Energy-Momentum Tensors with Worldline Numerics

Marco Schäfer

Theoretisch-Physikalisches Institut
Research Training Group (1523) 'Quantum and Gravitational Fields'
Friedrich-Schiller-University Jena
with Holger Gies and Idrish Huet Hernandez

QFEXT 2011
Benasque, 22 September, 2011

Research Training Group
Quantum and Gravitational Fields

seit 1558

massless scalar field Φ in $D = d + 1$ dimensions defined on domain $\mathcal{D}$ with background potential σ has the canonical EMT

$$\hat{T}_{\mu\nu}(x,t) := \lim_{x \to x'} \left[\partial_\mu \Phi \partial'_\nu \Phi' - \frac{1}{2} g_{\mu\nu} \left(\partial_\alpha \Phi \partial'^\alpha \Phi' - \sigma(x) \Phi \Phi' \right) \right]$$

$\left\langle \hat{T}_{\mu\nu}(x,t) \right\rangle$ is part of the source term in the semi-classical Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa \left\langle \hat{\Theta}_{\mu\nu}(x,t) \right\rangle = \kappa \left\langle \hat{T}_{\mu\nu} + \Delta \hat{T}_{\mu\nu} \right\rangle$$

Negative energy densities are undesirable !

classically: use energy conditions to avoid negative energy densities

but: QFT can violate (some) energy conditions [1] [2] [3]

So far ANEC is obeyed !

[1] D. Schwartz-Perlov, K.D. Olum, Phys. Rev. D **68** 065016 (2003)

[2] N. Graham, K.D. Olum, D. Schwartz-Perlov, Phys. Rev. D **70** 105019

[3] C.J. Fewster, K.D. Olum, M.J. Pfenning, Phys. Rev. D **75** 025007 (2007)

massless scalar field Φ in $D = d + 1$ dimensions defined on domain $\mathcal{D}$ with background potential σ has the canonical EMT

$$\hat{T}_{\mu\nu}(x,t) := \lim_{x \to x'} \left[\partial_\mu \Phi \partial'_\nu \Phi' - \frac{1}{2} g_{\mu\nu} \left(\partial_\alpha \Phi \partial'^\alpha \Phi' - \sigma(x) \Phi \Phi' \right) \right]$$

$\left\langle \hat{T}_{\mu\nu}(x,t) \right\rangle$ is part of the source term in the semi-classical Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa \left\langle \hat{\Theta}_{\mu\nu}(x,t) \right\rangle = \kappa \left\langle \hat{T}_{\mu\nu} + \Delta \hat{T}_{\mu\nu} \right\rangle$$

Negative energy densities are undesirable !

classically: use energy conditions to avoid negative energy densities

but: QFT can violate (some) energy conditions [1] [2] [3]

So far ANEC is obeyed !

[1] D. Schwartz-Perlov, K.D. Olum, Phys. Rev. D **68** 065016 (2003)

[2] N. Graham, K.D. Olum, D. Schwartz-Perlov, Phys. Rev. D **70** 105019

[3] C.J. Fewster, K.D. Olum, M.J. Pfenning, Phys. Rev. D **75** 025007 (2007)

Averaged Null Energy Condition: $\int \left\langle \hat{T}_{\mu\nu} V^\mu V^\nu \right\rangle \mathrm{d}\lambda \geq 0$

expand $\Phi(x,t)$: $\Phi(x,t) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{\sqrt{2E_p}} \left(\psi_p(x) e^{iE_p t} \hat{a}_p + h.c. \right)$

$$\Longrightarrow \left(-\vec{\nabla}^2 - p^2 + \sigma(x) \right) \psi_p(x) = 0 \qquad G(x,x',k) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\psi_p(x)\psi_p^*(x')}{p^2 - k^2 - i\varepsilon}$$

- choose z-direction: $V^\mu = (1,0,\ldots,0,1)$
- let σ impose Dirichlet BC on $\partial\mathcal{D}$

$$\langle \hat{T}_{00}(\vec{x},t) \rangle = \lim_{\vec{x}\to\vec{x}'} \int \frac{\mathrm{d}k}{\pi} \left(k^2 + \frac{1}{2} \vec{\nabla} \cdot \left(\vec{\nabla} + \vec{\nabla}' \right) \right) \mathrm{Im}\left[(G - G_0)(\vec{x},\vec{x}',k) \right]$$

$$\langle \hat{T}_{zz}(\vec{x},t) \rangle = \lim_{\vec{x}\to\vec{x}'} \int \frac{\mathrm{d}k}{\pi} \left(\partial_z \partial_{z'} - \frac{1}{2} \vec{\nabla} \cdot \left(\vec{\nabla} + \vec{\nabla}' \right) \right) \mathrm{Im}\left[(G - G_0)(\vec{x},\vec{x}',k) \right]$$

- use worldline formalism for evaluation of $(G - G_0)(\vec{x},\vec{x}',k)$

What is the worldline formalism?

Feynman 1950[4] - alternative description of Klein-Gordon field

idea: map QFT amplitudes on quantum mechanical path integrals over paths of quantum fluctuations

$\Longrightarrow$ this is particularly suitable for the treatment of the influence of external conditions[5] [6]

What is the advantage of the worldline formalism?

- works for arbitrary background potentials
- spacetime remains continuous
- numerics only needed for renormalized quantities[7] [8]

[4] R.P. Feynman, Phys. Rev. **80** (1950) 440

[5] C. Schubert, Phys. Rep. **355** (2001) 73-234

[6] C. Schubert, AIPConf. Proc. **917** 178-194 (2007)

[7] H. Gies, K. Langfeld, L. Moyaerts, JHEP **06** (2003) 018

[8] H. Gies, J. Sanchez-Guillen, R.A. Vazquez, JHEP **08** (2005) 067

Introduction
worldline formalism

- 1-loop effective actions, e.g. for Klein-Gordon field (Euclidean spacetime)

$$\Gamma_{1l}[\sigma] = \int_0^\infty \frac{\mathrm{d}T}{2T}\, e^{-Tm^2} \int \mathrm{d}x_{cm} \oint_{x_{cm}} \mathcal{D}y(\tau) e^{-\int_0^T \mathrm{d}\tau \frac{\dot{y}^2}{4}} \left(e^{-\int_0^T \mathrm{d}\tau\, \sigma(y)} - 1 \right)$$

- Green's functions of fields, e.g. for Helmholtz equation

$$G(x, x', k) = i \int_0^\infty \mathrm{d}s e^{isk^2} \int_{x'}^{x} \mathcal{D}y(\tau) e^{i \int_0^s \mathrm{d}\tau \left(\frac{\dot{y}^2}{4} - V(y) \right)}$$

in numerics:

1. approximate integrals with sums:
   infinitely many paths $\rightarrow$ finite ensemble of loops/lines
2. discretize paths:
   infinitely many points on path $\longrightarrow$ finitely many points per loop/line (ppl)
3. rescaling $y \rightarrow \sqrt{T} y$ makes weight factor independent of T (unit loops)

The Casimir effect on the worldline

- numerical calculations for a massless Klein-Gordon-field

$$\mathcal{E}_{Casimir} = \frac{\Gamma_{1l}[\sigma]}{\int \mathrm{d}x^0} = \frac{1}{2(4\pi)^{\frac{D}{2}}} \int_0^\infty \frac{\mathrm{d}T}{T^{\frac{D+2}{2}}} \int \mathrm{d}^{D-1}x_{cm} \langle e^{-\int_0^T \mathrm{d}\tau\, \sigma(y)} - 1 \rangle_{x_{cm}}$$

- for Dirichlet BCs[9] [10]: $\sigma = \lambda \cdot \delta(x_{plate})$ with $\lambda \to \infty$

$$e^{-\int_0^T \mathrm{d}\tau\, \sigma(y)} - 1 = \left\{ \begin{array}{ll} -1 & y \text{ intersects all boundaries} \\ 0 & \text{else} \end{array} \right\} = -\Theta\left(T - \hat{T}\right)$$

- curvature and edge effects can be easily investigated
- examples: (semi-)infinite plate(s), cylinder/sphere above plate, perpendicular plates (Casimir comb), ...[11]

[9] N. Graham, R.L. Jaffe, et al., Nucl. Phys. B **677** (2004) 379-404

[10] N. Graham, R.L. Jaffe, et al., Phys. Lett. B **572** (2003) 196-201

[11] H. Gies, K. Klingmüller, Phys.Rev. D **75** 045002 (2006)

Green's function for Helmholtz equation

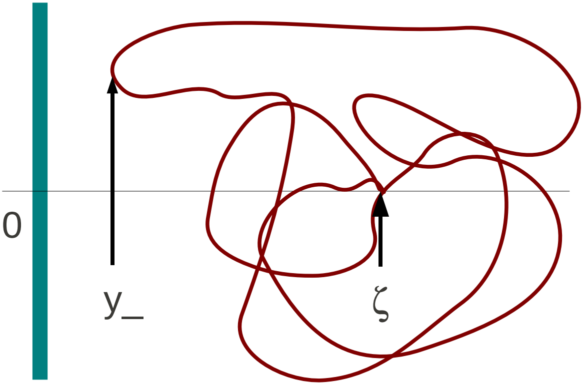
$$G(x,x',k) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\psi_p(x)\psi_p^*(x')}{p^2 - k^2 - i\varepsilon}$$
$$= i \int_0^\infty \mathrm{d}s\, e^{isk^2} \int_{x'}^{x} \mathcal{D}y(\tau) e^{i\int_0^s \mathrm{d}\tau \left(\frac{\dot{y}^2}{4} - \sigma(y)\right)}$$

$$\langle \hat{T}_{00}(\vec{x},t)\rangle = \frac{1}{(4\pi)^{\frac{d+1}{2}}} \int_0^\infty \frac{\mathrm{d}T}{2T^{\frac{d+1}{2}}} \left(\frac{1}{T} - \frac{1}{2}\vec{\nabla}^2\right) e^{-\frac{(\vec{x}-\vec{x}')^2}{4T}} \left\langle \Theta\left(T - \hat{T}\right)\right\rangle$$

$$\langle \hat{T}_{zz}(\vec{x},t)\rangle = -\frac{1}{(4\pi)^{\frac{d+1}{2}}} \int_0^\infty \frac{\mathrm{d}T}{2T^{\frac{d+1}{2}}} \left(\partial_z\partial_{z'} - \frac{1}{2}\vec{\nabla}^2\right) e^{-\frac{(\vec{x}-\vec{x}')^2}{4T}} \left\langle \Theta\left(T - \hat{T}\right)\right\rangle$$

with $\hat{T} = \hat{T}(z,z')$

- plate at origin of z-axis, measure distances in dimensionless variable ζ
- y_- is point of loop closest to the plate
- $\sqrt{T} y_- + \zeta \leq 0 \implies \Theta\left(T - \hat{T}\right) = \Theta\left(T - \frac{\zeta^2}{y_-^2}\right)$



energy density for a single plate in $d = 2$ at $\zeta = 0$

$$\left\langle \hat{T}_{00} \right\rangle = \frac{1}{(4\pi)^{\frac{3}{2}}} \frac{1}{(\mathrm{a}\zeta)^3} \left(\frac{\langle |y_-|^3 \rangle}{3} - \langle |y_-| \rangle \right)$$

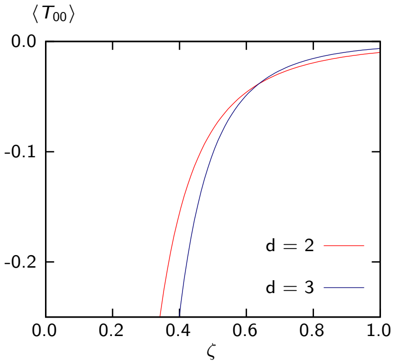
$$= \frac{1}{(4\pi)^{\frac{3}{2}}} \frac{1}{(\mathrm{a}\zeta)^3} \left(\frac{\sqrt{\pi}}{4} - \frac{\sqrt{\pi}}{2} \right)$$

ppl, N	$\langle \lvert y_- \rvert \rangle$	$\frac{\sqrt{\pi}}{2}$	$\langle \lvert y_- \rvert^3 \rangle$	$\frac{3\sqrt{\pi}}{4}$
2^{14}, 10^4	0.8789 ± 0.0046	0.8862	1.3018 ± 0.0210	1.3293
2^{18}, $5 \cdot 10^5$	0.8841 ± 0.0007	0.8862	1.3236 ± 0.0029	1.3293

energy density for a single plate in $d = 3$ at $\zeta = 0$

$$\left\langle \hat{T}_{00} \right\rangle = \frac{1}{(4\pi)^2}\frac{1}{(\mathrm{a}\zeta)^4}\left(\frac{\langle|y_-|^4\rangle}{4} - \frac{3\langle|y_-|^2\rangle}{2}\right)$$
$$= \frac{1}{(4\pi)^2}\frac{1}{(\mathrm{a}\zeta)^4}\left(\frac{1}{2} - \frac{3}{2}\right)$$

ppl, N	$\langle\|y_-\|^2\rangle$	$\langle\|y_-\|^4\rangle$
2^{14}, 10^4	0.9841 ± 0.0100	1.9610 ± 0.0488
2^{18}, $5 \cdot 10^5$	0.9964 ± 0.0014	1.9901 ± 0.0063



$$\left\langle \hat{T}_{00} \right\rangle_{\mathrm{d}=2} = -\frac{1}{32\pi} \frac{1}{(\mathrm{a}\zeta)^3}$$

$$\left\langle \hat{T}_{00} \right\rangle_{\mathrm{d}=3} = -\frac{1}{16\pi^2} \frac{1}{(\mathrm{a}\zeta)^4}$$

conclusions

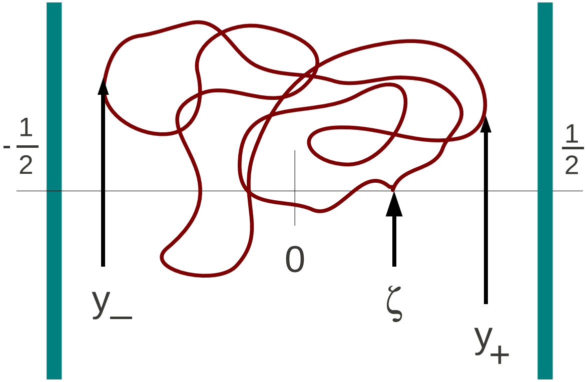
- negative energy density, divergent as $\zeta \to 0$
- $\langle |y_-|^p \rangle \to \Gamma\left[\frac{p+2}{2}\right]$ (compatible with results for Casimir effect[11])
- null energy condition:

$$\left\langle \hat{T}_{00} + \hat{T}_{zz} \right\rangle = \frac{1}{(4\pi)^{\frac{d+1}{2}}} \frac{1}{(\mathrm{a}\zeta)^{d+1}} \left(\frac{\langle |y_-|^{d+1} \rangle}{d+1} - \frac{d}{2} \langle |y_-|^{d-1} \rangle \right)$$

- NEC violated: $\left\langle \hat{T}_{00} + \hat{T}_{zz} \right\rangle < 0$ everywhere, also diverges as $\zeta \to 0$
- but: ANEC might still be obeyed,
  contribution of the plate $\to +\infty$ as $\sigma \to \infty$

[11] H. Gies, K. Klingmüller, Phys.Rev. D **75** 045002 (2006)

- plates at z$= \pm\frac{a}{2}$, i.e. at $\zeta = \pm\frac{1}{2}$
- y_- and y_+ are points of loop closest to the plates
- $\sqrt{T} y_\pm + (\zeta \mp \frac{1}{2}) \leq 0 \implies \Theta\left(T - \hat{T}\right) = \Theta\left(T - \min\left[\left(\frac{\zeta \pm \frac{1}{2}}{y_\mp}\right)^2\right]\right)$



evaluate $T_{00}(\mathrm{I})$ in $d = 2$ and $d = 3$

$$\left\langle \hat{T}_{00} \right\rangle (\mathrm{I}) = \frac{1}{(4\pi)^{\frac{d+1}{2}}} \int_0^\infty \frac{\mathrm{d}T}{2T^{\frac{d+1}{2}}} \frac{1}{T} e^{-\frac{(\vec{x}-\vec{x}')^2}{4T}} \left\langle \Theta\left(T - \hat{T}\right)\right\rangle$$

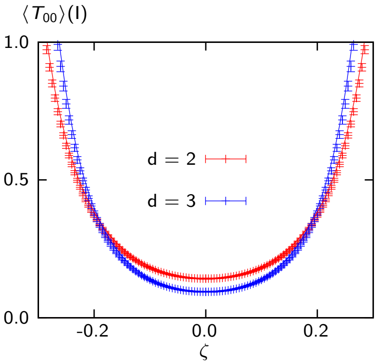
$$\left\langle \hat{T}_{00} \right\rangle_{\mathrm{d}=2} (\mathrm{I}) = -\frac{1}{32\pi} \frac{1}{\mathrm{a}^3} \left(2\zeta_R(3) - \zeta_H\left(3, \frac{1}{2} + \zeta\right) - \zeta_H\left(3, \frac{1}{2} - \zeta\right)\right)$$

$$\left\langle \hat{T}_{00} \right\rangle_{\mathrm{d}=3} (\mathrm{I}) = -\frac{1}{32\pi^2} \frac{1}{\mathrm{a}^4} \left(2\zeta_R(4) - \zeta_H\left(4, \frac{1}{2} + \zeta\right) - \zeta_H\left(4, \frac{1}{2} - \zeta\right)\right)$$

- divergent as $\zeta \to \pm\frac{1}{2}$ (canonical EMT !)
- adding *Huggins*-term $\Delta T_{\mu\nu}$ renders EMT finite and constant[12] [13]

[12] K. Tywoniuk, F. Ravndal, arXiv:quant-ph/0408163v2

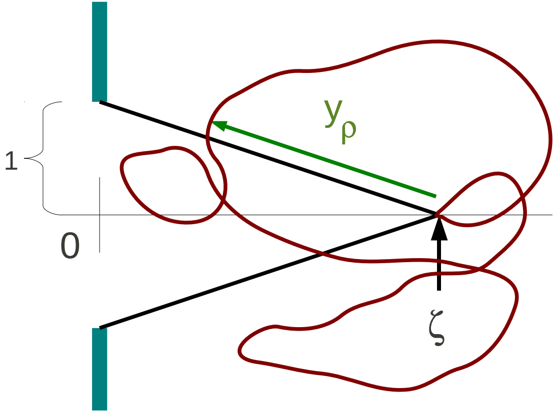
[13] K. Milton, Phys. Rev. D **68** 065020 (2003)



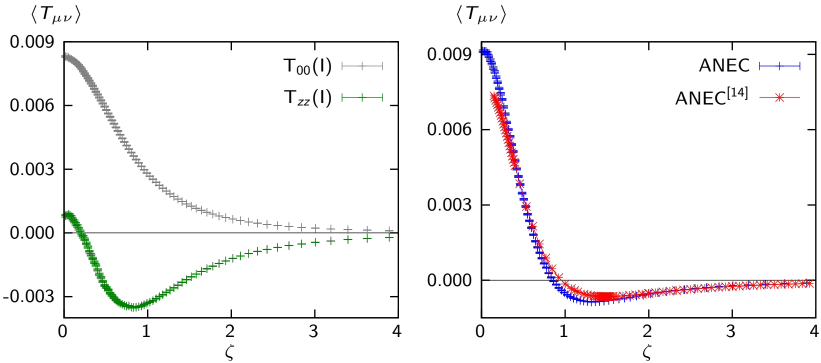
conclusions

- our algorithms work
- for ANEC choose setup where geodesic does not pass through boundary

- two-part boundary at z=0 in x-direction ($\partial\mathcal{D} = x \in (-\infty, -a) \cup (a, \infty)$)
- preliminary algorithm: loop always intersects closest point first
- distance to closest point: $\rho = \sqrt{\zeta^2 + 1}$
- $\sqrt{T} y_\rho + \rho = 0 \implies \Theta\left(T - \hat{T}\right) = \Theta\left(T - \frac{\rho^2}{y_\rho^2}\right)$



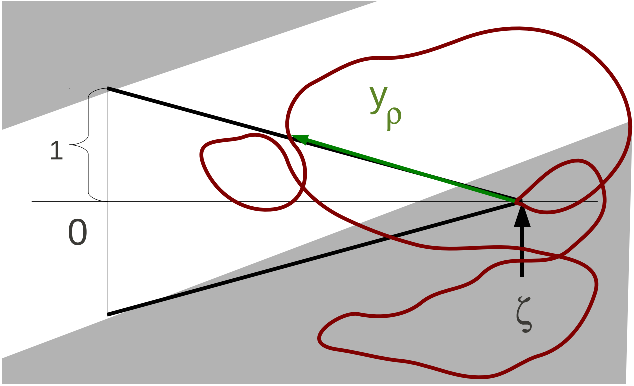
ANEC for configurations with holes
ANEC for punctured plate in d=2



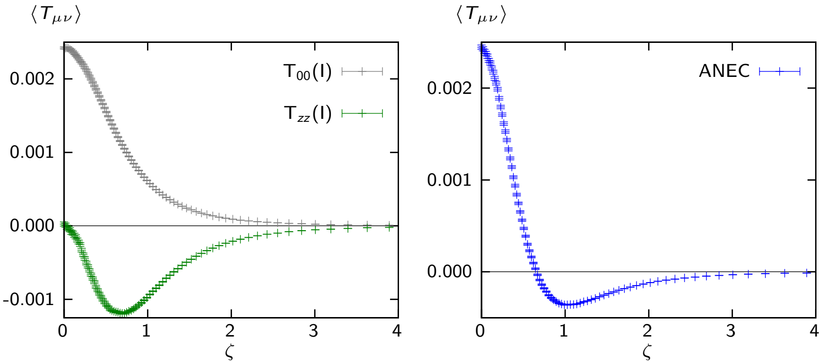
$$\left\langle \hat{T}_{zz} \right\rangle (\mathrm{I}) = -\frac{1}{(4\pi)^{\frac{d+1}{2}}} \int_0^\infty \frac{\mathrm{d}T}{2T^{\frac{d+1}{2}}} \partial_z \partial_{z'} e^{-\frac{(\vec{x}-\vec{x}')^2}{4T}} \left\langle \Theta \left(T - \hat{T} \right) \right\rangle$$

[14] N. Graham, K.D. Olum, Phys. Rev. D **72** 025013 (2005)

- two-part boundary at z=0 in x-y-plane
  $(\partial\mathcal{D} = (x, y) \in (-\infty, -a) \cup (a, \infty) \times (-\infty, \infty))$
- preliminary algorithm: loop always intersects closest point first
- distance to closest point: $\rho = \sqrt{\zeta^2 + 1}$
- $\sqrt{T} y_\rho + \rho = 0 \implies \Theta\left(T - \hat{T}\right) = \Theta\left(T - \frac{\rho^2}{y_\rho^2}\right)$



ANEC for configurations with holes
ANEC for plates with a slit in d=3



Summary/Conclusions

1. worldline formalism generalized to composite operators, e.g. EMT
2. systematic exploration of geometry dependence of EMT components
3. investigation of energy conditions in different setups

future projects

- improve/extend results for ANEC for punctured plate
- ANEC for 2 punctured Casimir plates
- What happens to ANEC when plates are thick? What happens when edges are rounded? ...

Thank you for your attention!