complicates the analysis but introduces no new points of principle. Section 4 then outlines the truly interesting scenario with v and τ both nonzero (though small). The final section 5 makes some general comments, and voices some related questions that to the writer's knowledge are open, and largely unaddressed. They concern the frequency-distribution of the frictionally generated energy, heat exchange between stationary half-spaces, and the applicability of predictions featuring temperature-dependent γ.

2. Nondissative system at zero temperature: $\gamma = 0 = \tau$

The potentials Φ and surface-charge densities Σ due to each half-space on its own are

$$(\Phi_{L,R})_{nd} = -\beta \int d^2k \sqrt{\frac{\hbar\omega_S}{4\pi k}}\,(a_{L,R})_{\mathbf{k}} \exp\left[i\mathbf{k}\cdot(\mathbf{s}\pm\boldsymbol{\sigma}/2) - k\,|z\pm\zeta/2|\right] + Hc, \quad (2.1)$$

$$(\Sigma_{L,R})_{nd} = -\frac{\beta}{2\pi} \int d^2k \sqrt{\frac{\hbar\omega_S k}{4\pi}}\,(a_{L,R})_{\mathbf{k}} \exp[i\mathbf{k}\cdot(\mathbf{s}\pm\boldsymbol{\sigma}/2)] + Hc. \quad (2.2)$$

Hc stands for Hermitean conjugate. The a and a^+ are the annihilation and creation operators for surface plasmons:

$$\left[a_{L\mathbf{k}}, a^+_{L\mathbf{k}'}\right] = \left[a_{R\mathbf{k}}, a^+_{R\mathbf{k}'}\right] = \delta(\mathbf{k}-\mathbf{k}'), \quad (2.3)$$

while a_L and a_L^+ commute with a_R and a_R^+. Then

$$(H_L)_{nd} = \int d^2k \frac{1}{2}\hbar\omega_S \left(a_{L\mathbf{k}}a^+_{L\mathbf{k}} + a^+_{L\mathbf{k}}a_{L\mathbf{k}}\right), \quad (2.4)$$

similarly for R, and

$$V_{nd} = \int d^2s\,[\Phi_L(z=\zeta/2)]_{nd}\,[\Sigma_R]_{nd} \quad (2.5)$$

$$= \frac{\hbar\beta^2\omega_S}{2} \int d^2k \exp(-\kappa)\exp(i\mathbf{k}\cdot\boldsymbol{\sigma})\left(a_{L,\mathbf{k}} + a^+_{L,-\mathbf{k}},\right)\left(a_{R,-\mathbf{k}} + a^+_{R,-\mathbf{k}},\right), \quad \boldsymbol{\kappa} \equiv \zeta\mathbf{k}. \quad (2.6)$$

Because H features only bilinearly coupled oscillators, it is easily diagonalized. When L, R are stationary the eigenmodes are $e$ven or $o$dd in z,

$$a_{e,o} = (a_L \pm a_R)/\sqrt{2}, \qquad \omega_{e,o}(\kappa) = \omega_S\sqrt{1\pm\exp(-\kappa)} \equiv \omega_{p,m}(\kappa), \quad (2.7)$$

$$H_{nd}(u=0) = \int d^2k \frac{1}{2}\left\{\hbar\omega_p(\kappa)\left(a_{p\mathbf{k}}a^+_{p\mathbf{k}} + a^+_{p\mathbf{k}}a_{p\mathbf{k}}\right) + \hbar\omega_m(\kappa)\left(a_{m\mathbf{k}}a^+_{m\mathbf{k}} + a^+_{m\mathbf{k}}a_{m\mathbf{k}}\right)\right\}. \quad (2.8)$$

The alternative subscripts p, m (for "$p$lus" and "$m$inus") are introduced by hindsight.

Crucially, it is the fluctuations of the exact normal modes that are mutually independent: the fluctuations in L, R are not, because of the correlations between L, R visible from (2.7a). Just how strong they can be is illustrated in appendix A. Because of these correlations the forces cannot be calculated perturbatively in V. The key to $\mathbf{F}$ is the adiabatic method, using not V but $\partial H/\partial t = \partial V/\partial t$ as a perturbation inducing transitions between the exact eigenmodes appropriate to the instantaneous relative displacement $\boldsymbol{\sigma}(t)$. To find these eigenmodes, labelled p, m, one must diagonalize the exact time-dependent Hamiltonian $H(\boldsymbol{\sigma})$, as is done in JPC. Technically, the problem reduces to diagonalizing, in L, R space, certain submatrices of H having the form

$$\begin{bmatrix} 1, & e^{-k\zeta} \exp\left(it\mathbf{u}\cdot\mathbf{k}\right) \\ e^{-k\zeta} \exp\left(-it\mathbf{u}\cdot\mathbf{k}\right), & 1 \end{bmatrix}. \tag{2.9}$$

Thus motion ($u \neq 0$) affects the eigenstates (which no longer have definite parity), but neither the eigenvalues of H nor the eigenfrequencies.

At $T = 0$ the system is in the no-plasmon state. One must start, as in JPC, by expressing $H_{nd} = \int d^2k \left[H_{\mathbf{k}}\right]_{nd}$, and then its time-derivative, in terms of the $a_{p,m}$. One finds

$$\frac{\partial \left[H_{\mathbf{k}}\right]_{nd}}{\partial t} = i\hbar \left(\beta\omega_S\right)^2 \mathbf{k}\cdot\mathbf{u} \left\{ \frac{e^{-\kappa+it(\mathbf{k}\cdot\mathbf{u})}}{4\sqrt{\omega_p\omega_m}} \left(a^+_{p,\mathbf{k}} + a_{p,-\mathbf{k}}\right)\left(a_{m,\mathbf{k}} + a^+_{m,-\mathbf{k}}\right) - Hc \right\}; \tag{2.10}$$

then the adiabatic version of the Golden Rule (Landau & Lifshitz 1977, Schiff 1968), in an obvious notation, reads

$$|\psi(t)\rangle \simeq |0,0\rangle + \int\int d^2k_p d^2k_m \exp\left(-i\Omega t\right) c(t) \left|\mathbf{k}_p, \mathbf{k}_m\right\rangle, \qquad \Omega \equiv \omega_p + \omega_m, \tag{2.11}$$

$$\frac{\partial c}{\partial t} = \frac{\left\langle \mathbf{k}_p, \mathbf{k}_m \right| \partial H_{\mathbf{k}}/\partial t \left|0,0\right\rangle}{\hbar\Omega} \exp\left(i\Omega t\right), \qquad c(0) = 0. \tag{2.12}$$

At times $t \gg 1/\Gamma$, but not so large as to invalidate the Golden Rule, plasmon excitation-probabilities grow linearly with t, whence

$$\begin{aligned} P &= \frac{1}{A} \lim_{t\to\infty} \frac{1}{t} \int\int d^2k_p d^2k_m \left|c(t)\right|^2 \hbar\Omega \\ &= \frac{\hbar\beta^4\omega_S^2}{16\pi^2\zeta^2} \int d^2\kappa \left(\mathbf{v}\cdot\boldsymbol{\kappa}\right) \frac{e^{-2\kappa}}{\sqrt{1-e^{-2\kappa}}} \delta\left(\Omega/\omega_S - \mathbf{v}\cdot\boldsymbol{\kappa}\right), \end{aligned} \tag{2.13}$$

where A is the (nominally infinite) total cross-sectional area. The asymptotics for $v \ll 1$ read

$$P \simeq \frac{\hbar\beta^4\omega_S^2 \exp(-4/v)}{\sqrt{8\pi} v^{3/2}\zeta^2} = \frac{\hbar\beta^4\omega_S^{7/2}}{\sqrt{8\pi} u^{3/2}\zeta^{1/2}} \exp\left(-\frac{4\omega_S\zeta}{u}\right), \tag{2.14}$$

tallying with a result obtained quite differently by Pendry (2010). Without dissipation friction at low speeds is exponentially weak, because of the gap in the frequency spectrum, for all $\mathbf{k}$ in ω_p, and in ω_m for all $\mathbf{k}$ except the end-point $\mathbf{k} = \mathbf{0}$.

3. Dissipative system at zero temperature: $\gamma \neq 0$, $\tau = 0$

The vdW version of the Huttner-Barnett model[1] for a dissipative heat bath, featuring a continuous spectrum of otherwise unspecified oscillators, has been discussed at length elsewhere (Barton 1997, 2000, and JPC). It amounts to treating each nondissipative normal mode according to the standard theory of a single damped harmonic oscillator (cf Grabert et al 1984, Tatarskiĭ 1987, Weiss 2008): each such discrete mode is as it were dissolved in the continuum, where it shows up as a resonance. For instance, the Hamiltonian $(H_L)_{nd}$ and the potential $(\Phi_L)_{nd}$ from (2.4) and (2.1) are replaced by

$$H_L = \int d^2k \int_0^\infty \hbar\omega_S \frac{1}{2}\left(a_{L\mathbf{k}\omega}a^+_{L\mathbf{k}\omega} + a^+_{L\mathbf{k}\omega}a_{L\mathbf{k}\omega}\right), \tag{3.1}$$

$$\Phi_L(\mathbf{s}, z) = -\omega_S\beta \int d^2k \int_0^\infty d\omega \sqrt{\frac{\hbar}{4\pi k\omega}} \frac{g_\omega a_{L\mathbf{k}\omega}}{[\omega_S^2 - \omega^2 - i\omega\Gamma]} e^{[i\mathbf{k}\cdot(\mathbf{s}+\boldsymbol{\sigma}/2)-k|z+\zeta/2|]} + Hc, \tag{3.2}$$

where $g_\omega = \sqrt{2\omega^2\Gamma/\pi}$ and $\left[a_{L\mathbf{k}\omega}, a^+_{L\mathbf{k}'\omega'}\right] = \delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')$.

Fortunately, given an expression waiting to be evaluated in the nd model, there are simple rules for writing down the dissipative version. For L or R alone, one substitutes

$$(a_{L,R})_\mathbf{k} \to \int_0^\infty d\omega \sqrt{\frac{\omega_S}{\omega}} \frac{g_\omega}{[\omega_S^2 - \omega^2 - i\omega\Gamma]} (a_{L,R})_{\mathbf{k}\omega}\,. \tag{3.3}$$

For L and R jointly, one merely replaces the discrete frequency ω_S by one of the discrete frequencies $\omega_{p,m}$, and $a_{L,R}$ by $a_{p,m}$; and extends final sums $\int d^2k...$ to $\sum_{e,o}\int d^2k...$. Using these rules plus the nd expressions from section 2, and in terms of dimensionless variables

$$x \equiv \omega/\omega_S, \qquad x_{p,m} \equiv \omega_{p,m}/\omega_S \equiv \sqrt{1 \pm \exp(-\kappa)}\,, \tag{3.4}$$

one obtains

$$P = \frac{\hbar\beta^4\omega_S^2}{2\pi^3\zeta^2} \int d^2\kappa\,(\mathbf{v}\cdot\boldsymbol{\kappa})\,e^{-2\kappa} \int_0^\infty \int_0^\infty \frac{dxdx'xx'\gamma^2\delta\left(x + x' - \mathbf{v}\cdot\boldsymbol{\kappa}\right)}{\left[\left(x^2 - x_p^2\right)^2 + \gamma^2x^2\right]\left[(x'^2 - x_m^2)^2 + \gamma^2x'^2\right]}. \tag{3.5}$$

[1]See Huttner & Barnett 1992; more references are given in Barton 1997 and in JPC. Recently, Philbin (2010, 2011) has derived an elegant alternative Hamiltonian, whose predictions to date are the same as Huttner and Barnett's.

To evaluate such integrals we write $\int d^2\kappa ... = \int_0^\infty d\kappa\kappa \int_{-\pi}^{\pi} d\phi \cos(\phi) ...$ and $\mathbf{v}\cdot\boldsymbol{\kappa} = v\kappa\cos(\phi)$.

To find the asymptotics of (3.5) it suffices to observe that by virtue of the factor $\exp(-2\kappa)$ we can take $\kappa \lesssim \mathcal{O}(1)$. Then, for $\gamma \ll 1$ as we assume, and in view of the delta function,

$$v \ll 1 \quad \Rightarrow \quad (x, x') \ll 1 \quad \Rightarrow \quad \frac{1}{[...]\,[...]} \rightarrow \frac{1}{x_p^4 x_m^4} = \frac{1}{(1-e^{-2\kappa})^2}\,, \tag{3.6}$$

which makes all the integrations trivial, and eventually yields

$$v \ll 1 \quad \Rightarrow \quad P \simeq \frac{15\zeta(5)}{2^8\pi^2}\left[\frac{\hbar\beta^4\omega_S^2\gamma^2 v^4}{\zeta^2}\right] = \frac{15\zeta(5)}{2^{10}\pi^4}\left[\frac{\hbar\beta^4 u^4}{\bar{\sigma}^2\zeta^6}\right]. \tag{3.7}$$

Pendry (1997) gives a result smaller by a factor 12. The reasons for the discrepancy are past recall.

Remarkably, P here is proportional to u^4, and not to u^2 as might reasonably have been expected. By contrast, for an atom moving parallel to the surface of such a half-space (Barton 2010), P does start at order u^2.

4. Dissipative system at finite temperature: $\gamma \neq 0$, $\tau \neq 0$

4.1. The general formula for P

By given τ we mean that, initially, the occupation numbers n of the exact eigenmodes having frequency ω are Boltzmann-distributed with thermal averages (identified as such by overbars)

$$\bar{n}(\tau, x) = \frac{1}{\exp(x/\tau) - 1}\,, \qquad \bar{N}(\tau, x) \equiv \bar{n} + \frac{1}{2} = \frac{1}{2}\coth\left[\frac{x}{2\tau}\right], \tag{4.1}$$

$$\bar{N}(-x) = -\bar{N}(x), \quad \lim_{\tau\to 0}\bar{N}(x) = \frac{1}{2}sign(x). \tag{4.2}$$

Our aim is to determine P to orders v^2, v^4, and $v^2\tau^2$, treating v and τ as comparably small, and dissipation as weak ($\gamma \ll 1$). It proves convenient to introduce a dimensionless function f and to scale

$$\Pi \equiv \gamma^2\beta^4\hbar\omega_S^2/\zeta^2, \qquad P = \Pi f(\tau, v). \tag{4.3}$$

For instance, from (3.7),

$$f(0, v) \simeq \frac{15\zeta(5)}{2^8\pi^2}\cdot v^4. \tag{4.4}$$

At $\tau > 0$, friction changes the energy through changes ± 1 in the occupation numbers of two normal modes, one p and one m. Some donkey-work finds that

(3.5) is adapted to transitions in the same or in opposite directions, respectively, by changing $\mathbf{v}\cdot\boldsymbol{\kappa}\delta(x+x'-\mathbf{v}\cdot\boldsymbol{\kappa})$ to $\delta(x\pm x'-\mathbf{v}\cdot\boldsymbol{\kappa})(x\pm x')\{N'\pm N\}$. Remarkably, by exploiting the antisymmetry of $\bar{N}(x)$ both types of contributions can be combined into

$$f=\frac{\mathcal{J}}{2\pi^3}\,,\quad \mathcal{J}(\tau,v)=\frac{1}{2}\int_0^\infty d\kappa\kappa^2 e^{-2\kappa}\mathcal{L}, \tag{4.5}$$

$$\mathcal{L}\equiv v\int_{-\pi}^{\pi}d\phi\cos(\phi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dxdx'xx'\delta\left(x+x'-\mathbf{v}\cdot\boldsymbol{\kappa}\right)\left[\bar{N}(x)+\bar{N}(x')\right]}{\left[\left(x_p^2-x^2\right)^2+x^2\gamma^2\right]\left[(x_m^2-x'^2)^2+x'^2\gamma^2\right]}\,, \tag{4.6}$$

with the integrals now running over all x, x' from $-\infty$ to $+\infty$.

4.2. Approximations

We expand the delta function in (4.6) by powers of $\mathbf{v}\cdot\boldsymbol{\kappa}=v\kappa\cos\phi$. This is an asymptotic approximation that cannot reach beyond the first few terms; but where it works it is relatively simple, and liberates one from Matsubara expansions. The explicit cosϕ already in the integrand ensures that odd powers of v vanish, as they must because P cannot depend on the direction of $\mathbf{v}$. Derivatives of order j are indicated by superscripts (j), and we implement $\delta^{(1,3)}(x+x')$ as $[\partial/\partial x]^{1,3}\,\delta(x+x')$. In an obvious notation

$$f\simeq f_2+f_4,\qquad \mathcal{J}\simeq\mathcal{J}_2+\mathcal{J}_4,\qquad \mathcal{L}\simeq\mathcal{L}_2+\mathcal{L}_4. \tag{4.7}$$

The coefficient of v^2 is required up to order τ^2, and the coefficient of v^4 only at $\tau=0$. The calculations, fairly tedious, proceed through repeated integrations by parts; they are given in JPC.

To order v^2 one finds

$$\mathcal{L}_2=-v\int_{-\pi}^{\pi}d\phi\cos(\phi)\,(\mathbf{v}\cdot\boldsymbol{\kappa})\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dxdx'xx'\delta^{(1)}\left(x+x'\right)\left[\bar{N}(x)+\bar{N}(x')\right]}{\left[\left(x^2-x_p^2\right)^2+\gamma^2x^2\right]\left[(x'^2-x_m^2)^2+\gamma^2x'^2\right]}$$

$$\simeq\frac{4v^2\kappa}{\left[1-\exp(-2\kappa)\right]^2}\int_{-\pi}^{\pi}d\phi\cos^2(\phi)\int_0^\infty\frac{dx}{\exp(x/\tau)-1}=\frac{\nu^2\tau^2\left(2\pi^3/3\right)\kappa}{\left[1-\exp(-2\kappa)\right]^2}\,. \tag{4.8}$$

The approximation in the second step follows because $\tau\ll 1$ ensures that the integral is dominated by $|x|\,,|x'|\ll 1$. Accordingly

$$\mathcal{J}_2=\frac{1}{2}\int_0^\infty d\kappa\kappa^2e^{-2\kappa}\mathcal{L}_2\simeq\frac{\pi^2\zeta(3)}{8}\cdot v^2\tau^2,\qquad \zeta(3)\simeq 1.202. \tag{4.9}$$

To order v^4 one finds

$$\mathcal{L}_4 = \frac{v^4\kappa^3}{3!}\int_{-\pi}^{\pi} d\phi \cos^4(\phi) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dxdx'\delta\left(x+x'\right)\left(\partial/\partial x\right)^3\left\{xx'\left[\bar{N}(x)+\bar{N}(x')\right]\right\}}{\left[\left(x^2-x_p^2\right)^2+\gamma^2x^2\right]\left[(x'^2-x_m^2)^2+\gamma^2x'^2\right]} .$$

By virtue of (4.2b) this yields

$$\lim_{\tau\to 0}\mathcal{L}_4 = \frac{v^4\kappa^3}{3!\left[1-\exp(-2\kappa)\right]^2}\int_{-\pi}^{\pi} d\phi\cos^4(\phi) = \frac{v^4\kappa^3\left(\pi/8\right)}{\left[1-\exp(-2\kappa)\right]^2} , \tag{4.10}$$

whence

$$\mathcal{J}_4 = \frac{1}{2}\int_0^{\infty} d\kappa\kappa^2 e^{-2\kappa}\mathcal{L}_4 = \frac{15\pi\zeta(5)}{2^7}\cdot v^4, \qquad \zeta(5)\simeq 1.034. \tag{4.11}$$

Correspondingly, $f_4(\tau,0) = \mathcal{J}_4/2\pi^3 = 15\zeta(5)v^4/\pi^2 2^8$, confirming (4.4) and thereby (3.7).

The end-result follows from (4.3, 4.5a, 4.7, 4.9, 4.11):

$$P = \Pi\cdot\frac{v^2}{16}\left\{\frac{15\zeta(5)}{16\pi^2}\cdot v^2 + \zeta(3)\tau^2 + \mathcal{O}\left(v^4, v^2\tau^2, \tau^4\right)\right\} , \tag{4.12}$$

$$P \simeq \Pi v^4 \frac{15\zeta(5)}{256\pi^2}\left\{1+\frac{16\pi^2\zeta(3)}{15\zeta(5)}\left(\frac{\tau}{v}\right)^2\right\} . \tag{4.13}$$

We recall

$$\Pi v^4 = \frac{\hbar\gamma^2\beta^4u^4}{\omega_S^2\zeta^6} , \qquad \left[\frac{\gamma^2\beta^4}{\omega_S^2}\right]_{Drude} = \frac{1}{4\pi^2\bar{\sigma}^2} , \qquad \left(\frac{\tau}{v}\right)^2 = \left(\frac{k_BT\zeta}{\hbar u}\right)^2 , \tag{4.14}$$

and note $15\zeta(5)/256\pi^2 \simeq 0.006156$, $16\pi^2\zeta(3)/15\zeta(5)\simeq 12.20$.

4.3. The changes in occupation numbers

Looking more closely one can examine the changes $\Delta N(x)$ in the mean occupation numbers over times Δt short enough to be covered by the Golden Rule. They are governed by the dynamics of the excitation mechanism, and are readily found for any initial distribution N: inspecting the integrands of (4.5 - 4.6), and writing $\mathbf{v}\cdot\boldsymbol{\kappa} = a$, it is easily seen that

$$\Delta N_p/\Delta t = R(x,a-x)\left[\bar{N}_p(x)+\bar{N}_m(a-x)\right]_{t=0} = R(x,a-x)\left[\bar{N}(x)+\bar{N}(a-x)\right] , \tag{4.15}$$

$$\Delta N_m/\Delta t = R(a-x,x)\left[\bar{N}_p(a-x)+\bar{N}_m(x)\right]_{t=0} = R(a-x,x)\left[\bar{N}(a-x)+\bar{N}(x)\right] , \tag{4.16}$$

with the rate constant

$$R(x, a-x) = \frac{\left(\gamma^2\beta^4\omega_S/\pi\right)\exp(-2\kappa)x(a-x)}{\left[\left(x^2-x_p^2\right)^2+\gamma^2x^2\right]\left[\left((a-x)^2-x_m^2\right)^2+\gamma^2(a-x)^2\right]}\ . \tag{4.17}$$

The rightmost versions of (4.15, 4.16) apply because initially $N_p = N_m = \bar{N}$. The approximation (3.6) leads to

$$x, a-x \ll 1: \qquad R(x, a-x) \simeq \frac{\gamma^2\beta^4\omega_S}{4\pi\sinh^2(\kappa)}x(a-x). \tag{4.18}$$

We see that in this approximation, but not otherwise, $\Delta N_p \simeq \Delta N_m$ because $R(x, a-x) \simeq R(a-x, x)$.

Evidently the changes $\Delta N_{p,m}$ are nonthermal, ie they differ from the changes, call them $\tilde{\Delta}N = -x\hbar\omega_S\Delta B/4\sinh^2(x\hbar\omega_S B/2)$, the same for p and m, that would ensue from any change $\Delta B = -\Delta T/k_BT^2$. This becomes obvious on observing (a) that, through R, the ΔN depend on κ, while $\tilde{\Delta}N$ does not; and (b) that even for given κ the ΔN vary with x quite differently from $\tilde{\Delta}N$. In other words, it is impossible to reproduce $\tilde{\Delta}N_{p,m}$ by any choice of ΔB independent, as it would have to be, of κ and of x.

5. Comments and some open questions

(i) As υ rises from zero at fixed nonzero τ, we see from (4.12) that initially it is the temperature-dependent term that is the larger. This is our most remarkable conclusion: it reflects the apparently fortuitous vanishing of the term of order υ^2, which one would have expected to be the leading term in the expansion of P by powers of υ and of τ. The temperature-dependent component ceases to dominate when $\upsilon \sim \tau$.

(ii) To overall second order in V, ie perturbatively, $\mathcal{J}_2$ in (4.9) would lose the factor $\zeta(3)$, a reduction by 20%. Such differences are yet another measure of the importance of the correlations between fluctuations in L and in R: they are by no means negligibly weak (cf appendix A), least of all for the small ζ appropriate in the van der Waals regime. They suggest caution regarding current theories of *heat exchange* insofar as they assume that the statistical fluctuations in L and R are mutually independent (see eg Levin et al 1980, Pendry 1999, Janowicz et al 2003). In effect, such theories treat the gap as a very weak link: the review by Dubi & Di Ventra (2011) illustrates the snags arising in heat-flow problems when analogous assumptions are abandoned. Similar caution might be appropriate regarding theories of the attractive (conservative) Casimir force between L and

R, insofar as they assume that all of L is kept at a uniform temperature T_L and all of R at a different uniform temperature T_R (see eg Krüger et al 2011).

(iii) The writer thinks it likely that in the real world friction acting over any experimentally relevant time-scale will cause occupation numbers to evolve conformably to some temperature rise governed by the total energy already dissipated, rather than non-thermally according to section 4.3. This would require some secondary redistributive mechanism among the oscillators, of a kind not built into Huttner-Barnett-type models, not yet explored, and not at all easy to visualize. The problem stems from the fact that systems of bilinearly coupled oscillators are not ergodic: to understand how thermal equilibrium is maintained, or reached to begin with, one needs some not-too-unrealistic version of the speck of dust proverbially invoked for black-body radiation (see eg Tatarskiĭ 1987). The difficulty is that the interaction with an anodyne speck of dust is now replaced by the interaction with the Huttner-Barnett heat reservoir, whose detailed dynamics apparently begin to matter. For instance, JPC speculates that the mechanism in question might feature more complicated couplings of the zero-order bath oscillators to the zero-order surface and bulk plasmons, and of the latter to each other; in any case one would need to consider to what extent and how the dissipated energy diffuses from the surface into the bulk of the material.

(iv) In the theory we have used, Γ and $\bar{\sigma}$ are independent of the temperature, because they parametrize the Hamiltonian, which is independent of T by the nature of things. The same is true of the dissipative parameter(s) featuring in Philbin's model, which reproduces (but on an intelligible Hamiltonian basis) the predictions of what is commonly called Lifshitz theory. This leaves open the question of what physical significance might attach to predictions of low-temperature behaviour, when the parameters they feature are assigned their observed T - dependence.

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A. Correlations between L and R

To illustrate how strong these can be we consider the Fourier transforms of the surface-charge densities $\Sigma_{\mp}(\mathbf{s}) = \int d^2k \Sigma_{\mp}(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{s})$ on L and R respectively (when *both* are present), in the simple case of zero-temperature nondissipative metals ($T = 0$, $\Gamma = 0$, $\omega_0 = 0$). We define correlation functions G by

$$\langle \Sigma_-(\mathbf{k})\Sigma_-(\mathbf{k}')\rangle = \delta(\mathbf{k}+\mathbf{k}')\frac{k\hbar\omega_S}{(2\pi)^3}G_{--}, \qquad \langle \Sigma_-(\mathbf{k})\Sigma_+(\mathbf{k}')\rangle = \delta(\mathbf{k}+\mathbf{k}')\frac{k\hbar\omega_S}{(2\pi)^3}G_{-+}, \tag{A.1}$$

and compare the G's for $\kappa = k\zeta \ll 1$ (noting that $k \to 0$ and $\zeta \to \infty$ are incompatible). Formulae from Barton (1997) yield

$$G_{--} = (1/\sqrt{\kappa})\left\{2 - 3\kappa/2 + 49\kappa^2/48 + \kappa^3/2\sqrt{2} + ...\right\}, \tag{A.2}$$

$$G_{-+} = (1/\sqrt{\kappa})\left\{-2 + 3\kappa/2 - 49\kappa^2/48 + \kappa^3/2\sqrt{2} + ...\right\}. \tag{A.3}$$

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