Casimir forces via stochastic quantization

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1. Casimir forces and stress tensor
2. Parisi-Wu formalism of stochastic quantization
3. Ultraviolet divergences
4. Applications
   - Piston of arbitrary cross section
   - Force fluctuations
   - Numerical calculation in the torus-sphere geometry
5. Conclusions

The Casimir force can be calculated averaging the the stress tensor, $\mathbb{T}$ on the quantum-thermal probability distribution of the fields ϕ.

The stress tensor is a bilinear form $\mathbb{T} = \boldsymbol{\mathcal{T}}[\phi, \phi, \mathbf{r}]$.

With a single scalar field satisfying Dirichlet boundary conditions.

$$T_{ik} = \frac{1}{2}\delta_{ik}\left(\nabla\phi\right)^2 - \frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_k}$$

and therefore the bilinear form is

$$\boldsymbol{\mathcal{T}}_{ik}[\psi, \varphi, \mathbf{r}] = \frac{1}{2}\delta_{ik}\left(\nabla\psi \cdot \nabla\varphi\right) - \frac{\partial\psi}{\partial x_i}\frac{\partial\varphi}{\partial x_k}$$

The extension to vectorial Electromagnetism is direct, considering the Transverse Electric and the Transverse Magnetic decomposition.

Casimir force via the stress tensor

The fields ϕ display the quantum-thermal probability distribution

$$P[\phi] = Z^{-1} e^{-S[\phi]/\hbar}$$

where $S[\phi]$ is the action, Wick-rotated in the time variable ($t = i\tau$).

In the case of a scalar field with zero mass

$$S[\phi] = -\frac{1}{2} \int_0^{\beta\hbar} d\tau \int d\mathbf{r}\, \phi \left(\frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} + \nabla^2 \right) \phi$$

$$Z = \int D\phi\, e^{-S[\phi]/\hbar}$$

is the partition function.

For the bosonic case, $\phi(\tau + \beta\hbar, \mathbf{r}) = \phi(\tau, \mathbf{r})$.

Parisi-Wu formalism

The probability distribution can be built via a fictitious stochastic process. A Langevin equation is written in an auxiliary time s: $\phi(\tau, \mathbf{r}) \to \phi(\tau, \mathbf{r}; s)$

$$\begin{aligned} \frac{\partial \phi(\tau, \mathbf{r}; s)}{\partial s} &= -\frac{\delta S[\phi]}{\delta \phi} + \eta(\tau, \mathbf{r}; s) \\ &= \left(\frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} + \nabla^2 \right) \phi + \eta(\tau, \mathbf{r}; s) \end{aligned}$$

The term $\eta(\tau, \mathbf{r}; s)$ is a Gaussian white noise

$$\begin{aligned} \langle \eta(\tau, \mathbf{r}; s) \rangle &= 0 \\ \langle \eta(\tau, \mathbf{r}; s) \eta(\tau', \mathbf{r}'; s') \rangle &= 2 k_B T \delta(\tau - \tau') \delta(\mathbf{r} - \mathbf{r}') \delta(s - s') \end{aligned}$$

The solution of the Langevin equation in the limit $s \to \infty$ reproduces the probability distribution.

For a given geometry (and BC), the field ϕ and the noise are expanded

$$\phi(\tau, \mathbf{r}; s) = \sum_{n,m} \phi_{nm}(s) g_m(\tau) f_n(\mathbf{r})$$

with

$$\nabla^2 f_n(\mathbf{r}) = -\lambda_n^2 f_n(\mathbf{r}), \quad \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2} g_m(\tau) = -\omega_m^2 g_m(\tau)$$

Considering a bosonic field that obeys periodic boundary conditions in τ, the eigenvalues are the Matsubara frequencies $\omega_m = 2\pi m/\beta\hbar c$, $m \in \mathbb{Z}$ and $g_m(\tau) = \exp(-i\omega_m \tau)$.

The equation

$$\frac{\partial\phi(\tau,\mathbf{r};s)}{\partial s} = \left(\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2} + \nabla^2\right)\phi + \eta(\tau,\mathbf{r};s)$$

reduces to

$$\frac{d\phi_{nm}(s)}{ds} = -\left[\lambda_n^2 + \omega_m^2\right]\phi_{nm}(s) + \eta_{nm}(s)$$

which can be integrated to give

$$\phi_{nm}(s) = \int_{-\infty}^{s} d\sigma e^{(\lambda_n^2+\omega_m^2)(\sigma-s)}\eta_{nm}(\sigma),$$

Finally, the field

$$\phi(\tau,\mathbf{r};s) = \sum_{n,m}\phi_{nm}(s)\exp(-i\omega_m\tau)f_n(\mathbf{r})$$

reproduces the probability distribution.

Substituting and computing in the limit $s \to \infty$.

$$\begin{aligned}\langle \mathbb{T}(\mathbf{r}) \rangle &= \langle \mathcal{T}[\phi, \phi, \mathbf{r}] \rangle = \sum_{n_1, m_1, n_2, m_2} \langle \phi_{n_1 m_1} \phi^*_{n_2, m_2} \rangle \mathcal{T}[f_{n_1}, f^*_{n_2}, \mathbf{r}] \\ &= \frac{1}{\beta} \sum_{nm} \frac{\mathcal{T}_{nn}(\mathbf{r})}{\lambda_n^2 + \omega_m^2}\end{aligned}$$

where $\mathcal{T}_{nm}(\mathbf{r}) = \mathcal{T}[f_n, f^*_m, \mathbf{r}]$
Summing over ω_m

$$\langle \mathbb{T}(\mathbf{r}) \rangle = \frac{\hbar c}{2} \sum_n \frac{\mathcal{T}_{nn}(\mathbf{r})}{\lambda_n} \left[1 + \frac{2}{e^{\beta \hbar c \lambda_n} - 1} \right]$$

The total force over a body is

$$\mathbf{F}_C = \oint_\Omega \langle \mathbb{T}(\mathbf{r}) \rangle \cdot d\mathbf{S}$$

$$\langle \mathbb{T}(\mathbf{r}) \rangle = \frac{\hbar c}{2} \sum_n \frac{\mathcal{T}_{nn}(\mathbf{r})}{\lambda_n} \left[1 + \frac{2}{e^{\beta \hbar c \lambda_n} - 1} \right]$$

In the limit of vanishing temperature or high temperature the stress tensor reduces to

$$\lim_{T \to 0} \langle \mathbb{T}(\mathbf{r}) \rangle = \frac{\hbar c}{2} \sum_n \frac{\mathcal{T}_{nn}(\mathbf{r})}{\lambda_n}$$

$$\lim_{\hbar \to 0} \langle \mathbb{T}(\mathbf{r}) \rangle = \frac{1}{\beta} \sum_n \frac{\mathcal{T}_{nn}(\mathbf{r})}{\lambda_n^2}$$

To compute the Casimir force, the spectral decomposition of the Laplacian is needed.

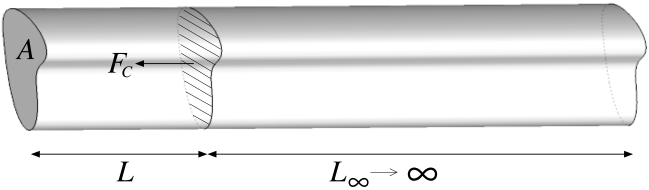
Regularization of the ultraviolet divergences

Eigenvalues of the Laplacian: $\lambda \sim |\vec{k}|$
Contribution of each mode to the stress tensor: $\mathcal{T} \sim k^2$
Therefore:

$$\langle \mathbb{T}(\mathbf{r}) \rangle = \frac{\hbar c}{2} \sum_n \frac{\mathcal{T}_{nn}(\mathbf{r})}{\lambda_n} \left[1 + \frac{2}{e^{\beta \hbar c \lambda_n} - 1} \right]$$

is divergent for large wavevectors.
The ultraviolet divergence does not contribute to net forces.

Needs regularization.



The plates and the mantle are metallic.
On each cylinder (L finite and $L_\infty \to \infty$) there are transverse electric and transverse magnetic modes.

$$\int_{1\ \text{side}} \langle \mathbb{T}_{xx} \rangle dS_x = \frac{2}{\beta L} \sum_{m\in\mathbb{Z}} \sum_{n_x=1}^{\infty} \sum_{n} \frac{k_x^2}{\omega_m^2 + k_x^2 + \lambda_n^2}$$

where $k_x^2 = (n_x \pi / L)^2$ and λ_n^2 are the 2D Laplace eigenvalues with Dirichlet and Neumann BC on the perimeter.

Cylinder of arbitrary cross section

The sum is divergent. Using the Chowla-Selberg summation formula: the divergent L-independent contribution is separated from the convergent L-dependent part.

Subtracting the contributions from both sides of he plate

$$F_C = -\frac{1}{\beta} \sum_p \sum_{m \in \mathbb{Z}} \frac{\sqrt{\omega_m^2 + \lambda_p^2}}{e^{2L\sqrt{\omega_m^2+\lambda_p^2}} - 1}; \qquad \omega_m = 2\pi m/\beta\hbar c$$

In the limit $T \to 0$, the sum over m can be replaced by an integral

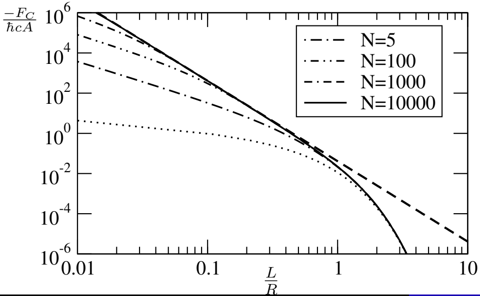
$$\lim_{T\to 0} F_C = -\frac{\hbar c}{2\pi} \sum_p \sum_{n=1}^{\infty} \lambda_p^2 \left[K_0(2nL\lambda_p) + K_2(2nL\lambda_p)\right]$$

At short distances, the sum over p can be replaced by an integral using the Laplacian density of states $\rho(\lambda_p) = \frac{A}{2\pi}\lambda_p$

$$\lim_{T\to 0} F_C = -\frac{\hbar c \pi^2}{240 L^4} A$$

At large distances, larger eigenvalues are exponentially suppressed

$$\lim_{T\to 0} F_C = -\frac{\hbar c}{2\sqrt{\pi L}} g_1 \lambda_1^{3/2} e^{-2L\lambda_1}$$



At intermediate distances (summing numerically the eigenvalues for a circular piston)

$$\sigma_F^2 = \oint_\Omega \oint_\Omega \langle [\mathbb{T}(\mathbf{r}_1)\cdot d\mathbf{S}_1][\mathbb{T}(\mathbf{r}_2)\cdot d\mathbf{S}_2]\rangle - F_C^2; \quad F_C = \oint_\Omega \langle \mathbb{T}(\mathbf{r})\cdot d\mathbf{S}\rangle$$

Gaussian noise allows factorization of the four-field terms.

$$\langle \mathbb{T}(\mathbf{r}_1)\mathbb{T}(\mathbf{r}_2)\rangle = \frac{(\hbar c)^2}{4}\sum_{nm} P(\lambda_n)P(\lambda_m)\left[\mathcal{T}_{nn}(\mathbf{r_1})\mathcal{T}_{mm}(\mathbf{r_2}) + 2\mathcal{T}_{nm}(\mathbf{r_1})\mathcal{T}_{mn}(\mathbf{r_2})\right]$$

where

$$P(\lambda) = \frac{1}{\lambda}\left[1 + \frac{2}{e^{\beta\hbar c\lambda} - 1}\right]$$

In the case of planar geometry

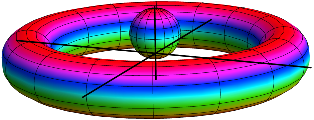
$$\boxed{\sigma_F^2 = 2F_C^2; \quad \forall T}$$

The variance is finite, independent of the regularizing procedure.

Sphere aligned with a torus

Torus: Large radius R_1, small radius R_2

Sphere: Radius R_3, height H.



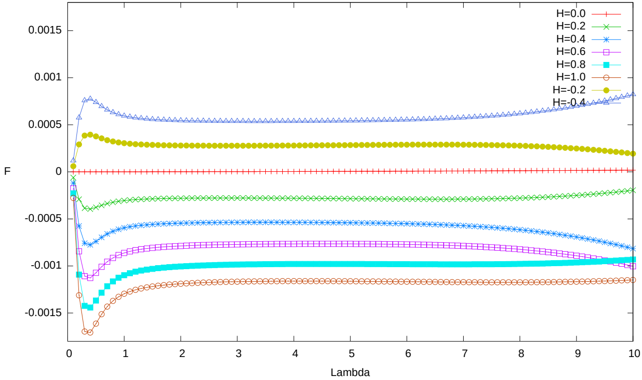
Using the FreeFEM++ software, the eigenvalues and eigengunctions are computed numerically in cylindrical coordinates.

A kernel regularization is applied to compute the force on each object

$$F_C^{\mathrm{reg}} = \frac{\hbar c}{2} \sum_n \frac{K(\lambda_n/\Lambda)}{\lambda_n} \oint_\Omega \mathcal{T}_{nn}(\mathbf{r}) \cdot d\mathbf{S}$$

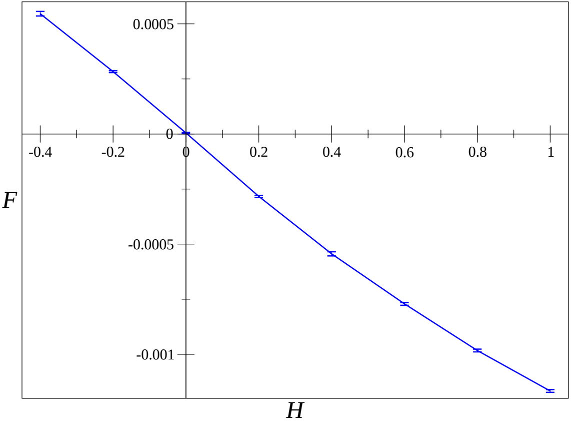
where $K(x)$ is a regularizing kernel (e.g. $K(x) = e^{-x^2}$) and Λ is the cutoff eigenvalue.

The sum converges when $\Lambda \to \infty$.
There are numerical errors at at large eigenvalues.



Numerical details: Grid size: 100×100.
Eigenvalues by ARPACK (Implicitly Restarted Arnoldi Method).

The force is obtained in the *plateau*



Geometrical parameters:

Torus: $R_1 = 1,\ R_2 = 5$

Sphere: $R_3 = 1,\ H = 0, \ldots, 1{,}0.$

- The stochastic quantization method allows to compute the average stress due to quantum-thermal fluctuations
- Integrating the stress over the surface bodies gives the Casimir force
- The computation needs only the spectral decomposition of the Laplacian in a given geometry
- The force on a piston of arbitrary cross section is obtained at any temperature as a function of the 2D Laplace eigenvalues
- The force fluctuations can be computed; in the case of a planar geometry a universal result is obtaine
- The method is amenable for numerical computations

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